Lecture 7:

Recall $f(x) = sgn(x) =$

$$f(x) = \left\{ \begin{array}{ll} 1 & x > 0 \\ -1 & x < 0 \end{array} \right\}$$

Q: Does $\lim_{x \to 0} sgn(x)$ exist?

A: No. Informally, you get a different limit coming from the right $(+1)$ than coming from the left (-1).

Informal defn: *right limit*: $\lim_{x \to a^+} f(x) = L$ means that as x approaches a from the right, then $f(x)$ approaches L.

Formally:
$\forall \epsilon > 0 \exists \delta > 0$ s.t. if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$.

Note: for existence of right limit f needs to be defined only on a one-sided punctured nbhd. $(a, a + \delta)$ for some $\delta > 0$.

Formal Defn: *left limit*: $\lim_{x \to a^-} f(x) = L$ means same thing but replace $0 < x - a < \delta$ with $0 < a - x < \delta$.

Theorem 1 (p. 68): $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^+} = L$ and $\lim_{x \to a^-} = L$.

Can prove using $\delta - \epsilon$.

Example:

$$\lim_{x \to 0^+} sgn(x) = +1, \quad \lim_{x \to 0^-} sgn(x) = -1$$

So, $\lim_{x \to 0} sgn(x)$ does not exist. Can verify with $\delta - \epsilon$.

Example:

$$\lim_{x \to 0^+} \frac{|x|}{x^2 + 3x} = \lim_{x \to 0^+} \frac{x}{x(x + 3)} = 1/3.$$

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$$\lim_{x \to 0^-} \frac{|x|}{x^2 + 3x} = \lim_{x \to 0^-} \frac{-x}{x(x+3)} = -1/3.$$

Can verify with $\delta - \epsilon$.

So, $\lim_{x \to 0} \frac{|x|}{x^2+3x}$ does not exist.

Limits at ∞:

Note: in the text $\infty = +\infty$.

Informally, $\lim_{x \to \infty} f(x) = L$ means that as x gets arbitrarily large (and positive), $f(x)$ approaches L.

Graphically, this means that $y = L$ is a horizontal asymptote.

Computing these limits:

Example: $\lim_{x \to \infty} 1/x = 0$.

Example: $\lim_{x \to \infty} \frac{x+2}{x+3} = 1$:

$\frac{x+2}{x-3} = \frac{1+2/x}{1-3/x} \to 1$ as $x \to \infty$.

Defn: *polynomial*: a function of the form $a_n x^n + \ldots + a_0$, $a_i \in \mathbb{R}$, $a_n \neq 0$, *degree* $= n$

Defn: *rational function:* $R(x) := \frac{f(x)}{g(x)}$ where f and g are polynomials.

To compute $\lim_{x \to \infty} R(x)$ divide numerator and denominator by highest degree in denominator. This yields:

If $deg(num) > deg(denon)$, then limit is 0.

If $deg(num) < deg(denon)$, then limit does not exist (because it goes to ∞)

If $deg(num) = deg(denon)$, then limit is ratio of highest coefficients.

Another Example: $\lim_{x \to \infty} \sqrt{x^2 + x} - x = 1/2$.

Rationalize:

$$\sqrt{x^2 + x} - x = \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} = \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x}$$

$$\frac{x}{\sqrt{x^2 + x} + x} = \frac{1}{\sqrt{1 + 1/x} + 1} \to 1/2.$$

Can also consider $\lim_{x \to -\infty} f(x)$

Formal Defn: $\lim_{x \to \infty} f(x) = L$ means:

$\forall \epsilon > 0 \exists R$ s.t. if $x > R$, then $|f(x) - L| < \epsilon$.

Example: $\lim_{x \to \infty} \frac{x+2}{x-3} = 1$.

Let $\epsilon > 0$.

$$|\frac{x + 2}{x - 3} - 1| = |\frac{5}{x - 3}| = \frac{5}{x - 3} \text{ for } x > 0$$

We want $\frac{5}{x-3} < \epsilon$, equivalently, $x > 5/\epsilon + 3$ (a very large number).

