NOTES ON ‘RICCI FLOW ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS’

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Abstract. This note presents Yu Li’s “Ricci Flow on Asymptotically Euclidean Manifolds” [17] with enough detail and background to (hopefully) make it accessible to any student familiar with basic knowledge of differential and Riemannian manifolds. The main result is a proof of the positive mass conjecture assuming non-negative scalar curvature and long-time existence of the Ricci flow. We mostly follow the order and nature of the original work, while introducing necessary background information and filling in the details of most of the arguments within. When appropriate, theorems, lemmas, and remarks are numbered to match those in the original work to allow readers to most easily follow along. Though much of the added detail is my own, I claim no original intellectual property within.

Definition. A smooth orientable Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is called an Asymptotically Euclidean (AE) manifold if for some compact \(K \subset M\), \(M \setminus K\) consists of a finitely many number of disjoint components \(E_1, E_2, \ldots, E_k\) such that for each \(E_i\) there exists a diffeomorphism \(\Phi_i : E_i \to \mathbb{R}^n \setminus B(0, A_i)\) \((A_i > 0)\) such that under this identification, the metric \(g_{ij}\) is asymptotically Euclidean in the following sense:

\[
g_{ij} = \delta_{ij} + O(r^{-\sigma_i}), \quad \partial^\alpha g_{ij} = O(r^{k-\sigma_i})
\]

for any multiindex \(\alpha\) of order \(k\) as \(r \to \infty\). Here \(\sigma_i > 0\) is the (asymptotic) order of the end \(E_i\) and \(r(x) = |\Phi_i(x)|\) is the Euclidean distance function on the end \(E_i\).

From now on, fix some complete AE manifold \(M^n\) with a single end \(E\) (and corresponding diffeomorphism \(\Phi\)), and non-negative integrable scalar curvature \(R\). We will identify \(x \in E\) with \(\Phi(x) \in \mathbb{R}^n\) without explicitly mentioning \(\Phi\). Moreover, we fix some positive smooth function \(r : M \to (0, \infty)\) such that \(r(x) = |\Phi(x)|\) for all \(x \in E\). Finally, we assume that the order \(\sigma\) of the end \(E\) satisfies

\[
\frac{n-2}{2} < \sigma \leq n-2.
\]

The requirement that \(\sigma \leq n-2\) is somewhat artificial since any AE manifold of order \(\geq n-2\) is also of order \(n-2\). We only enforce this condition now because it will be useful in the proof of theorem 4.4.

The importance of assuming \(R \geq 0\) and \(\sigma > \frac{n-2}{2}\) will make themselves apparent in the following.

Definition. The ADM mass (cf. [11]) of an AE manifold \((M, g)\) (with one end \(E\)) is defined as

\[
m(g) = \lim_{r \to \infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) dA^j
\]

where \(dA^j = \partial_j dV_{gE}\) and \(g_E\) is the canonical Euclidean metric on \(\mathbb{R}^n\).
The definition of mass involves a choice of asymptotic coordinates, so apriori, it is unclear if it is even well-defined. But it follows (due to Robert Bartnik \([2]\)) that if the order \(\sigma > (n - 2)/2\) and the scalar curvature \(R\) is integrable, then the mass is finite and independent of the choice of AE coordinates. In other words, \(m(g)\) depends only on the metric \(g\). The general positive mass conjecture is the following.

**Conjecture** (Positive Mass Conjecture). Let \((M^n, g)\) be an AE manifold of dimension \(n \geq 3\) with the order \(\sigma > (n - 2)/2\), and nonnegative integrable scalar curvature. Then \(m(g) \geq 0\) with equality if and only if \((M, g)\) is isometric to \((\mathbb{R}^n, g_E)\).

The condition that \(R \geq 0\) is physical in nature; A 3-dimensional time-symmetric space-like hypersurface \((M, g)\) in a 4-Lorentzian manifold \((L, h)\) has non-negative scalar curvature \(R_g \geq 0\) if and only if \(L\) satisfies the dominant energy condition. Colloquially, the dominant energy condition is the condition that energy can never be observed to be flowing faster than light. For more details about the general relativity motivations, see [31].

In dimension three, the positive mass conjecture was first proved by Schoen and Yau in 1979 [27] by constructing a stable minimal surface and considering its stability inequality. In addition, Schoen and Yau showed that their method could be extended to the case when the dimension was less than eight [28]. In 1981, Witten [32] proved the positive mass conjecture for spin manifolds of any dimension. In 2001, Huisken and Ilmanen [15] and Bray [3] proved the stronger Riemannian Penrose inequality in dimension three by using the inverse mean curvature flow. In 2015, Hein and LeBrun [14] gave a proof of the positive mass conjecture for Kähler AE manifolds. There is (so far) no proof of the positive mass conjecture in general dimension.

Since the Positive Mass Conjecture has been proven (at least in part) using the mean curvature flow, a natural question arises; can we prove the positive mass conjecture by using other geometric flows? One of the most powerful geometric flows is the Ricci Flow which was introduced by Hamilton in the 1980s as part of a lofty goal of proving the famed Poincaré conjecture which is the conjecture that every simply connected closed 3-manifold is homeomorphic to the 3-sphere. Though Hamilton made much progress towards this goal, it was Grigori Perelman who completed Hamilton’s program. In unpublished (even to this day!) arXiv preprints released in 2002 and 2003 (see [20], [21], [22]), Perelman shocked the world by not only by proving the Poincaré conjecture, but by completely solving Thurston’s Geometrization Conjecture (which categorizes 3 manifolds). Perelman’s work was heavily scrutinized in the years to come, but was eventually recognized as being correct, and he was subsequentially awarded the Fields medal in 2006 as well as the first Clay Millenium Prize in 2010. Though as an interesting historical note, he turned down both awards noting that he was “not interested in money or fame” and also felt that he deserved no more credit than Richard Hamilton himself who pioneered the program.

In this write up, we are interested in how the Ricci flow interacts with AE manifolds and the ADM mass. Recall that Ricci flow is a geometric flow such that a family of metrics \(g(t)\) on a smooth manifold \(M\) are evolved under the PDE

\[
\partial g(t) = -2Rc(g(t)).
\]

One should think of the Ricci Flow as a sort of “geometric heat equation” since (in harmonic coordinates)

\[
Rc_{ij} = -\frac{1}{2} \Delta g_{ij} + \text{lower order terms}.
\]
Though as a word of warning; one should only consider the Ricci flow to be like the heat equation for intuition’s sake. Indeed, the harmonic coordinates in which one can write \( \partial_t g = \Delta g + \) lower order terms depends on the time \( t \). The Ricci flow is known to only be weakly parabolic which historically has made it difficult to prove even simple things like short time existence. We’ll of course be interested in the case when \( g(0) = g \). That is, \((M, g(0))\) is AE.

Our main theorem is as follows.

**Theorem 1.2.** If there exits a long time solution \( g(t) \) (that is, \( 0 \leq t < \infty \)) of the Ricci flow with \( g(0) = g \), then the mass \( m(g) \geq 0 \) with equality if and only if \((M^n, g)\) is isometric to \((\mathbb{R}^n, g_E)\).

The steps in the proof are as follows. The two points marked with a * are quite involved.

