Computing the entropy of two-dimensional shifts of finite type

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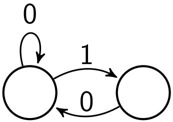
- An SFT is a "constraint" on the set of allowable words.

Examples

- Example 1: the **golden mean shift,** $(G^{(1)})$, $A = \{0, 1\}$:

$$\mathcal{F} = \{11\}.$$

Typical allowed sequence: $\ldots 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0 \ldots$



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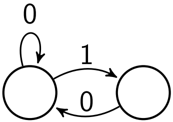
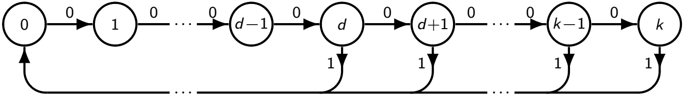


- Example 2: the **run-length-limited shift** (RLL(d, k)), $A = \{0, 1\}$

$$\mathcal{F} = \{11, 101, 1001, \ldots, 10^{d-1}1, 0^{k+1}\}$$



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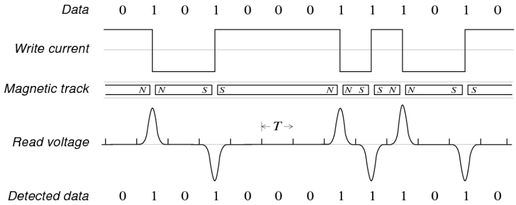

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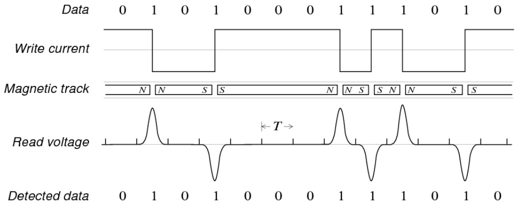

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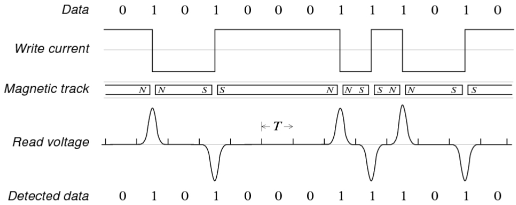

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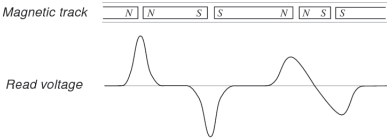

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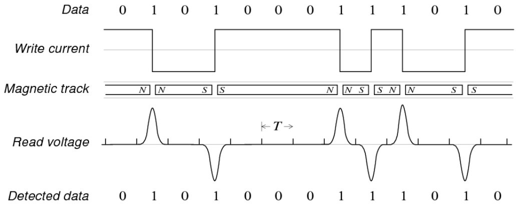

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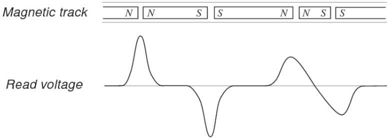

- Hence an RLL constraint on allowed stored sequences.

Encoding

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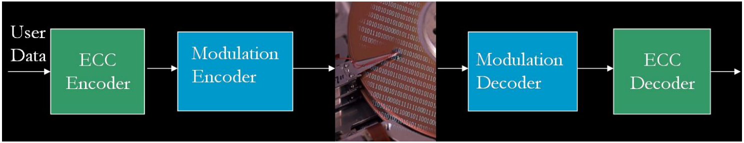

Topological entropy of 1-D SFT's (a.k.a. entropy, noiseless capacity)

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- The entropy is the maximal rate of encoder from the set of all arbitrary data sequences into X.

Computation of entropy

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- So, we can compute entropies of 1-dimensional SFT's.
- And we can characterize the set of numbers that occur as entropies of 1-dimensional SFT's:
  - **Theorem** (Lind, 1983)): A number h is the entropy of a one-dimensional SFT if and only if h is the log of a root of a Perron number (special kind of algebraic integer).