Choose $R := 5/\epsilon + 3$. If $x > R$, then

$$|\frac{x + 2}{x - 3} - 1| = \frac{5}{x - 3} < \epsilon.$$

□

Infinite Limits

Informal defn: $\lim_{x \to a} f(x) = \infty$ means that as x gets closer and closer to a, then $f(x)$ gets arbitrarily large (and positive).

Technically, the limit does not exist (as a real number).

Graphically, this corresponds to a vertical asymptote.

Example: $f(x) = 1/x^2$: $\lim_{x \to 0} 1/x^2 = \infty$ (but does not exist).

Example: $f(x) = 1/x$: $\lim_{x \to 0} 1/x$ does not exist because

$\lim_{x \to 0^+} 1/x = \infty$

$\lim_{x \to 0^-} 1/x = -\infty$

Example: $\lim_{x \to 2} \frac{x-3}{x^2-4x+4} = -\infty$ (both one-sided limits yield $-\infty$.

Formal defn: Defn 11, p. 92.

Lecture 8:

Open interval: (a, b) or (a, ∞) or $(-\infty, a)$.

Closed interval: (a, b) or (a, ∞) or $(-\infty, a)$.

Let f be a function. We assume that the domain of f contains a neighbourhood of a, i.e. $(a - \alpha, a + \alpha)$ for some $\alpha > 0$.

Defn: f is *continuous* at a if $\lim_{x \to a} f(x) = f(a)$.

Defn: $f(x)$ is *right continuous* at a if $\lim_{x \to a^+} f(x) = f(a)$.

Defn: $f(x)$ is *left continuous* at a if $\lim_{x \to a^+} f(x) = f(a)$.

If a is an endpoint of the domain, we require that the limit holds only as a one-sided limit.

For example, if the domain is $[a, b]$, then:

$f(x)$ is *continuous* at a if $\lim_{x \to a^+} f(x) = f(a)$.

Similarly, if the domain is $[c, a]$, we require only that $\lim_{x \to a^-} f(x) = f(a)$.

No holes, finite jumps or infinite jumps.

Types of discontinuities:

Removable discontinuity: limit exists but does not equal $f(a)$

$$f(x) = \left\{ \begin{array}{cc} \frac{x^2 - 9}{x - 3} & x \neq 3 \\ 7 & x = 3 \end{array} \right\}$$

Note that in this case we can always redefine $f(a) := \lim_{x \to a} f(x)$, and then f becomes continuous at a. In this case, redefine $f(3) = 6$.

Technically, the redefined function is a different function, called the *continuous extension* of f.

Jump discontinuity: limit does not exist but all one-sided limits exist.

Example: (extension of $sgn(x)$)

$$f(x) = \left\{ \begin{array}{cc} 1 & x \leq 0 \\ -1 & x < 0 \end{array} \right\}$$

You cannot redefine $f(a)$ to make f continuous at a.

Infinite discontinuity: at least one one-sided limit is infinite.

$$f(x) = \left\{ \begin{array}{cc} 1/x & x \neq 0 \\ 0 & x = 0 \end{array} \right\}$$

You cannot redefine $f(a)$ to make f continuous at a.

At least one one-sided limit does not exist and is not infinite

$$f(x) = \left\{ \begin{array}{cc} \sin(1/x) & x > 0 \\ 0 & x \leq 0 \end{array} \right\}$$

You cannot redefine $f(a)$ to make f continuous at a.

Recall Limit Theorems (limit of sum is sum of limits, etc.). We get corresponding continuity theorems:

$f(x) = c$ and $f(x) = x$ are continuous,.

If f and g are continuous at a, then so are $f + g$, $f - g$, fg and f/g (for the latter, provided $g(a) \neq 0$).

So, polynomials are continuous at all a.

Lecture 9:

Theorem 6 (p. 81): Sum, diff, product, quotient of function continous at a are continuous at a (except that in the case of $a \neq 0$ must assume that denominator is not 0 at a.