1. **Prove the AE condition and the ADM mass is preserved along the flow (Theorem 2.2).**
2. **Use the strong maximum principle to conclude that either \( R(t) > 0 \) for all \( t \in [0, \infty] \) or else \( R(t) \equiv 0 \).**
3. **Prove that if \((M, g)\) is AE and Ricci flat, then \((M, g) \cong (\mathbb{R}^n, g_E)\).**
4. **Introduce Perelman’s \( \mu \)-functional and argue that minimizers of \( \mu \) satisfy an elliptic equation (so we have regularity).**
5. **Show that \( \lim_{\tau \to \infty} \mu(g, \tau) = 0 \) (Theorem 3.4*) and that if \( \mu(g, \tau) \leq 0 \) for all \( \tau \) with equality if and only if \((M, g) \cong (\mathbb{R}^n, g_E)\) (Theorem 3.3).**
6. **Discuss injectivity radius bound along the flow using \( \kappa \)-noncollapsed property and results of Perelman (Theorem 3.9).**
7. **Argue that we can assume \( |Rm| \leq \frac{C}{1+t} \) by analyzing the singularity at time infinity.**
8. **Prove that \( |Rm| \leq \frac{C}{(1+t)^{1+\tau}} \) (Theorem 4.5*).**
9. **Prove that \( |\nabla^k Rm| \leq \frac{C}{(1+t)^{1+\tau+k}} \) (Theorem 4.7).**
10. **Prove that \( g(t) \) converges smoothly to \( g_E \) (in strong sense) and in particular, \( M \approx \mathbb{R}^n \).**
11. **Prove the theorem by showing that \( m(g) = \lim_{t \to \infty} \int R(t) dV_{g(t)} \) which is non-negative (we’ll see that \( R(t) \geq 0 \) by maximum principle). Show equality only occurs iff \((M, g) \cong (\mathbb{R}^n, g_E)\).**

Notationally, let \( B_g(p, r) \) denote the ball of radius \( r \) centered at \( p \) under the metric \( g \) (the manifold should be clear from contest) and let \( \cong \) denote a Riemannian manifold isomorphism. Additionally, \( \Delta = \Delta_{g(t)}, \nabla = \nabla_{g(t)}, dV = dV_{g(t)} \), etc. Moreover, \( C \) may vary line-by-line though we’ll attempt to make it clear in the context what variables \( C \) may depend on.
Remark. Yu Li’s original paper also provides an alternative proof of the positive mass conjecture in dimension $n = 3$ by considering the Ricci flow with surgery on an arbitrary 3-dimensional AE manifold. This part of the paper (chapters 6 and 7) are not included in this write-up.

CHAPTER 2: MASS UNDER RICCI FLOW

One of the main tools we’ll use throughout is the following version of the maximum principle for the Ricci flow on noncompact manifolds [9]. It essentially says that if you have a subsolution $u$ to a parabolic operator $L$, then (under mild conditions on $L$ and $u$) the subsolution $u$ is forever bounded by the solution to the ODE that is given purely by the reaction term in $L$.

**Theorem 2.1.** Suppose that $g(t), t \in [0, T]$, is a complete solution to the Ricci flow on $M$ with bounded curvature. Let

$$Lu = u_t - \Delta u - \langle X(t), \nabla u \rangle - G(u, t)$$

where $X(t)$ is a smooth family of bounded vector fields and $G : \mathbb{R} \times [0, T] \to \mathbb{R}$ is locally Lipschitz in the $\mathbb{R}$ factor and continuous in the $[0, T]$ factor. Suppose that $u$ is a smooth function such that $Lu \leq 0$ and $|u(x, t)| \leq \exp\left(b(d_{g(t)}(O, x) + 1)\right)$ for some constant $b$ and some $O \in M$. For any $c \in \mathbb{R}$, let $U(t)$ be the solution to the ODE

$$\frac{dU}{dt} = G(U, t), \quad U(0) = c.$$ 

If $u(x, 0) \leq c$ for all $x \in M$, then $u(x, t) \leq U(t)$ for all $x \in M$ and $t \in [0, T]$ as long as the ODE exists.

The first step in the proof of Theorem 1.2 is to show that the AE coordinate system and ADM mass is preserved along the Ricci flow. Unlike the proof in [10] (see also [19]), we will fix the AE coordinate system along the flow.

**Theorem 2.2.** Let $(M, g(t))$ be a Ricci flow solution on $[0, T]$ with $|Rm(g(t))| < S$ on all of $M \times [0, T]$. If $(M, g(0))$ is an AE manifold of order $\sigma > 0$, then

(i) The AE condition is preserved along the flow with the same AE coordinates and order $\sigma$.

(ii) If the order $\sigma > (n - 2)/2$, then the mass is unchanged along the flow. That is, $m(g(t)) = m(g(0))$.

**Proof.** (i) Since $(M, g(0))$ is an AE manifold with one end $E$, let $\Phi : E \to \mathbb{R}^n \setminus B_A(0)$ be the diffeomorphism given in the definition of AE. Under this coordinate system, we have

$$g_{ij}(0) = \delta_{ij} + O(r^{-\sigma}), \quad \partial^\alpha g_{ij}(0) = O(r^{-\sigma-k})$$

for all multi-indices $|\alpha| = k$ and any $k \in \mathbb{N}$. Therefore

$$\Gamma^k_{ij} = \frac{1}{2} g^{mk}[g_{mi;j} + g_{mj;i} - g_{ij;m}] = O(r^{-\sigma-1})$$
and subsequently

$$|\text{Rm}_{ijkl}| = |g_{lp} R^p_{ijkl}| = |g_{lp} \left[ \frac{\partial \Gamma^p_{ik}}{\partial x^j} - \frac{\partial \Gamma^p_{jk}}{\partial x^i} + \Gamma^q_{ik} R^p_{jq} - \Gamma^q_{jk} R^p_{iq} \right]|$$

Then

$$|\nabla (\text{Rm})_{ijklm}| = |\text{Rm}_{ijkl;m} - \text{Rm}_{ajkl} \Gamma^a_{mi} - \text{Rm}_{akskl} \Gamma^a_{mj} - \text{Rm}_{ajkl} \Gamma^a_{mk} - \text{Rm}_{ijkl} \Gamma^a_{ml}|$$

The pattern continues for higher covariant derivatives:

$$|\nabla^k \text{Rm}| = O(r^{-\sigma-k-2}).$$

Now by the evolution equation for $|\text{Rm}|$ (standard formula, but one reference is on page 153 [5]), we have

$$\partial_t |\text{Rm}|^2 = \Delta |\text{Rm}|^2 - 2|\nabla \text{Rm}|^2 + \text{Rm} \ast \text{Rm} \ast \text{Rm} \leq \Delta |\text{Rm}|^2 + 16|\text{Rm}|^3 \leq \Delta |\text{Rm}|^2 + 16|\text{Rm}|^2.$$  

Note that the $A \ast B$ is just notation for some linear sum of contractions of tensors $A$ and $B$. Let $u = |\text{Rm}|^2 e^{-16St}$. Then

$$\partial_t u = -16S|\text{Rm}|^2 e^{-16St} + \partial_t |\text{Rm}|^2 e^{-16St} \leq -16S|\text{Rm}|^2 e^{-16St} + |\Delta |\text{Rm}|^2 + 16S|\text{Rm}|^2 |e^{-16St} = \Delta u$$

on $M \times [0, T]$. Let $h(x) = t^{4+2\sigma}$ and let $w = hu$. Notice that $h$ has no dependence on $t$. Then

$$(\partial_t - \Delta)w = h \partial_t u - |\Delta hu + 2\nabla h \nabla u + h\Delta u| \leq -\Delta hu - 2\nabla h \nabla u$$

$$= \frac{2|h|\nabla h|^2}{h^2} hu - \frac{2\nabla h}{h} h \nabla u - \frac{2\nabla h}{h} u \nabla h$$

$$= \frac{2|h|\nabla h|^2}{h^2} - h \Delta h - 2\nabla \log h \nabla w =: Bw - 2\nabla \log h \nabla w.$$

We will show that $B$ is uniformly bounded on $M \times [0, T]$. 