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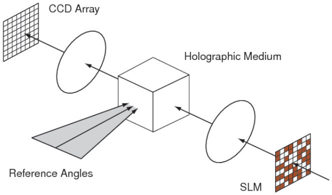
- Typical allowed configuration:

$$
\begin{array}{cccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \cdot \\
\cdot & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdot \\
\cdot & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdot \\
\cdot & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Motivation for 2-dimensional SFT's: Holographic storage



More examples of 2-dimensional SFT's

- NAK (Non-attacking kings): $\mathcal{F} = \left\{ 11,\ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix},\ \begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix},\ \begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix} \right\}$.

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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}$$

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. & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & . \\
. & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & . \\
. & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
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- exact value of entropy is known for only a handful of 2-D SFT's (unknown even for $G^{(2)}$).
- Even worse: given $\mathcal{F}$, it is algorithmically undecidable whether or not $X = \varnothing$!

Computing entropy

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- Current best estimates (Friedland, 2007): $0.58789116177534 \leq h(G^{(2)}) \leq 0.58789116177535$.

Strip systems

- Define H_n to be the set of configurations on an n-high strip which do not include any of the forbidden neighbours in $\mathcal{F}$.

.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\uparrow$	...	0	1	0	0	0	0	0	0	1	0	0	0	1	0	...
n	...	0	0	1	0	0	1	0	1	0	0	1	0	0	0	...
$\vert$	...	0	0	0	1	0	0	0	0	0	1	0	1	0	0	...
$\downarrow$	...	0	1	0	0	0	1	0	1	0	0	0	0	1	0	...
..	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	

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$\vert$	...	0	0	0	1	0	0	0	0	0	1	0	1	0	0	...
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  - The pair $\begin{matrix} a_n \\ \vdots \\ a_2 \\ a_1 \end{matrix} \quad \begin{matrix} b_n \\ \vdots \\ b_2 \\ b_1 \end{matrix}$ may appear if and only if each $a_i b_i$ is admissible.

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- Fact: $h(X) = \lim_{n\to\infty} \frac{h_n}{n}$.

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- $h_n = \log(\lambda(M_n))$
- $\lambda(M_n)$ is lower bounded by *Rayleigh quotient*:
  Let $\mathbf{1}_n$ denote the vector of all 1's. For any p

$$\lambda((M_n)^p) \geq \frac{\mathbf{1}_n (M_n)^p \mathbf{1}_n^{\mathrm{t}}}{\mathbf{1}_n \cdot \mathbf{1}_n^{\mathrm{t}}},$$

  where numerator is a count of admissible $n \times p$ patterns.

- (Markley and Paul, 1981)

$$h(X) = \lim_{n\to\infty} \frac{h_n}{n} = \lim_{n\to\infty} \frac{\log(\lambda(M_n))}{n} \geq \lim_{n\to\infty} \frac{1}{pn} \log \frac{\mathbf{1}_n (M_n)^p \mathbf{1}_n^{\mathtt{t}}}{\mathbf{1}_n \cdot \mathbf{1}_n^{\mathtt{t}}}$$

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$\uparrow$	...	0	1	0	0	0	0	0	0	1	0	0	0	1	0	...
n	...	0	0	1	0	0	1	0	1	0	0	1	0	0	0	...
$\vert$	...	0	0	0	1	0	0	0	0	0	1	0	1	0	0	...
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	$\longleftarrow$	—	—	—	—	p	—	—	—	—	$\longrightarrow$
$\uparrow$	1	0	0	0	0	0	0	1	0	0	0
n	0	1	0	0	1	0	1	0	0	1	0
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- Letting V_p denote a vertical transition matrix of width p,

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- All above used $\mathbf{1}_n$ so that the limit above may be computed as the log of largest eigenvalue of a vertical transition matrix.