Proof of continuity of sum.

Corollary: polynomials are continuous at all a.

Corollary: Rational functions are continuous at all a s.t. denominator(a) $\neq 0$.

Will later show that all trig functions, exponential function and logarithmic function are continuous at all points in their domains. We will show that they are actually differentiable and that differentiability implies continuity.

Example: $f(x) = \sqrt{x}$ is continuous at all $a \geq 0$.

Proof: Case 1: $a \neq 0$.

Let $\epsilon > 0$. We want: $|f(x) - f(a)| < \epsilon$.

$$|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = |\frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}}|$$

$$= |\frac{x - a}{\sqrt{x} + \sqrt{a}}| \leq |\frac{x - a}{\sqrt{a}}|$$

Want $|f(x) - f(a)| < \epsilon$. For this, it is enough to have:

$$\frac{|x - a|}{\sqrt{a}} < \epsilon$$

Let $\delta := \sqrt{a}\epsilon$. If $0 < |x - a| < \delta = \sqrt{a}\epsilon$, then

$$|f(x) - f(a)| \leq \frac{|x - a|}{\sqrt{a}} < \epsilon.$$

Case 2: $a = 0$. Let $\epsilon > 0$. Choose $\delta = \epsilon^2$.

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If $0 \leq x < \delta = \epsilon^2$, then $f(x) = \sqrt{x} < \epsilon$.

Theorem 7 (p. 82): If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Proof: Let $\epsilon > 0$.

Choose $\delta > 0$ s.t. if $0 < |z - g(a)| < \delta$, then $|f(z) - f(g(a))| < \epsilon$. This can be done because f is continuous at $g(a)$.

Choose $\gamma > 0$ s.t. if $0 < |x - a| < \gamma$, then $|g(x) - g(a)| < \delta$. This can be done because g is continuous at a.

If $0 < |x - a| < \gamma$, then letting $z = g(x)$, we see that $|z - g(a)| < \delta$ and so

$$|f \circ g(x)) - f \circ g(a)| = |f(z) - f(g(a))| < \epsilon.$$

$\square$

Example: $f(x) = e^{\sin(x)}$ is continuous at all a.

Example: $f(x) = \sqrt{x^2 + 1}$ is continuous at all a.

Defn (just terminology): f is continuous on an interval (open, closed, finite, infinite, etc.) if it is continuous at all points of the interval.

Remember that continuity at an endpoint of the domain of f means that it is right- or left-continuous.

Graphically, continuity means on an interval that the graph of $f(x)$ on the interval is unbroken.

Theorem 9 (the intermediate value theorem (IVT)): Let f be continuous a continuous function on an interval I. Let $a, b \in I$ such that $a < b$. Let s be between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ s.t. $f(c) = s$.

Theorem 9 may seem obvious because the graph is unbroken, but the proof requires the "completeness" of the real numbers.

IVT is (may be) false if f is not continuous:

Example: consider the function $f(x)$ on $[a, b] = [0, 1]$:

$$f(x) = \left\{ \begin{array}{ll} x & x \neq 1/2 \\ 3/4 & x = 1/2 \end{array} \right\}$$

Finding roots of equations (Example 11, p. 80): Show that $x^3 - x - 1 = 0$ for some $x \in [1, 2]$.

Proof: $f(1) = -1$, $f(2) = 5$. Since $-1 \leq 0 \leq 5$, there exists $c \in [1, 2]$ s.t. $f(c) = 0$ (by IVT; here $s = 0$).

Theorem 8 (Max-Min Theorem): Let f be continuous on a closed finite interval $[a, b]$. Then f has an absolute maximum and an absolute minimum on $[a, b]$. That is, there exist $p, q \in [a, b]$ s.t. for all $x \in [a, b]$, we have

$$f(p) \leq f(x) \leq f(q).$$

Max-Min theorem is (may be) false on open intervals (a, b): consider $f(x) = x$ on $(a, b) = (0, 1)$.