Note that we have

$$\partial_t |\nabla h|^2 = \partial_t (g^{ij} \partial_t h \partial_j h) = \partial_t g^{ij} \partial_t h \partial_j h = 2\text{Rc}_{ik} g^{ij} \partial_j h \partial_l h = 2\text{Rc}(\nabla h, \nabla h).$$

And another identity is regarding the evolution of the Laplacian under Ricci flow:

$$\partial_t (\Delta h) = \partial_t (\text{tr} \nabla^2 h) = \partial_t [g^{ij} (\partial_i \partial_j h - \Gamma^k_{ij} \partial_k h)] = (\partial_t g^{ij}) \nabla_i f \nabla_j f - g^{ij} (\partial_t \Gamma^k_{ij}) \partial_k f = 2(\text{Rc}, \nabla^2 h)$$

where we used that $\partial_t g^{ij} = 2\text{Rc}_{pq} g^{ip} g^{jq}$ and that $g^{ij} \partial_t \Gamma^k_{ij} = 0$. The latter equation follows from the contracted second Bianchi identity and the evolution of the Christoffel symbols:

$$g^{ij} \partial_t \Gamma^k_{ij} = g^{ij} [-g^{kl} (\nabla_i R_{jlt} + \nabla_j R_{ilt} - \nabla_t R_{ij})] = -g^{ij} g^{kl} (2\nabla_i R_{jlt} - \nabla_t R_{ij})$$

$$= -g^{kl} (g^{ij} 2\nabla_i R_{jlt} - \nabla_t R) = -[2\text{tr} g(\nabla \text{Rc}) - \nabla R]^# = 0.$$ 

Since $\partial_t g = -2\text{Rc}$ and $\text{Rc}$ is uniformly bounded, the metrics $g(t)$ are uniformly comparable to $g(0)$. Which is to say there is some $C > 0$ such that

$$C^{-1} g(0) \leq g(t) \leq C g(0).$$

Then from the bound on the curvature and the comparability of the metrics, we have

$$|\partial_t |\nabla h|^2| \leq C |\nabla h|^2, \text{ and } |\partial_t \Delta h| \leq C |\nabla^2 h|^2 \leq C |\nabla g(0)| h|_{g(0)}. $$

So by integrating wrt $t$, we can subsequently bound

$$|\nabla h|^2|_{g(t)} \leq C |\nabla g(0)| h|_{g(0)}, \text{ and } |\Delta g(t)| h| \leq C |\nabla g(0)| h|_{g(0)}.$$
where $C$ now depends on $T$. To estimate $|\nabla g_0 h|_g^2$ and $|\nabla^2 g_0 h|_g$, we use the given coordinate system of $g(0)$ in the end $E$. We have

$$|\nabla g_0 h|_g^2 = g^{ij} (\partial_i r^{4+2\sigma}) (\partial_j r^{4+2\sigma}) \lesssim g^{ij} r^{6+4\sigma} \frac{x_i x_j}{r^2} = O(r^{6+4\sigma}).$$

Similarly

$$|\nabla^2 g_0 h|_g = O(\partial_i \partial_j h - \Gamma^k_{ij} h) = O(r^{2+2\sigma}) + O(r^{2+\sigma}) = O(r^{2+2\sigma}).$$

Therefore

$$|B| = \left| \frac{2|\nabla h|^2 - h \Delta h}{h^2} \right| \leq C \left| \frac{|\nabla g_0 h|_{g(0)}^2 - h \Delta g_0 h}{h^2} \right| \leq C \frac{r^{6+4\sigma} + r^{4+2\sigma} r^{2+2\sigma}}{r^{2(4+2\sigma)}} = C r^{-2} \leq C$$

where the last equality is true because $r$ is continuous, strictly positive and $\gg 1$ off of a compact set (indeed $r(x) \to \infty$ as $|x| \to \infty$). Thus we have shown that $B$ is uniformly bounded.

Now since $w = hu$ and $u = |\text{Rm}|^2 e^{-16St}$, we see that $w \geq 0$ and so

$$0 \geq \partial_t w - \Delta w + \langle 2\nabla \log h, \nabla w \rangle - B w \geq \partial_t w - \Delta w + \langle 2\nabla \log h, \nabla w \rangle - Cw.$$

Note that since $h = r^{4+2\sigma}$ is a positive smooth function (that stays away from 0), $X := 2\nabla \log h$ is a bounded smooth vector field on $M$.

Furthermore, note that

$$w = hu = r^{4+2\sigma} |\text{Rm}|^2 e^{-16St} = r^{4+2\sigma} O(1) = O(r^{4+2\sigma}).$$

So $w$ satisfied the condition

$$|w(x,t)| \leq \exp \left( b(d_{g(t)}(O,x) + 1) \right)$$

for some constant $b$. Also note that when $t = 0$,

$$w(x,0) = r^{4+2\sigma} |\text{Rm}|^2 = O(r^{4+2\sigma}) O(r^{-4-2\sigma}) = O(1)$$

is bounded.

Of course, the solution to the IVP

$$\frac{dU}{dt} = G(U,t) = CU, \ U(0) = c$$

is $U(t) = ce^{Ct}$ which is bounded on $[0,T]$ by $ce^{CT}$.

So by Theorem 2.1 $w$ is bounded on $M \times [0,T]$ (by a constant possibly depending on $T$). Therefore

$$C \geq w = hu = r^{4+2\sigma} |\text{Rm}|^2 e^{-16St} \geq r^{4+2\sigma} |\text{Rm}|^2 \implies |\text{Rm}| \leq Cr^{-2-\sigma}.$$

claim: $|\nabla^k \text{Rm}| \leq Cr^{-2-k-\sigma}$.

Proof of the claim: We proved the base case of $k = 0$ above. By strong induction, assume the claim holds for all $0 \leq l < k$. 


From the evolution equation of $|\nabla^k Rm|^2$ (page 153 of [8]) we have
\[
\partial_t |\nabla^k Rm|^2 = \Delta |\nabla^k Rm|^2 - 2|\nabla^{k+1} Rm|^2 + \sum_{l=0}^{k} |\nabla^l Rm | |\nabla^k Rm|. \\
\leq \Delta |\nabla^k Rm|^2 + C \sum_{l=0}^{k} |\nabla^l Rm | |\nabla^{k-l} Rm | |\nabla^k Rm|.
\]

Now let $h_k = r^{4+2\sigma+2k}$, $u_k = |\nabla^k Rm|^2$, and $w_k = h_k u_k$. We have
\[
(\partial_t - \Delta) w_k = h_k \partial_t u_k - [\Delta h_k u_k + 2\\nabla h_k \nabla u_k + h_k \Delta u_k] \\
\leq -\Delta h_k u_k - 2\nabla h_k \nabla u_k + C \sum_{l=0}^{k} h_k u_{l} u_{k-l} u_k \\
= 2|\nabla h_k|^2 - h_k \Delta h_k u_k - 2\nabla h_k h_k u_k - 2\nabla h_k u_k \nabla h_k + C \sum_{l=0}^{k} h_k u_{l} u_{k-l} u_k \\
= 2|\nabla h_k|^2 - h_k \Delta h_k u_k - 2\nabla \log h_k w_k + C \sum_{l=0}^{k} h_k u_{l} u_{k-l} u_k \\
=: B_k w_k - 2\nabla \log h_k \nabla w_k + C \sum_{l=0}^{k} h_k |\nabla^l Rm| |\nabla^{k-l} Rm| |\nabla^k Rm|.
\]