Improved Lower bounds

- (Louidor and Marcus, 2009) *Improved Rayleigh Method:* Replace $\mathbf{1}_n$ with sequence of vectors $\mathbf{y}_n$ such that $\mathbf{y}_n(M_n)^p\mathbf{y}_n^{\mathsf{t}}$ represents weighted counts of patterns; incorporate $\mathbf{y}_n$ into a vertical transition matrix $\tilde{V}_p$ and find $\mathbf{x}_p$ such that

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NAK	0.4250636891	0.4250767745	0.4250767997
RWIM	0.5350150	0.5350151497	0.5350428519

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- In the 80's and 90's, data suggested that $\lim_{n\to\infty} h_{n+1} - h_n = h(G^{(2)})$, and that the error is exponentially small.
- However, a proof of convergence of $h_{n+1} - h_n$ for any nondegenerate $\mathbb{Z}^2$ SFT has been an open problem.

An excellent approximation (but not quite the Holy Grail)

- **Theorem** (Pavlov, 2009): There exist positive constants A and B so that $|h_{n+1} - h_n - h(G^{(2)})| < Ae^{-Bn}$ for any n.

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  - Thus, $h(G^{(2)})$ is much "nicer" than the typical entropy.

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- Introduce a stationary process μ_n on each H_n of maximal measure-theoretic (Shannon) entropy: $h_{\mu_n} = h(H_n)$.

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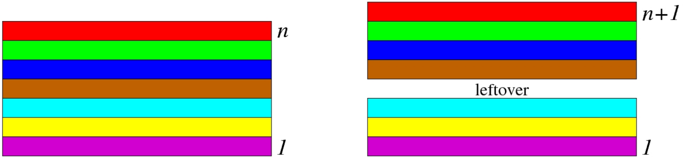

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  - On the Z^2 lattice, a site is "open" with probability p and closed with probability $1 - p$, independent from site to site.
  - For $p < p_c$, the critical probability, the probability of an "open" path from the origin to the boundary of an $n \times n$ square decays exponentially fast in n.

Generalizations

Theorem (Marcus and Pavlov, 2009):

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- Exponential approximations (differences of strip entropies) to measure-theoretic entropy for a class of Markov Random Fields (2-dimensional analogue of 1-dimensional Markov chain and probabilistic analogue of 2-dimensional SFT)

1-dimensional sofic shifts

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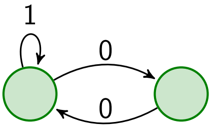

More examples of 1-dimensional sofic, non-SFT, shifts

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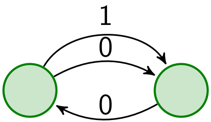

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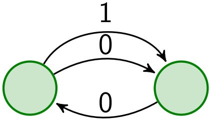
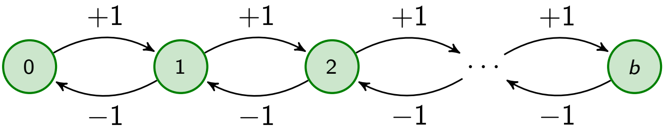


- The CHG(b) shift
  $A = \{+1, -1\}$:



$$w_1 \dots w_m \in B_m(X) \iff \text{for all } 1 \leq s \leq t \leq m, \left| \sum_{i=s}^{t} w_i \right| \leq b$$

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Computing entropy of 2-dimensional sofic shifts

- (Louidor and Marcus, 2009): applied improved Rayleigh method to estimate entropies of sofic shifts $\mathrm{EVEN}^{\otimes^2}$ and $\mathrm{CHG}(3)^{\otimes^2}$:

Constraint	Old lower bound	New lower bound	Upper bound
$\mathrm{EVEN}^{\otimes 2}$	0.4385027973	0.4402086447	0.4452873312
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Why is $h(\mathrm{CHG}(2)^{\otimes D}) \geq 1/2^D$?