Max-Min theorem is (may be) false on infinite intervals $[a, \infty)$: consider $f(x) = x$ on $[a, \infty) = [0, \infty)$ (has an absolute min but not an absolute max).

Max-Min theorem is (may be) false for discontinuous functions on closed intervals $[a, b]$:

Example: consider the function f on $[-1, 1]$ defined by

$$f(x) = \left\{ \begin{array}{ll} |x| & x \neq 0 \\ 1/2 & x = 0 \end{array} \right\}$$

Lecture 10:

Which of the following functions are cts. at 0?

1. $f(x) = |x|$

Yes: $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$

So, right and left hand limits agree and equal $f(0)$. So, $\lim_{x \to 0} f(x) = f(0)$.

2. $f(x) = \sin(1/x), x \neq 0$.

No: No matter how $f(0)$ is defined, f is not cts at 0 because $\lim_{x \to 0} \sin(1/x)$ d.n.e.

3. $f(x) = x \sin(1/x), x \neq 0$?

If we define $f(0) = 0$, $\lim_{x \to 0} f(x) = 0 = f(0)$ and so f is continuous at 0.

For x_0 be an interior point of the domain of f. We say that f is *differentiable* at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists (and is finite). And the limit is called $f'(x_0)$.

What does this mean?

1. The slope of the chord from $(x, f(x))$ to $(x_0, f(x_0))$ converges (to a finite number $f'(x_0)$).

Equivalently,

2. There is a "good" linear approximation to $f(x)$ in a neighbourhood of x_0, more precisely, there is a linear function $L(x) = mx + b$ such that

$$\lim_{x \to x_0} \frac{f(x) - L(x)}{x - x_0} = 0.$$

Namely, $L(x) = f(x_0) + f'(x_0)(x - x_0)$ works:

$$\lim_{x \to x_0} \frac{f(x) - L(x)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0}$$

$$= f'(x_0) - f'(x_0) = 0.$$

The graph of $y = L(x)$ is the *tangent line* to $y = f(x)$ at the point $(x_0, f(x_0))$.

Note: not only is $\lim_{x \to x_0} f(x) - L(x) = 0$, but it approaches zero "faster" than $x - x_0$ approaches zero.

Linear functions are "simple." So, differential calculus is largely about approximating complicated functions by simple functions.

If $y = f(x)$ is interpreted as position and x as time, then $f'(x_0)$ can be interpreted as velocity at time x_0; many other interpretations depending on application.

Equivalent notation:

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Here, we are letting x_0 vary and calling it x, so that we get a new function f' whose domain is contained in the domain of f.

Examples: (in these computations we use the limit theorems, instead of $\delta - \epsilon$ proofs)

1. $f(x) = c$.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} 0 = 0.$$

2. $f(x) = x$.

$$f'(x) = \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1.$$

3. $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} \quad \lim_{h \to 0} 2x + h = 2x.$$

For the following functions, we are interested in differentiability at $x = 0$.

4. $f(x) = |x|$. Right-hand derivative:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0} 1 = 1.$$

Lert-hand derivative:

$$f'(x) = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0} -1 = -1.$$

These do not agree, and so f is not diffble. at 0.

5. $f(x) = x \sin(1/x)$.

$$f'(0) = \lim_{h \to 0} \frac{h \sin(1/h)}{h} = \lim_{h \to 0} \sin(1/h)$$

d.n.e. So, $f(x)$ is not diffble. at $x = 0$.

(we will later see that $f(x)$ is diffble. at all $x \neq 0$).

If x_0 is an endpoint of the domain of f (say a left endpoint), we say that f is diffble at x_0 if $f'_+(x_0)$ exists and is finite.

6. $f(x) = \sqrt{x}$, $x = 0$.

$$f'_+(0) = \lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}} = \infty$$

So, f is not diffble. at $x = 0$.

Note that the graph $y = f(x)$ has a vertical tangent line at the origin.

Examples 4,5,6 are cts. but not diffble. at $x = 0$.