Lecture 11:

Recall examples that are cts. at 0 but not diffble. at 0.

Theorem 1 (p. 109): If f is diffble at x, then f is cts at x.

Proof:

$$\lim_{h\to 0} f(x+h) - f(x) = \lim_{h\to}\left(\frac{f(x+h)-f(x)}{h}\right)h = f'(x)0 = 0.$$

This is equivalent to saying $\lim_{h\to 0} f(x+h) = f(x)$ which is equivalent to continuity of f at x:

Compare with $\lim_{x\to x_0} f(x) = f(x_0)$ $\square$

Continuity means the graph is unbroken.

Differentiability means the graph is smooth.

Right and left handed derivatives:

Defn: $f'_{\pm}(x) = \lim_{h\to 0^{\pm}} \frac{f(x+h)-f(x)}{h}$

Fact: f is diffble. at x iff $f'_{+}(x) = f'_{-}(x)$ (exist as a real number).

Defn: If a is an endpoint (say left endpoint) of an interval in the domain of f, we define $f'(x) = f'_{+}(x)$

Differentiability Rules: Let f and g be diffble. at x.

Constants: $f(x) = c, f'(x) = 0$. (already done)

Identity: $f(x) = x, f'(x) = 1$. (already done)

Sum/Difference: $(f \pm g)'(x) = f'(x) \pm g'(x)$

Product Rule: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Proof:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) + f(x)g(x)}{h}$$

$$= lim_{h\to 0} \frac{f(x+h) - f(x)}{h}) g(x+h) + lim_{h\to 0} f(x) \frac{g(x+h) - g(x)}{h})$$

$$= lim_{h\to 0} \frac{f(x+h) - f(x)}{h}) lim_{h\to 0} g(x+h) + f(x) lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

(since diffble $\Rightarrow$ cts) $\square$

Power Rule: Let $f(x) = x^n$ where n is a positive integer. Then $f'(x) = nx^{n-1}$.

Proof: by induction on n. True for $n = 0$ by derivative of constant.

Assume true for n. Then $(x^n)' = nx^{n-1}$.

By induction hypoth, $(x^{n+1})' = (x^n x)' = (nx^{n-1})x + (x^n)1 = (n+1)x^n$. $\square$

You can also start the induction off with $n = 1$.

Note: we will later prove power rule for all real n.

Reciprocal rule: If $f(x) \neq 0$, then $(\frac{1}{f(x)})' = \frac{-f'(x)}{f(x)^2}$.

Proof:

$$(\frac{1}{f(x)})' = lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= lim_{h\to 0} \frac{-1}{f(x+h)f(x)} \frac{f(x+h) - f(x)}{h}$$

$$= (\frac{-1}{(f(x))^2}) f'(x)$$

(again since diffble $\Rightarrow$ cts) $\square$

Corollary: For a positive integer n, hen $(x^{-n})' = -nx^{-n-1}$.

(so, $(x^n)' = nx^{n-1}$ holds for all integers, positive, negative and zero).

Proof: Let $(x^{-n})' = (\frac{1}{x^n})' = (\frac{-1}{x^{2n}})(nx^{n-1}) = -nx^{-n-1}$.

Quotient rule: If $g(x) \neq 0$, then $(\frac{f(x)}{g(x)})' = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$.

Proof: Write $\frac{f(x)}{g(x)} = f(x)(\frac{1}{g(x)})$ and apply product rule.

$$(\frac{f(x)}{g(x)})' = f'(x)(\frac{1}{g(x)}) = f(x)(\frac{-g'(x)}{(g(x))^2} + f'(x)(\frac{1}{g(x)})$$

□

Examples:

Leibniz notation: If you think of $y = f(x)$, then can write $f'(x)$, *as a function* in many ways:

$f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x) = D_x f(x)$ and others.

For the derivative evaluated at a particular value of $x = x_0$, we write

$$f'(x_0) = \frac{dy}{dx}|_{x=x_0} = \frac{dy}{dx}|_{x_0}$$

Example: $f(x) = x^2$; $f'(x) = 2x$; $f'(3) = 6$.

Differentiability on an interval means dibble. at each point of the interval (remember that diffible at an endpoint means as a one-sided derivative).