- $X = \mathrm{CHG}(2)^{\otimes D}$.
- $D=2$. Consider the two "checkerboard" 2×2 arrays, $\Gamma^{(0)}$, $\Gamma^{(1)}$

$$\Gamma^{(0)} = \begin{pmatrix} + & - \\ - & + \end{pmatrix} \qquad \Gamma^{(1)} = \begin{pmatrix} - & + \\ + & - \end{pmatrix}$$

- Any tiling consisting of $n \times n$ copies of $\Gamma^{(0)}$ or $\Gamma^{(1)}$ is a $2n \times 2n$ array that satisfies X.

$$\begin{pmatrix} \Gamma^{(i_{1,1})} & \Gamma^{(i_{1,2})} & \dots & \Gamma^{(i_{1,n})} \\ \Gamma^{(i_{2,1})} & \Gamma^{(i_{2,2})} & \dots & \Gamma^{(i_{2,n})} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^{(i_{n,1})} & \Gamma^{(i_{n,2})} & \dots & \Gamma^{(i_{n,n})} \end{pmatrix}, \quad i_{s,t} \in \{0, 1\}$$

Why is $h(\mathrm{CHG}(2)^{\otimes D}) \geq 1/2^D$? (cont.)

- Generally, for arbitrary D, consider the two $2 \times 2 \times \ldots \times 2$ checkerboard arrays:

$$\Gamma^{(0)}_{i_1,\ldots,i_D} = (-1)^{\sum i_j} \qquad \Gamma^{(1)}_{i_1,\ldots,i_D} = (-1)^{1+\sum i_j}$$

- Any tiling of $n \times n \times \ldots \times n$ copies of $\Gamma^{(0)}$ or $\Gamma^{(1)}$ is a $2n \times 2n \times \ldots \times 2n$ array that satisfies X.

$$\implies |B_{2n \times 2n \times \ldots \times 2n}(X)| \geq 2^{n^D}$$

$$\implies \frac{\log |B_{2n \times 2n \times \ldots \times 2n}(X)|}{(2n)^D} \geq \frac{n^D}{(2n)^D}$$

$$\implies h(X) \geq \frac{1}{2^D}.$$

Why is $h(\mathrm{CHG}(2)^{\otimes D}) \leq 1/2^D$?

- For $D=1$ every legal word of X is essentially such a tiling of checkerboard arrays:

Lemma

$x_0 \ldots x_{n-1}$ satisfies $\mathrm{CHG}(2)$, iff
$x_i = -x_{i+1}$ for all <u>even</u> $i \in \{0, \ldots, n-2\}$ or
$x_i = -x_{i+1}$ for all <u>odd</u> $i \in \{0, \ldots, n-2\}$.

x_0	x_1	x_2	x_3	x_4	$\ldots$	x_{n-3}	x_{n-2}	x_{n-1}

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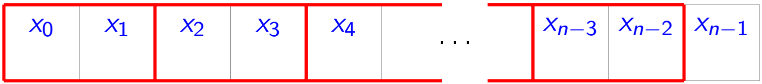

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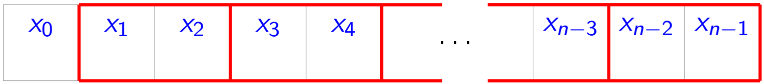

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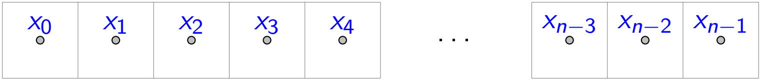

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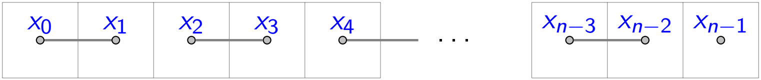

<u>Phase-0</u> sequence

$$T_0(i) = \begin{cases} i+1 & \text{if } i \text{ is even} \\ i-1 & \text{if } i \text{ is odd} \end{cases}$$

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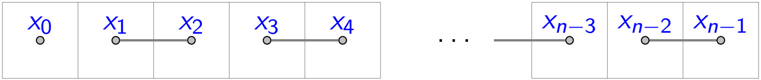

<u>Phase-1</u> sequence

$$T_1(i) = \begin{cases} i-1 & \text{if } i \text{ is even} \\ i+1 & \text{if } i \text{ is odd} \end{cases}$$

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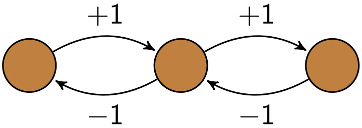
- Proof:



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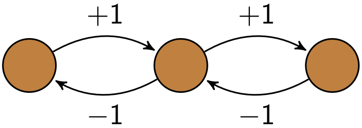

- Unfortunately, the previous Lemma does not generalize to larger dimension:

+	−	−	+
+	+	−	−
−	+	+	−
−	−	+	+

Why is $h(\mathrm{CHG}(2)^{\otimes D}) \leq 1/2^D$? (cont.)