Using (3), we can estimate $|\nabla g(0) h|_{g(0)}^2$ and $|\nabla^2 g(0) h|_{g(0)}$ just as before. Namely
\[
|\nabla g(0) h|_{g(0)}^2 = g^{ij}(\partial_i r^{4+2\sigma+2k})(\partial_j r^{4+2\sigma+2k}) \lesssim g^{ij} r^{6+4\sigma+4k} x_i x_j = O(r^{6+4\sigma+4k})
\]
and
\[
|\nabla^2 g(0) h|_{g(0)} = O(\partial_i \partial_j h_k - \Gamma^l_{ij} h_k) = O(r^{2+2\sigma+2k}) + O(r^{2+\sigma+2k}) = O(r^{2+2\sigma+2k}).
\]

Therefore using the comparability of metrics (2) just as before, we have
\[
|B_k| \leq \left\{ \frac{2|\nabla h_k|^2 - h_k \Delta h_k}{h_k^2} \right\} \leq C \left\{ \frac{|\nabla g(0) h|_{g(0)}^2 - h_k \Delta g(0) h_k}{h_k^2} \right\} \\
\leq C r^{6+4\sigma+4k} + r^{4+2\sigma+2k} \frac{r^{2+2\sigma+2k}}{r^{2+2\sigma+2k}} = C r^{-2} \leq C
\]
where the last equality is true as $\inf r > 0$. Thus we have shown that $B_k$ is uniformly bounded. Moreover, by our inductive hypothesis, we have
\[
h_k |\nabla^l Rm| |\nabla^{k-l} Rm| |\nabla^k Rm| = h_k |\nabla^l Rm|^2 \leq C w_k \quad \text{for} \ l = 0, k
\]
and
\[
h_k |\nabla^l Rm| |\nabla^{k-l} Rm| |\nabla^k Rm| \leq r^{4+2\sigma+2k} \frac{1}{r^{2-\sigma-2-(k-l)-\sigma}} |\nabla^k Rm| \\
= r^k |\nabla^k Rm| \leq C r^{2+\sigma+k} |\nabla^k Rm| = C w_k^{1/2} \quad \text{for} \ 0 < l < k
\]
In the last inequality, we again used the fact that $\inf r > 0$. Therefore we have
\[
(\partial_t - \Delta) w_k \leq B_k w_k - 2\nabla \log h_k \nabla w_k + C \sum_{l=0}^{k} h_k |\nabla^l Rm| |\nabla^{k-l} Rm| |\nabla^k Rm| \\
\leq C w_k - 2\nabla \log h_k \nabla w_k + C (w_k + w_k^{1/2}) = -2\nabla \log h_k \nabla w_k + C (w_k + w_k^{1/2})
\]
Now we can prove by induction that
\[ \partial_t w_k - \Delta w_k + 2\nabla \log h_k \nabla w_k - C(w_k + w_k^{1/2}) \geq \partial_t w_k - \Delta w_k + 2\nabla \log h_k \nabla w_k - C(w_k + 1). \]

Just as before, \( 2\nabla \log h_k =: X_k \) is a smooth bounded vector field (since \( \inf r > 0 \)) and the function \( G(u) = C(u + 1) \) is locally Lipschitz. Of course, the solution to the IVP
\[
\frac{dU}{dt} = G(U, t) = C(U + 1), \quad U(0) = c
\]
is \( U(t) = (c + 1)e^{Ct} - 1 \) which is bounded on \([0, T] \) by \((c + 1)e^{CT}\). Since \( |\nabla^k Rm(0)|^2 = O(r^{-4-2\sigma-2k}) \), we can see that \( w_k = r^{4+2\sigma+2k}|\nabla^k Rm(0)|^2 \leq c \) for an appropriate choice of \( c \). Therefore, by Theorem 2.1, we have
\[
w_k = r^{4+2\sigma+2k}|\nabla^k Rm|^2 \leq C \implies |\nabla^k Rm| \leq Cr^{-2-\sigma-k}.
\]
This proves the claim. \( \square \)

Now for any vector field \( V \) on \( M \) we have (by the FTOC)
\[
\left| \log \frac{g(x, t)(V, V)}{g(x, 0)(V, V)} \right| = \left| \log \frac{g(x, t)(V, V) - g(x, 0)(V, V)}{g(x, 0)(V, V)} \right| = \left| \int_0^t \frac{\partial_s g(x, s)(V, V)}{g(x, s)(V, V)} \frac{ds}{g(x, s)(V, V)} \right|
\leq \left| \int_0^t \frac{2\text{Re}(x, s)(V, V)}{g(x, s)(V, V)} \frac{ds}{g(x, s)(V, V)} \right| \leq C \left| \int_0^t \text{Rm}(ds) \right| \leq Cr^{-2-\sigma}.
\]
where we used monotonicity of the integral and the bound on \( \text{Rm} \). Thus we have
\[
\left| \log \frac{g(x, t)(V, V)}{g(x, 0)(V, V)} \right| \leq Cr^{-2-\sigma} \implies \left| \frac{g(x, t)(V, V)}{g(x, 0)(V, V)} \right| \leq e^{Cr^{-\sigma} - 2} = (e^C)^{r^{-\sigma} - 2} = 1 + O(r^{-\sigma})
\]
and in particular
\[
g_{ii}(t) = g_{ii}(0)(1 + O(r^{-\sigma-2})) = (1 + O(r^{-\sigma}))(1 + O(r^{-\sigma-2})) = 1 + O(r^{-\sigma}).
\]

Now use the polarization identity (with \( V = \partial_i - \partial_j \) and \( i \neq j \)), and the asymptotically Euclidean condition of \( g(0) \):
\[
g_{ij}(t) = \frac{1}{2} [g_{ii}(t) + g_{jj}(t) - g(t)(V, V)] = \frac{1}{2} [g_{ii}(t) + g_{jj}(t) - (1 + O(r^{-\sigma-2}))g(0)(V, V)]
\leq \frac{1}{2} [1 + O(r^{-\sigma}) + 1 + O(r^{-\sigma}) - (1 + O(r^{-\sigma-2}))g(0)_{ii} + g(0)_{jj} - 2g(0)_{ij}]
\leq \frac{1}{2} [2 + O(r^{-\sigma}) - (1 + O(r^{-\sigma-2}))(1 + O(r^{-\sigma}) + 1 + O(r^{-\sigma} + O(r^{-\sigma})) = O(r^{-\sigma}).
\]
Recall the evolution equation of the Christoffel symbols
\[
\partial_t \Gamma^k_{ij} = -g^{kl}(\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij})
\]
and that \( |\nabla^k Rm| \leq Cr^{-2-k-\sigma} \) and \( \Gamma^k_{ij}(0) = O(r^{-1-\sigma}) \) to get
\[
\partial_t \Gamma^k_{ij}(t) = (-\delta^{kl} + O(r^{-\sigma}))O(r^{-3-\sigma}) = O(r^{-3-\sigma})
\]
which (after integrating) yields
\[
\Gamma^k_{ij}(t) - \Gamma^k_{ij}(0) = O(r^{-3-\sigma}) \implies \Gamma^k_{ij}(t) = O(r^{-1-\sigma}).
\]
Hence
\[
\partial_t R_{jk} = \nabla_i R_{jk} + \Gamma^l_{ij} R_{lk} + \Gamma^l_{ik} R_{jl} = O(r^{-3-\sigma}) + O(r^{-1-\sigma})O(r^{-2-\sigma}) = O(r^{-3-\sigma}).
\]
Since \( \partial_t(\partial_i g_{jk}) = -2\partial_t R_{jk} = O(r^{-\sigma-3}) \), and \( \partial_t g_{ij}(0) = O(r^{-1-\sigma}) \), we have \( \partial_t g_{jk} = O(r^{-\sigma-1}) \). Now we can prove by induction that \( \partial^k g_{ij} = O(r^{-\sigma-k}) \) for all \( k \) which would make \( \langle E, g_{ij}(t) \rangle \)
an AE coordinate system of order $\sigma$. But that induction is not a lot of fun! I therefore only do the next case (of $k = 2$): First, from the coordinate expression for the covariant derivative,