Chain Rule: If g is diffble at x, and f is diffble. at $g(x)$, then $f \circ g$ is diffble. at x and $(f \circ g)'(x) = f'(g(x))g'(x)$.

Example: Find the derivative of $(x^2 + 1)^{10}$.

This function is $f \circ g(x)$ where $g(x) = x^2 + 1$ and $f(x) = x^{10}$. So,

$$((x^2 + 1)^{10})' = (f'(g(x)))g'(x) = (10(x^2 + 1)^9)(2x) = 20(x^2 + 1)^9 x$$

You could alternatively expand it out and differentiate term by term. So, chain rule is a labor-saving device. But there are other applications where there is no alternative.

Lecture 12:

Chain Rule: If g is diffble at x, and f is diffble. at $g(x)$, then $f \circ g$ is diffble. at x and $(f \circ g)'(x) = f'(g(x))g'(x)$.

Proof of Chain rule: Fix x. Let $k(h) = g(x+h) - g(x)$.

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x) + k(h)) - f(g(x))}{k(h)} \frac{g(x+h) - g(x)}{h}$$

Use the limit rule for products. Makes some sense since $k(h) \to 0$ as $h \to 0$. Problem: $k(h)$ may be zero.

Instead: fix u in domain of f.

Define $E(k) := \frac{f(u+k) - f(u)}{k} - f'(u)$ if $k \neq 0$ and $E(0) = 0$.

$E(k)$ is the "error" in the approximation of $f'(u)$ by the *difference quotient*.

Note: for all k, (including $k = 0$),

$$f(u+k) - f(u) = (f'(u) + E(k))k$$

Fix x and let $u = g(x)$.

Let $k = k(h) = g(x+h) - g(x)$. We have

$$f(g(x+h)) - f(g(x)) = (f'(g(x)) + E(k(h)))(g(x+h) - g(x)).$$

Now, divide by h:

$$\frac{f(g(x+h)) - f(g(x))}{h} = (f'(g(x)) + E(k(h)))\frac{(g(x+h) - g(x))}{h}.$$

Let $h \to 0$. Then,

$$(f \circ g)'(x) = (f'(g(x)) + \lim_{h \to 0} E(k(h)))g'(x)$$

So, it suffices to show:

$$\lim_{h \to 0} E(k(h)) = 0.$$

This follows from:

a) By continuity of g (which we have since g is diffble.), $\lim_{h \to 0} k(h) = 0$.

b) By diffble. of f, $\lim_{k \to 0} E(k) = 0$. $\square$

Derivatives of Trig functions

Will compute derivatives of all trig functions.

Need some Lemmas:

Theorem 7: $\sin(x)$ and $\cos(x)$ are cts.

Proof: Exercise 62 of section 2.5.

Theorem 8: $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

Note: this follows from LHopital's Rule once you know $(\sin(x)' = \cos(x)$. But we haven't prove that yet.

Proof of Theorem 8: Let $\theta > 0$, measured in radians.

Area(Δ OAP) $<$ Area (sector OAP) $<$ Area(Δ OAT)

Thus,

$$\frac{\sin(\theta)}{2} < \theta/2 < \frac{\tan(\theta)}{2} = \frac{\sin(\theta)}{2\cos(\theta)}$$

$$1 < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos(\theta)}$$

$$1 > \frac{\sin(\theta)}{\theta} > \cos(\theta)$$

Apply Squeeze Theorem. $\square$

Example 1, p. 122: $\lim_{h\to 0} \frac{\cos(h)-1)}{h} = 0.$

Proof: Use half-angle formula: $\cos(h) = 1 - 2\sin^2(h/2)$. Then

$$\lim_{h\to 0} \frac{\cos(h) - 1)}{h} = -\lim_{h\to 0} \frac{\sin^2 h/2}{h/2}$$

$$= -\lim_{h\to 0}(\frac{\sin h/2}{h/2}) \lim_{h\to 0} \sin(h/2) = -1 \cdot 0 = 0, .$$

Lecture 13:

Inputs to trig functions are measured in radians, not degrees.

Theorem 9: Let $f(x) = \sin(x)$. Then $f'(x) = \cos(x)$.