- $\Gamma \in B_{n\times n\times \ldots \times n}(X)$ iff every row of Γ is either a phase-0 or a phase-1 sequence.
- $\mathbf{r} = (r_i)$: binary vector with an entry for each row of $\{0, \ldots, n-1\}^D$.
- $A(\mathbf{r}) = \{\Gamma \in B_{n\times n\times \ldots \times n}(X) \;:\; \text{row } i \text{ of } \Gamma \text{ has phase } r_i\}$

$$\stackrel{\text{Lemma 1}}{\Longrightarrow} B_{n\times n\times \ldots \times n}(X) = \bigcup_{\mathbf{r}} A(\mathbf{r}).$$

Why is $h(\mathrm{CHG}(2)^{\otimes D}) \leq 1/2^D$? (cont.)

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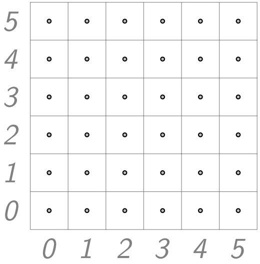

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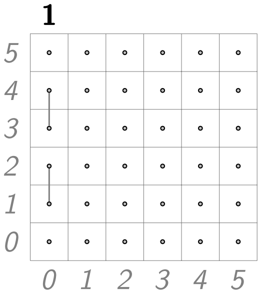

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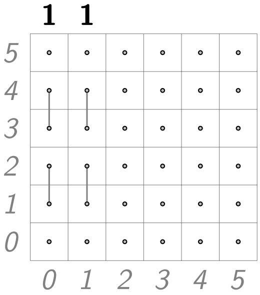

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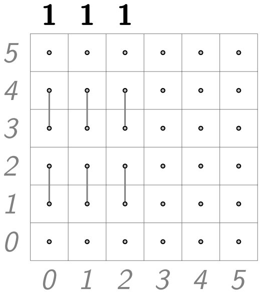

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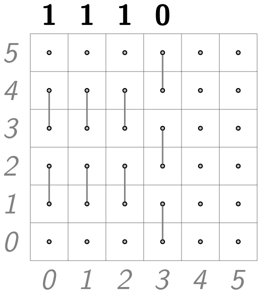

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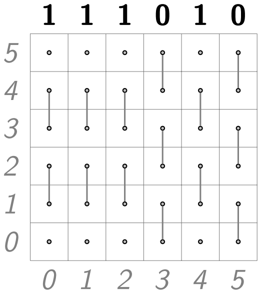

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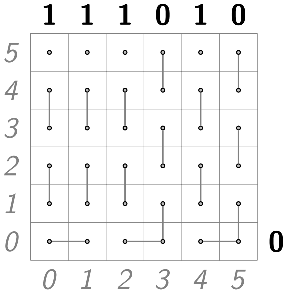

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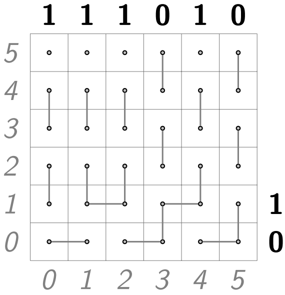

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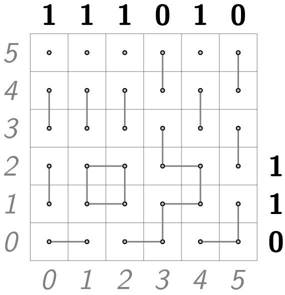

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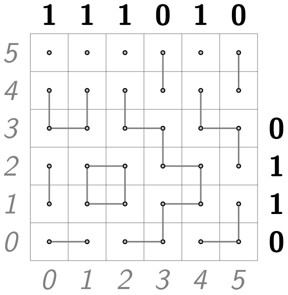