$$\partial_i(\nabla_p R_{ij}) = \nabla_i \nabla_p R_{jk} + \Gamma^l_{ij} \nabla_p R_{lk} + \Gamma^l_{ik} \nabla_p R_{jl} = O(r^{-\sigma - 4}) + O(r^{-\sigma - 1}) O(r^{-\sigma - 3}) = O(r^{-\sigma - 4}).$$

On the other hand,

$$\partial_i(\nabla_p R_{ij}) = \partial_i \partial_p R_{jk} - \partial_i (\Gamma^l_{pj} R_{lk}) - \partial_i (\Gamma^l_{pk} R_{jl})$$

but $\partial_i R_{lk} = O(r^{-\sigma - 3})$ from above and $\partial_i \Gamma^l_{pj} = O(r^{-\sigma})$ is true because of

$$\partial_i \partial_i \Gamma^k_{ij} = \partial_i [-g^{kl} (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_i R_{ij})] = O(r^{-\sigma - 4})$$

and because $\partial_i \Gamma^k_{ij}(0) = O(r^{-\sigma - 2}).$

Conclusion: $(E, g(t)_{ij})$ is an AE coordinate system of order $\sigma$.

Proof of (ii) By the definition of ADM mass

$$m(g(t)) = \lim_{r \to \infty} \int_{S_r} (\partial_i g_{ij}(t) - \partial_j g_{ii}(t)) \, dA^j$$

which is well-defined since Ricci flow preserves the AE condition and integrable scalar curvature (by using our bound on the curvature). We just showed in (i) that we have a common AE coordinate system. So

$$m'(g(t)) = \frac{d}{dt} \lim_{r \to \infty} \int_{S_r} (\partial_i g_{ij}(t) - \partial_j g_{ii}(t)) \, dA^j$$

$$= \lim_{r \to \infty} \int_{S_r} (\partial_i \partial_i g_{ij}(t) - \partial_j \partial_j g_{ii}(t)) \, dA^j$$

$$= -2 \lim_{r \to \infty} \int_{S_r} (\partial_i R_{ij}(t) - \partial_j R_{ii}(t)) \, dA^j$$

$$= -2 \lim_{r \to \infty} \int_{S_r} (\nabla_i R_{ij}(t) + \Gamma^d_{ij}(t) R_{dj}(t) + \Gamma^d_{ij}(t) R_{di}(t)$$

$$- \nabla_j R_{ci}(t) - \Gamma^d_{ij}(t) R_{di}(t) - \Gamma^d_{ij}(t) R_{di}(t)) \, dA^j$$

$$= -2 \lim_{r \to \infty} \int_{S_r} (\nabla_i R_{ij}(t) + \Gamma^d_{ij}(t) R_{dj}(t) - \nabla_j R_{ci}(t) - \Gamma^d_{ij}(t) R_{di}(t) \, dA^j.$$
This limit is equal to 0 whenever $t > 0$ by Lemma 11 in [19]. We need only show that $m$ is continuous at $t = 0$. This is also shown in [19] but we include it here for the reader’s convenience. By Lemma 10 in [19], there is some $\epsilon$ which goes to 0 as $r \to \infty$ such that

$$\int_{M \setminus B_r} R(t) < \epsilon(r)$$

for all $t \in [0, T]$. Furthermore, by an integration by parts, we have

$$m(g(t)) = \int_{M \setminus B_r} R(t) \, dV + \int_{\partial B_r} (g(t)_{ij,i} - g(t)_{ii,j}) \, dS^i + O(r^{-\lambda}).$$

So

$$|m(g(t)) - m(g(0))| \leq \epsilon(r) + \left| \int_{\partial B_r} (g(t)_{ij,i} - g(0)_{ij,i}) \, dS^i \right| + \left| \int_{\partial B_r} (g(t)_{ii,j} - g(0)_{ii,j}) \, dS^i \right|.$$

First, set $r$ large enough so that $\epsilon(r)$ is small. Now we can require $t$ be small enough so that the latter two terms are small (since the $\partial_t g = \text{Rc}$ and the curvature is bounded). This completes the proof of Theorem 2.2.

Recall the evolution equation for scalar curvature:

$$\partial_t R = \Delta R + 2|\text{Rc}|^2 \geq \Delta R.$$ 

So by Theorem 2.1 (with $u = -R$ and $X \equiv 0 \equiv G$), we see that $R(t) \geq 0$ on all of $M \times [0, T]$. Now by the strong maximum principle under Ricci flow (see Lemma 6.57 in [9]), either $R(x, t) > 0$ for all $(x, t) \in M \times (0, T)$, or else $R \equiv 0$. In the first case, we simply ‘cut off $t = 0’$ so-to-say by redefining the flow to begin at some $t_0 \in (0, T)$ so that $R(x, t) > 0$ for all $x \in M$, $t \geq 0$. This isn’t a problem in proving our main theorem (that $m(g) \geq 0$) since the mass is unchanged along the flow. In the second case, the evolution equation of $R$ implies that $\text{Rc}(0) \equiv 0$. That is to say that $(M, g(0))$ is Ricci-flat. In that case, we have

**Theorem 2.4.** If $(M, g)$ is a Ricci-flat AE manifold, then $(M, g) \cong (\mathbb{R}^n, g_E)$.

Before proceeding with the proof of Theorem 2.4, we have two lemmas. Fix a point $p \in M$ and let $d(x) = d_g(x, p)$ be the distance function to $p$.

**Lemma 2.5.** $\lim_{|x| \to \infty} \frac{r(x)}{d(x)} = 1$.

**Proof.** From the defn of AE manifolds, there exists some (possibly large) $r_0$ such that

$$1 + Cr^{-\sigma})^{-1} g_E(x) \leq g(x) \leq (1 + Cr^{-\sigma}) g_E(x)$$

for all $r \geq r_0$. Now fix some $r_1 > r_0$ and some $x \in M$ with $r(x) \gg r_1$ and let $\{\gamma(t), t \in [0, d(x)]\}$ be the minimizing geodesic from $p$ to $x$. Note that such a $\gamma$ exists since

$$d(x) = d_g(x, p) = \inf_\alpha \{ \alpha : \text{curves from } x \text{ to } p \}$$

is achieved since we’re assuming completeness. Then there exists an $r_x \in [0, d]$ such that $r(\gamma(r_x)) = r_1$ and $r(\gamma(t)) \geq r_1$ for all $t \in [r_x, d(x)]$ since the metric is AE. Note that we may need to take $r_1$ to be quite large to make this happen, but that’s alright. We assume
Now we estimate the distance between $\gamma(r_x)$ and $x$ under the metric $g_E$. Since $r$ gives the Riemannian distance to the origin in $\mathbb{R}^n \setminus B(0, A)$, we have

$$r(x) - r_1 \leq d_E(x, \gamma(r_x)) \leq \int_{r_x}^{d(x)} |\gamma'(t)|_{g_E} \, dt$$

since $\gamma : [r_x, d(x)] \to M$ is a path from $\gamma(r_x)$ to $x$. Now use (4) to see

$$\int_{r_x}^{d(x)} |\gamma'(t)|_{g_E} \, dt \leq (1 + C r_1^{-\sigma}) \int_{r_x}^{d(x)} |\gamma'(t)|_g \, dt = (1 + C r_1^{-\sigma})(d(x) - r_x) \leq (1 + C r_1^{-\sigma}) d(x).$$