Proof: Use trig addition formula:

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$\sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= \cos(x)$$

$\square$

Theorem 10 (p. 123): $\frac{d}{dx}\cos(x) = -\sin(x)$.

Proof:
Use fact: $\cos(x) = \sin(\pi/2 - x)$ and $\sin(x) = \cos(\pi/2 - x)$.

Reason: say for an acute angle x; look at a right triangle, with angles $x, \pi/2 - x, \pi$. Then "opposite" for x is adjacent for $\pi/2 - x$ and vice versa.

Now apply chain rule:

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sin(\pi/2 - x) = -\cos(\pi/2 - x) = -\sin(x).$$

Examples using chain rule:

1. Let $f(x) = (\sin(x))^{10}$.

$$f'(x) = 10(\sin(x))^9 \cos(x).$$

2. Let $f(x) = \sin(\cos(x))$. Then,

$$f'(x) = \cos(\cos(x))(-\sin(x)) = -\cos(\cos(x))(\sin(x))$$

Derivatives of other trig functions are given on top of p. 125. For instance,

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

Proof: Apply quotient rule:

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cos(x) - ((\sin(x))(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= \sec^2(x).$$

Higher order derivatives

A diffble. function $f(x)$ gives rise to a new function $f'(x)$. So, why not differentiate again?

The *2nd derivative* of $f(x)$ is defined:

$$f''(x) := (f'(x))'$$

Other notation:

$$f''(x) = \frac{d^2}{dx^2}f(x)$$

Can consider *3rd derivative*

$$f'''(x) := (f''(x))'$$

Other notation:

$$f'''(x) = \frac{d^3}{dx^3}f(x)$$

For even higher derivatives, the "prime" notation gets too cumbersome. The *n-th derivative* is denoted:

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = \frac{d^n y}{dx^n}$$

the latter if y is understood as $f(x)$.

Major motivation: if $f(t)$ represents distance of a moving particle as a function of time t, then $f'(t)$ represents the (instantaneous) velocity of the particle.

And $f''(t)$ denotes acceleration.

Example: $f(x) = 3x^5 - 7x + 13$.

$f'(x) = 15x^4 - 7$.

$f^{(5)}(x) = (3)(5!) = 360$ and for all $m > 5$, $f^{(m)}(x) = 0$.

In general, for a polynomial of degree n, $f^{(m)}(x) = 0$ for all $m > n$.

Note that the domain of $f^{(n)}(x)$ may depend on n:

Example:

$$f(x) = \begin{matrix} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{matrix}$$

Then $f'(x) = 2|x|$. So, the domain of both f and f' is all of $\mathbb{R}$. But f' is not diffble. at $x = 0$ and so f'' does not exist at $x = 0$.

Example (special case of Example 5, p. 129): Let $f(x) = \sin(x)$. What is $f^{(n)}(x)$?

Claim: For n even, $f^{(n)}(x) = (-1)^{n/2} \sin(x)$.

For n odd, $f^{(n)}(x) = (-1)^{(n-1)/2} \cos(x)$.

Proof: by induction.

For $n = 1$, $f^{(n)}(x) = f'(x) = (-1)^{(n-1)/2} \cos(x)$.

Assume true for n even. Then $n+1$ is odd.

$f^{(n+1)}(x) = (-1)^{n/2}\cos(x) = (-1)^{((n+1)-1)/2}\cos(x)$ and so the result holds for $n+1$.

Assume true for n odd. Then $n+1$ is even.

$f^{(n+1)}(x) = (-1)^{(n-1)/2+1}\sin(x) = (-1)^{(n+1)/2+1}\sin(x)$ and so the result holds for $n+1$. $\square$

Lecture 14:

Angles are in radians!

MVT: Given a function $f(x)$ and $a < b$ such that f continuous on $[a, b]$ and f differentiable on (a, b), there exists c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

You are travelling on a toll road, which has a maximum speed limit of 120km/hour. The time and location that you entered the road are recorded on your toll ticket. A policeman stops you and asks to see your toll ticket, which shows that you entered the road 2 hours earlier. The policeman notes that you are exactly 242 km from where you entered. He then gives you a traffic ticket. You protest because you know that in the last several minutes you were driving well under the speed limit. The policeman replies: "Your average speed was 121 km/hour. Therefore by the Mean Value Theorem, at some point in your trip you were traveling at 121 km/hour."