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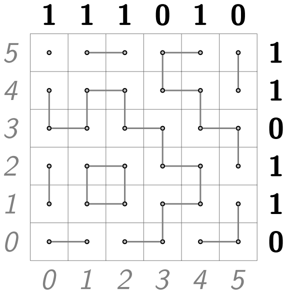

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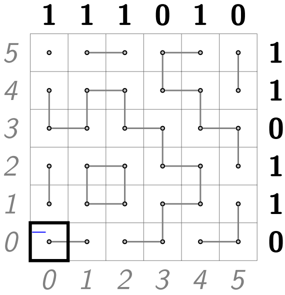

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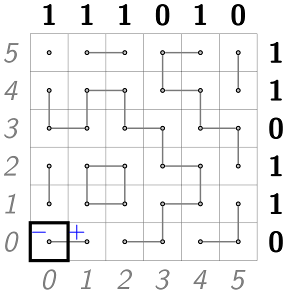

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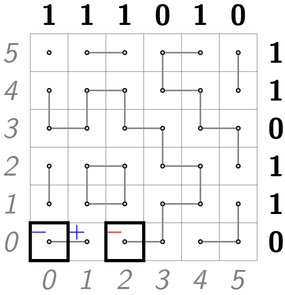

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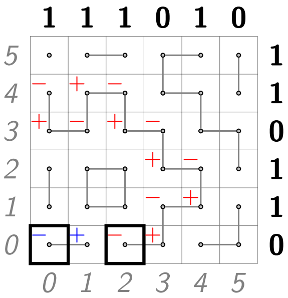

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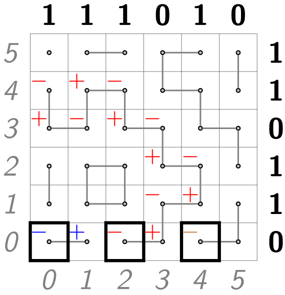

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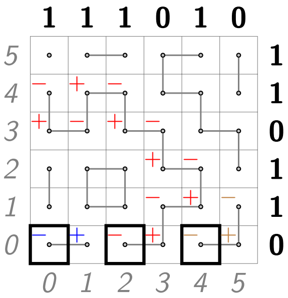

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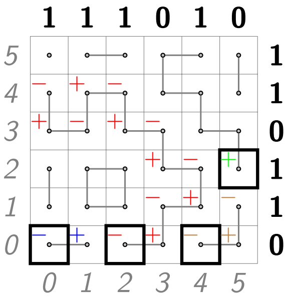

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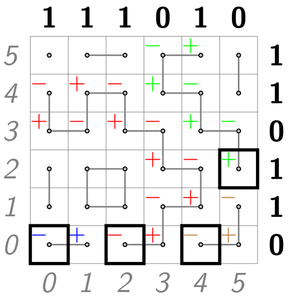

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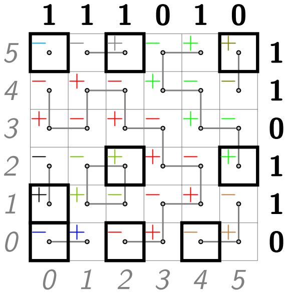

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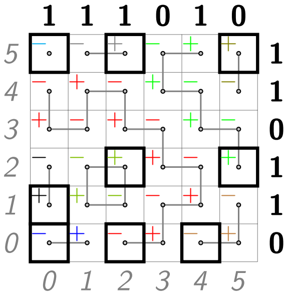

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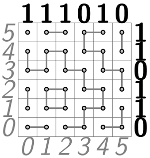

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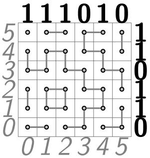

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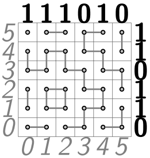

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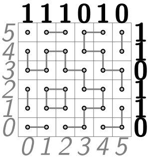