Therefore

$$r(x) \leq (1 + C r_1^{-\sigma}) d(x) + r_1$$

and so

$$\limsup_{r \to \infty} \frac{r(x)}{d(x)} \leq \frac{(1 + C r_1^{-\sigma}) d(x) + r_1}{d(x)} = (1 + C r_1^{-\sigma})$$

since $r_1$ is fixed. On the other hand, let $\{\gamma_1(t), \ t \in [0, a]\}$ be the minimizing geodesic from $\gamma(r_x)$ to $x$ under the Euclidean metric $g_E$. Since $d$ is the distance (from $p$) function under $g$, and since $\gamma(r_x)$ is a point lying on the minimizing geodesic from $p$ to $x$, we have

$$d(x) - r_x \leq d_g(x, \gamma(r_x)) \leq \int_0^a |\gamma_1'(t)|_g \, dt$$

since $\gamma_1 : [0, a] \to M$ is a path from $\gamma(r_x)$ to $x$. Now use (4) to see

$$\int_0^a |\gamma_1'(t)|_g \, dt \leq (1 + C r_1^{-\sigma}) \int_0^a |\gamma_1'(t)|_{g_E} \, dt = (1 + C r_1^{-\sigma})(d(x) - r_x) \leq (1 + C r_1^{-\sigma})(r(x) + r_1).$$

The last inequality is true because $\gamma(r_x)$ and $x$ may not ‘line up’ in the Euclidean sense... you may have to track all the way back to the ‘zero point’ (the $r_1$ contribution) and then go all the way out to $x$ (the $r(x)$ contribution). Then from $r_x \leq C r_1 r_1$, we have

$$d(x) \leq (1 + C r_1^{-\sigma})(r(x) + r_1) + r_x \leq (1 + C r_1^{-\sigma}) r(x) + (1 + C r_1^{-\sigma} + C_1) r_1.$$

Therefore

$$(1 + C r_1^{-\sigma})^{-1} = \liminf_{r \to \infty} \frac{r(x)}{(1 + C r_1^{-\sigma}) r(x) + (1 + C r_1^{-\sigma} + C_1) r_1} \leq \liminf_{r \to \infty} \frac{r(x)}{d(x)}.$$

Now combine (5) and (6) and recall that $r_1 \geq r_0$ was arbitrary, so we can take $r_1 \to \infty$ to conclude

$$1 \leq \liminf_{r \to \infty} \frac{r(x)}{d(x)} \leq \limsup_{r \to \infty} \frac{r(x)}{d(x)} \leq 1.$$

\[\square\]

**Lemma 2.6.**

$$\lim_{r \to \infty} \frac{\text{Vol}_g B(p, d(x))}{w_n r^n(x)} = 1,$$

where $w_n$ is the volume of the unit ball in $\mathbb{R}^n$. 
\begin{proof}
We again start by recalling that for an AE manifold, there is some \( r_0 > 0 \) sufficiently large such that \( M \) holds and therefore
\begin{equation}
(1 + Cr^{-\sigma})^{-1} \text{Vol}_{g_E}(x) \leq \text{Vol}_g(x) \leq (1 + Cr^{-\sigma})\text{Vol}_{g_E}(x)
\end{equation}
for any \( r(x) \geq r_0 \). By Lemma 2.5, there exists a function \( \epsilon(r) > 0 \) with \( \epsilon(r) \to 0 \) as \( r \to \infty \) such that
\begin{equation}
e^{-\epsilon(r)} \leq \frac{r(x)}{d(x)} \leq e^{\epsilon(r)}.
\end{equation}

Fix some \( r_1 \geq r_0 \). Then for any \( r(x) > r_1 \) with \( e^{-\epsilon(r)}r(x) \geq r_1 \) (which is possible since \( \epsilon(r) \to 0 \) and we’ve fixed \( r_1 \)), we have
\[
w_n \left( (e^{-\epsilon(r)}r)^n - r_1^n \right) = \text{Vol}_{g_E}(B_{g_E}(p, e^{-\epsilon(r)}r) \setminus B_{g_E}(p, r_1)).
\]
Now whenever \( r(x) \geq r_1 \), we have the volume comparison \( \text{Vol}_{g_E}(B_{g_E}(0, e^{-\epsilon(r)}r) \setminus B_{g_E}(0, r_1)) \leq (1 + Cr_1^{-\sigma})\text{Vol}_g(B_{g_E}(0, e^{-\epsilon(r)}r) \setminus B(0, r_1)). \)

But since \( r \) is the Euclidean distance function, the set
\[
B_{g_E}(0, e^{-\epsilon(r)}r) \setminus B(0, r_1)
\]
is exactly all points \( y \in M \) satisfying \( r_1 \leq r(y) \leq e^{-\epsilon(r(x))}r(x) \). Then since \( r_1 \geq r_0 \) and \( \text{(8)} \) holds for all \( y \) with \( r(y) \geq r_0 \), we have
\[
B_{g_E}(0, e^{-\epsilon(r)}r) \setminus B(0, r_1) \subset B_g(p, d).
\]
So we have
\[
w_n \left( (e^{-\epsilon(r)}r)^n - r_1^n \right) \leq (1 + Cr_1^{-\sigma})\text{Vol}(B_g(p, d)).
\]
Therefore
\[
(1 + Cr_1^{-\sigma})^{-1} = (1 + Cr_1^{-\sigma})^{-1} \lim_{r \to \infty} e^{-\epsilon(r)}
\]
\[
= (1 + Cr_1^{-\sigma})^{-1} \lim_{r \to \infty} \frac{w_n \left( (e^{-\epsilon(r)}r)^n - r_1^n \right)}{w_n r^n}
\]
\[
\leq \lim_{r \to \infty} \frac{\text{Vol}_g(B(p, d(x))}{w_n r^n}.
\]
Then for any \( x \) with \( d(x) \geq e^{\epsilon(r_1)}r_1 \), we have
\[
\text{Vol}_g(B_g(p, d) \setminus B_g(p, e^{\epsilon(r_1)}r_1)) \leq \text{Vol}_g(B_{g_E}(0, e^{\epsilon(r)}r) \setminus B(0, r_1))
\]
since
\[
r(x)e^{\epsilon(r)} \geq d(x) \geq d(y) \quad \text{and} \quad e^{\epsilon(r(y))}r(y) \geq e^{\epsilon(r_1)}r(y) \geq d(y) \geq e^{\epsilon(r_1)}r_1.
\]
Then using the volume comparison \( \text{(8)} \), we have
\[
\text{Vol}_g(B_{g_E}(0, e^{\epsilon(r)}r) \setminus B(0, r_1)) \leq (1 + Cr_1^{-\sigma})\text{Vol}_{g_E}(B_{g_E}(0, e^{\epsilon(r)}r) \setminus B(0, r_1)).
\]
Now since \( r \) is the Euclidean distance from the origin, we have
\[
(1 + Cr_1^{-\sigma})\text{Vol}_{g_E}(B_{g_E}(0, e^{\epsilon(r)}r) \setminus B(0, r_1)) = (1 + Cr_1^{-\sigma})w_n((e^{\epsilon(r)}r)^n - r_1^n)
\]
\[
\leq (1 + Cr_1^{-\sigma})w_n(e^{\epsilon(r)}r)^n.
\]
Therefore
\[
\limsup_{r \to \infty} \frac{\text{Vol}_g(B(p, d(x))}{w_n r^n} \leq \limsup_{r \to \infty} \frac{(1 + Cr_1^{-\sigma})w_n(e^{-\epsilon(r)}r)^n \text{Vol}_g(B(p, e^{\epsilon(r_1)}r_1))}{w_n r^n}
\]
\[
= (1 + Cr_1^{-\sigma}) \limsup_{r \to \infty} e^{-\epsilon(r)}r^n = (1 + Cr_1^{-\sigma}).
\]
\end{proof}
So combining our results, we have
\[
(1 + Cr_1^{-\sigma})^{-1} \leq \liminf_{r \to \infty} \frac{\Vol_g B(p, d(x))}{w_n r^n} \leq \limsup_{r \to \infty} \frac{\Vol_g B(p, d(x))}{w_n r^n} \leq (1 + Cr_1^{-\sigma})
\]
and the proof is completed by taking \(r_1 \to \infty\).