– MVT does not tell you an exact value of c.

– There may be more than one c that works.

What hypotheses are needed?

MVT hypotheses:

Give pictorial examples why these hypotheses are needed:
– discts at endpoint
– discts at interior point
– not diffble at interior point: $f(x) = x^{2/3}$ on $[-1, 1]$.

Proof of MVT postponed.

Example of MVT: Show that the MVT holds for $f(x) = \sqrt{x}$ and any $0 \leq a < b$.

– so differentiability at an endpoint is *not* needed.

Proof:

$$\frac{f(b)-f(a)}{b-a} = \frac{\sqrt{b}-\sqrt{a}}{b-a} = \frac{(\sqrt{b}-\sqrt{a})(\sqrt{b}+\sqrt{a})}{(b-a)(\sqrt{b}+\sqrt{a})}$$

$$= \frac{1}{\sqrt{b}+\sqrt{a}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

So, for the MVT to hold with intermediate point c, we would have

$$\frac{1}{2\sqrt{c}} = \frac{1}{\sqrt{b}+\sqrt{a}}$$

equivalently,

$$\sqrt{c} = \frac{\sqrt{b}+\sqrt{a}}{2}$$

So, define c as

$$c = (\frac{\sqrt{b}+\sqrt{a}}{2})^2.$$

It remains to check that $a < c < b$:

$$a = (\frac{\sqrt{a}+\sqrt{a}}{2})^2 < (\frac{\sqrt{b}+\sqrt{a}}{2})^2 < (\frac{\sqrt{b}+\sqrt{b}}{2})^2 = b.$$

□

– Note: In this example, we get an explicit value of c and it is unique.

– Note: When $a = 0, c = b/4$.

Application of MVT: $\sin(x) < x$ for all $x > 0$.

Draw graph.

Proof: For $x > 2\pi$, this clearly holds.

So assume $0 < x \leq 2\pi$.

$$\frac{\sin(x)}{x} = \frac{\sin(x) - \sin(0)}{x - 0} = \frac{d}{dx}|_{x=c} \sin(x) = \cos(c)$$

for some $0 < c < x \leq 2\pi$ and so $\cos(c) < 1$.

So, $\frac{\sin(x)}{x} < 1$. $\square$

In fact, one can show in the same way that: $-\sin(x) < x$ for all $x > 0$ and so

$$|\sin(x)| < x \; for \; all \; x > 0.$$

Corollary: For all $x \neq 0$, $\cos(x) > 1 - x^2/2$.

Draw graph.

Proof: Since $\cos(x)$ and x^2 are even functions, it suffices to prove then when $x > 0$.

From $|\sin(x)| < x$, we get

$$\sin^2(x/2) < x^2/4$$

But by double angle formula,

$$\sin^2(x/2) \;=\; \frac{1 - \cos(x)}{2}$$

Thus,

$$\frac{1 - \cos(x)}{2} < x^2/4$$

which is equivalent to the claim of the corollary. $\square$.

In Math 121, you will study infinite series and derive the formulas: for all x,

$$\sin(x) = x - x^3/(3!) + x^5/(5!) - x^7/(7!) + \ldots$$

$$\cos(x) = 1 - x^2/2 + x^4/(4!) - x^6/(6!) + \dots$$

The preceding results are consistent with this.

Defn: Let f be defined on an interval I.

f is *increasing* (or *strictly increasing*) if whenever $x_1, x_2 \in I$ and $x_2 > x_1$, then $f(x_2) > f(x_1)$.

– Note: I could be $\mathbb{R}$.

Another application of MVT:

Theorem 12, p. 140:

If f is cts. on $[a, b]$ and diffble. on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.

Proof: Let $a \leq x_1 < x_2 \leq b$. By MVT, there exists c s.t. $x_1 < c < x_2$ and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0.$$

Since $x_2 - x_1 > 0$, we have $f(x_2) - f(x_1) > 0$. $\square$

Q: If f is differentiable and (strictly) increasing on an interval, is $f'(x) > 0$ for all x in the interval?