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$$(x, y),\ M_{\mathbf{r},1}(x, y),\ M_{\mathbf{r},2}(x, y),\ M_{\mathbf{r},1}(M_{\mathbf{r},2}(x, y))$$

  - Are all in the connected component of (x, y).
  - Are all distinct.
- For general D:
  - For $\mathbf{x} \in \{1, 2, \ldots, n{-}2\}^D$ (not on the "border"), the 2^D entries:

$$M_{\mathbf{r},i_1}(M_{\mathbf{r},i_2}(\ldots(M_{\mathbf{r},i_s}(\mathbf{x}))\ldots)),$$
$$\text{For each } \{i_1, \ldots, i_s\} \subseteq \{1, 2, \ldots, D\},\ \ 1 \le i_1 < i_2 < \ldots < i_s \le D$$

  - Are all in the connected component of $\mathbf{x}$.
  - Are all distinct.
- $\implies$ Any component having a vertex in the interior has at least 2^D vertices.

Max # of Connected Components in $G_{\mathbf{r}}$.

- $D=2$.
  - For $(x,y) \in \{1,2,\ldots,n-2\}^2$ (not on the "border"):

$$(x,y),\ M_{\mathbf{r},1}(x,y),\ M_{\mathbf{r},2}(x,y),\ M_{\mathbf{r},1}(M_{\mathbf{r},2}(x,y))$$

  - Are all in the connected component of (x,y).
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- For general D:
  - For $\mathbf{x} \in \{1,2,\ldots,n-2\}^D$ (not on the "border"), the 2^D entries:

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  - Are all in the connected component of $\mathbf{x}$.
  - Are all distinct.
- $\implies$ Any component having a vertex in the interior has at least 2^D vertices.
- $\implies$ There are at most $n^D/2^D$ such components.

Max # of Connected Components in $G_{\mathbf{r}}$ (cont.).

$$\implies \begin{pmatrix} \text{\# of components having a vertex in } \{1, 2, \ldots, n-2\}^D \end{pmatrix} \le n^D / 2^D$$

Max # of Connected Components in $G_{\mathbf{r}}$ (cont.).

$$\Longrightarrow \begin{pmatrix}\text{\# of components having a vertex in } \{1,2,\ldots,n-2\}^D\end{pmatrix} \le n^D/2^D$$

$$\begin{pmatrix}\text{\# of components not having a vertex in } \{1,2,\ldots,n-2\}^D\end{pmatrix} \le \begin{pmatrix}\text{\# of vertices not in } \{1,2,\ldots,n-2\}^D\end{pmatrix} = n^D - (n-2)^D$$

$$\implies \begin{pmatrix}\text{\# of components} \\ \text{having a vertex in} \\ \{1,2,\ldots,n-2\}^D\end{pmatrix} \le n^D/2^D$$

$$\begin{pmatrix}\text{\# of components} \\ \text{not having a vertex} \\ \text{in } \{1,2,\ldots,n-2\}^D\end{pmatrix} \le \begin{pmatrix}\text{\# of vertices not in} \\ \{1,2,\ldots,n-2\}^D\end{pmatrix} = n^D - (n-2)^D$$

$$\implies (\text{Total \# of components}) \le n^D/2^D + n^D - (n-2)^D.$$

Max # of Connected Components in $G_{\mathbf{r}}$ (cont.).

$$\implies \begin{pmatrix} \text{\# of components having a vertex in } \{1,2,\ldots,n-2\}^D \end{pmatrix} \le n^D/2^D$$

$$\begin{pmatrix} \text{\# of components not having a vertex in } \{1,2,\ldots,n-2\}^D \end{pmatrix} \le \begin{pmatrix} \text{\# of vertices not in } \{1,2,\ldots,n-2\}^D \end{pmatrix} = n^D-(n-2)^D$$

$$\implies (\text{Total \# of components}) \le n^D/2^D + n^D - (n-2)^D.$$

$$\implies |A(\mathbf{r})| = 2^{(\text{Total \# of components})} \le 2^{n^D/2^D+n^D-(n-2)^D}$$