\[\square\]

**Proof of Theorem 2.4.** The Bishop-Gromov volume comparison theorem (see Corollary 1.134 in [9]) states that if \(M^n\) has \(Rc \geq 0\), then
\[
\Psi(r) := \frac{\Vol_g B_g(p, r)}{\Vol_{(\mathbb{R}^n, g_E)} B_{g_E}(0, r)}
\]
is a non-increasing function in \(r\), tends to 1 as \(r \to 0^+\). Moreover, if \(\Phi(r) = 1\) for any \(r > 0\), \(B_g(p, r) \equiv B_{g_E}(0, r)\). But by the previous two lemmas, we have
\[
\lim_{r \to \infty} \frac{\Vol_g B_g(p, r)}{\Vol_{g_E} B_{g_E}(0, r)} = \lim_{r \to \infty} \frac{\Vol_g B_g(p, r(x))}{\Vol_{g_E} B_{g_E}(0, r)} = 1
\]
Therefore, \(r \mapsto \Psi(r)\) is constant and thus \((M, g)\) is isometric to \((\mathbb{R}^n, g_E)\).

\[\square\]

As commented on immediately before the statement of Theorem 2.4, we may thus assume that \(R(t) > 0\) for all \(x \in M, t \in [0, \infty)\).

**CHAPTER 3: PERELMAN’S \(\mu\)-FUNCTIONAL**

Recall that Perelman’s \(W\) entropy [20] is defined as
\[
W(g, f, \tau) = \int (\tau(|\nabla f|^2_g + R_g) + f - n)) \frac{e^{-f}}{(4\pi \tau)^{n/2}} dV_g
\]
for a smooth function \(f\) and \(\tau > 0\). Let \(u^{-f/2}\) so that the above becomes
\[
\overline{W}(g, u, \tau) = \int (\tau(4u^{-2} |\nabla u|^2_g + R_g) - 2 \log u - n)) \frac{u^2}{(4\pi \tau)^{n/2}} dV_g
\]
\[
= \int (\tau(4 |\nabla u|^2_g + R_g u^2) - \log u^2 u^2 - nu^2) (4\pi \tau)^{-n/2} dV_g.
\]

Moreover, For a general (possibly incomplete) Riemannian manifold \((M, g)\), \(\mu\)-functional is defined as
\[
\mu(g, \tau) = \inf \{\overline{W}(g, u, \tau) | u \in W^{1,2}_0(M) \text{ and } \int_M u^2(4\pi \tau)^{-n/2} dV = 1\}.
\]

Note that when \(M\) is complete, may replace \(W^{1,2}_0(M)\) with \(W^{1,2}(M)\). Moreover, we have \(\mu_U(g, \tau) \geq \mu_M(g, \tau)\) for any open set \(U \subset M\) since we are infing over a small set of functions. Also, by the scaling properties of metrics, it is easy to show for any \(c > 0\) that
\[
(9) \quad \overline{W}(g, u, c\tau) = \overline{W}(c^{-1} g, u, \tau)
\]
which immediately implies
\[
\mu(g, c\tau) = \mu(c^{-1} g, \tau).
\]

We have the following monotonicity result [4] for the complete noncompact manifold,
\[
\mu(g(t_2), \tau(t_2)) \geq \mu(g(t_1), \tau(t_1))
\]
for all \(0 \leq t_1 \leq t_2 < \tau\) where \(\tau(t) = \tau - t\) for \(0 < \tau < T\). Here we assume that the Ricci flow exists for \([0, T]\) and \(|Rm|\) is uniformly bounded in space and time. The proof of this
monotonicity result is done as follows: Take \( t < s \) and let \( f_s \) to be a function that either minimizes \( W \) or else is arbitrarily close. Then flow backwards by the conjugate heat equation (which is equivalent to flowing forward by the standard heat equation which can always be done) and prove the monotonicity along this flow (this monotonicity is what is proved in \([4]\)). Then since \( \mu \) is the inf of all such functions, it is less than or equal to the value obtained from the function resulting from the flow. Since we started with a minimizer (or arbitrarily close), the result follows. We’ll use this monotonicity result mainly in the following form:

\[
\mu(g(t), \tau) \geq \mu(g(0), \tau + t).
\]

If \( g \) has bounded geometry (that is to say bounded curvature and positive injectivity radius), then \( \mu(g, \tau) \) is finite for any \( \tau \) (see \([24]\)).

Moreover, Zhang \([33]\) proved that if \( g \) has bounded geometry, then \( W(g, u, 1) \) has a smooth positive minimizer if \( \mu(g, 1) \) is less than the corresponding value at infinity. That is to say, \( W(g, u, 1) \) has a smooth positive minimizer if

\[
\mu(\tau^{-1}g, 1) < \lim_{r \to \infty} \mu(M \setminus B(p, r), \tau^{-1}g, 1).
\]

We’ll soon see that the corresponding value at infinity is 0 for any AE manifold (note that \( \tau^{-1}g \) is still an AE metric). It is worth noting that in \([33]\), Zhang works with a slightly different functional \( \lambda(g) \) which differs from \( \mu(g, 1) \) by a constant. Precisely,

\[
\mu(g, 1) = \lambda(g) - n - \frac{n}{2} \log 4\pi.
\]

On our way to proving our main theorem, we will prove the utmost important theorem 3.4 which is stated below. It may be of independant interest to readers as it is not a result which involves the Ricci flow, but rather is a statement about the behaviour of the \( \mu \)-functional when \( \tau \to \infty \). As an aside, I am particularly interested if this result holds when \( R \) is not assumed to be positive.

**Theorem 3.4.** If \((M^n, g)\) is an AE manifold with \( R > 0 \), then \( \lim_{\tau \to \infty} \mu(g, \tau) = 0 \).

The proof will go by contradiction in which we assume that there is a sequence of ‘times’ \( \tau_k \) such that \( \mu(g, \tau_k) \to c < 0 \). Thus, we will use the fact that \( \mu(\tau^{-1}_k g, 1) = \mu(g, \tau_k) < 0 \) which is the corresponding value at infinity. Thus \( \mu(g, \tau_k) \) has a smooth positive minimizer. The following lemma says that this minimizer satisfies a parabolic PDE called the Euler-Lagrange equation. This will allow us to gain extra insight on the regularity of this minimizer.