Max # of Connected Components in $G_{\mathbf{r}}$ (cont.).

$$\implies \begin{pmatrix} \text{\# of components having a vertex in} \\ \{1,2,\ldots,n-2\}^D \end{pmatrix} \le n^D/2^D$$

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$$\implies |A(\mathbf{r})| = 2^{(\text{Total \# of components})} \le 2^{n^D/2^D + n^D - (n-2)^D}$$

$$\implies |B_{n\times n\times\ldots\times n}(X)| \le \sum_{\mathbf{r}} |A(\mathbf{r})| \le 2^{Dn^{D-1}} 2^{n^D/2^D + n^D - (n-2)^D}$$

Max # of Connected Components in $G_{\mathbf{r}}$ (cont.).

$$\Longrightarrow \begin{pmatrix} \text{\# of components having a vertex in } \{1,2,\ldots,n-2\}^D \end{pmatrix} \le n^D/2^D$$

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$$\Longrightarrow |B_{n\times n\times\ldots\times n}(X)| \le \sum_{\mathbf{r}} |A(\mathbf{r})| \le 2^{Dn^{D-1}} 2^{n^D/2^D + n^D - (n-2)^D}$$

$$\Longrightarrow |B_{n\times n\times\ldots\times n}(X)| \le 2^{n^D/2^D + O(n^{D-1})}$$

Max # of Connected Components in $G_{\mathbf{r}}$ (cont.).

$$\Longrightarrow \begin{pmatrix} \text{\# of components} \\ \text{having a vertex in} \\ \{1,2,\ldots,n-2\}^D \end{pmatrix} \le n^D/2^D$$

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$$\Longrightarrow (\text{Total \# of components}) \le n^D/2^D + n^D - (n-2)^D.$$

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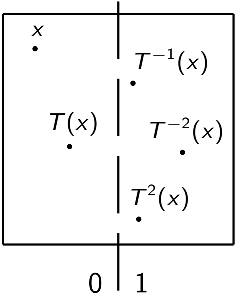
$$\Longrightarrow |B_{n\times n\times\ldots\times n}(X)| \le \sum_{\mathbf{r}} |A(\mathbf{r})| \le 2^{Dn^{D-1}} 2^{n^D/2^D+n^D-(n-2)^D}$$

$$\Longrightarrow |B_{n\times n\times\ldots\times n}(X)| \le 2^{n^D/2^D+O(n^{D-1})}$$

$$\Longrightarrow h(X) \le 1/2^D \quad \square$$

Motivation for 1-dimensional SFT's: Modelling dynamical systems

$T : \Omega \to \Omega$:



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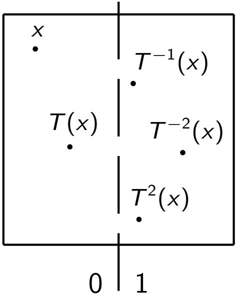

- Represent $x \in \Omega$ by binary itinerary sequence:

$$\begin{array}{ccc} x & \longleftrightarrow & \dots 11.001 \dots \\ T(x) & \longleftrightarrow & \dots 110.01 \dots \end{array}$$

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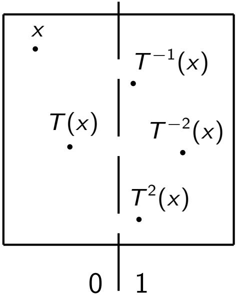

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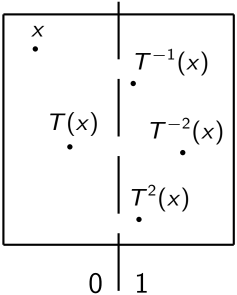

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Ω replaced by an SFT X

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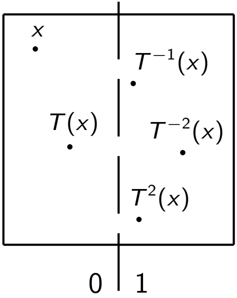

- Represent $x \in \Omega$ by binary itinerary sequence:

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Ω replaced by an SFT X
T replaced by the shift mapping.