**Lemma.** Suppose \( u \) is a smooth positive minimizer for \( \mu(g, \tau) \). Then \( u \) satisfies

\[
(\tau(-4\Delta u + Ru) - u \log u^2 - nu = \mu(g, \tau)u
\]

Equation \((10)\) is called the Euler-Lagrange equation for \( \mu(g, \tau) \).

**Proof.** Recall the scaling property of \( W \) from equation \([9]\) is that

\[
W(g, u, c\tau) = W(c^{-1}g, u, \tau).
\]

Thus if \( u \) is a smooth positive minimizer for \( \mu(g, c\tau) \), then it is also for \( \mu(c^{-1}g, \tau) \). Moreover, by the scaling property of metrics, we have

\[
\Delta c^{-1}g = c\Delta g, \quad \text{and} \quad R_{c^{-1}g} = cR_g.
\]
Therefore for any $\tau > 0$, let $c = (4\pi)^{1/2}$. Then assuming that the result holds for the special case $\tau = (4\pi)^{-1}$, we have

\[
\mu(g, \tau) = \mu(g, c(4\pi)^{-1}) = \mu(c^{-1}g, (4\pi)^{-1}) \\
= (4\pi)^{-1}(-4\Delta_{c^{-1}g}u + R_{c^{-1}g}u) - u \log u^2 - nu \\
= c(4\pi)^{-1}(-4\Delta_g u + R_g u) - u \log u^2 - nu \\
= \tau(-4\Delta_g u + R_g u) - u \log u^2 - nu.
\]

Of course $c^{-1}g$ is still an AE metric. Thus, it suffices to prove the result for this special case. That is, assume $\tau = (4\pi)^{-1}$ so that

\[
4\pi\tau = 1 \implies \int u^2 \, dV = (4\pi\tau)^{n/2} = 1.
\]

Then for any $\eta \in C^\infty_0(M)$, let $F : M \times (-1, 1) \to \mathbb{R}$ be defined as

\[
F(t) = F(x,t) = \frac{u + t\eta}{(\int (u + t\eta)^2 \, dV)^{1/2}}
\]

so that $F(0) = u$ and

\[
\int F(x,t)^2 \, dV = \frac{\int (u + t\eta)^2 \, dV}{(\int (u + t\eta)^2 \, dV)^2} = 1 = (4\pi\tau)^{n/2}.
\]

Thus $F(t) \in W^{1,2}(M) \cap C^\infty(M)$ for each $t$. By direct computation, we have

\[
\left. \frac{d}{dt} \right|_{t=0} F(t)^2 = \left. \frac{d}{dt} \right|_{t=0} \frac{(u + t\eta)^2}{(\int (u + t\eta)^2 \, dV)^2} \\
= \frac{\left( \int (u + t\eta)^2 \, dV \right) (2\eta u + 2t\eta^2) - (u + t\eta)^2 \left( \int 2\eta u + 2t\eta^2 \, dV \right)}{(\int (u + t\eta)^2 \, dV)^2} \bigg|_{t=0} \\
= \frac{2\eta u \left( \int u^2 \, dV \right) - 2u^2 \left( \int \eta u \, dV \right)}{(\int u \, dV)^2} = 2\eta u - 2u^2 \left( \int \eta u \, dV \right)
\]

and

\[
\left. \frac{d}{dt} \right|_{t=0} F(t)^2 \log F(t)^2 = \left. \frac{d}{dt} \right|_{t=0} \frac{(u + t\eta)^2}{\int (u + t\eta)^2 \, dV} \log \left( \frac{(u + t\eta)^2}{\int (u + t\eta)^2 \, dV} \right) \\
= \left. \frac{d}{dt} \right|_{t=0} \frac{(u + t\eta)^2}{\int (u + t\eta)^2 \, dV} \log \left( \frac{(u + t\eta)^2}{\int (u + t\eta)^2 \, dV} \right) + \log \left( \frac{(u + t\eta)^2}{\int (u + t\eta)^2 \, dV} \right) \frac{d}{dt} \left( \frac{(u + t\eta)^2}{\int (u + t\eta)^2 \, dV} \right) \\
= u^2 \frac{2\eta u - 2u^2 \left( \int \eta u \, dV \right)}{u^2} + \log(u^2) \left( 2\eta u - 2u^2 \left( \int \eta u \, dV \right) \right) \\
= \left( 2\eta u - 2u^2 \left( \int \eta u \, dV \right) \right) (1 + \log u^2)
\]
and
\[
\frac{d}{dt} \bigg|_{t=0} |\nabla F(t)|^2 = \frac{d}{dt} \bigg|_{t=0} \left| \nabla \left( \frac{u + t\eta}{(f(u + t\eta)^2)\frac{1}{2}} \right) \right|^2 \\
= \frac{d}{dt} \bigg|_{t=0} \frac{1}{f(u + t\eta)^2} \left| \nabla (u + t\eta) \right|^2 \\
= \frac{d}{dt} \bigg|_{t=0} \left( |\nabla u|^2 + 2t(\nabla u, \nabla \eta) + t^2 |\nabla \eta|^2 \right) \\
= 2(\nabla u, \nabla \eta) - 2|\nabla u|^2 \left( \int \eta u \, dV \right).
\]

Since \(u\) and \(\eta\) are both smooth and \(\eta\) decays nicely, we can pass the time derivative through the integral by DCT and use the above calculations
\[
0 = \frac{d}{dt} \bigg|_{t=0} \mathcal{W}(g, F(t), \tau) \\
= \frac{d}{dt} \bigg|_{t=0} \int (\tau(4|\nabla F(t)|^2 + RF(t)^2) - F(t)^2 \log F(t)^2 - nF(t)^2) \, dV \\
= \int \left[ 8\tau(\nabla u, \nabla \eta) - 8\tau|\nabla u|^2 \left( \int \eta u \, dV \right) + 2\tau R\eta u - 2\tau Ru^2 \left( \int \eta u \, dV \right) \\
- \left( 2\eta u - 2u^2 \left( \int \eta u \, dV \right) \right) (1 + \log u^2) - 2n\eta u + 2nu^2 \left( \int \eta u \, dV \right) \right) \, dV \\
= 2 \int \left[ \tau(4(\nabla u, \nabla \eta) + R\eta u) - \eta u \log u^2 - \eta u - n\eta u \right] \, dV \\
- 2 \left( \int \eta u \, dV \right) \int \left[ \tau(4|\nabla u|^2 + Ru^2 - u^2 - u^2 \log u^2 - nu^2) \right] \, dV.
\]

Now since \(u\) is the minimizer for \(\mu(g, \tau)\), it satisfies
\[
\int \left[ \tau(4|\nabla u|^2 + Ru^2 - u^2 - u^2 \log u^2 - nu^2) \right] \, dV = \mu(g, \tau).
\]

Now using that \(\int u^2 \, dV = 1\), and integrating by parts (the boundary term is zero because \(\eta \in \mathcal{C}_0^\infty(M)\)), we have
\[
0 = \int \left[ \tau(4(\nabla u, \nabla \eta) + R\eta u) - \eta u \log u^2 - \eta u - n\eta u \right] \, dV \\
- \left( \int \eta u \, dV \right) \int \left[ \tau(4|\nabla u|^2 + Ru^2 - u^2 - u^2 \log u^2 - nu^2) \right] \, dV \\
= \int \left[ \tau(4(\nabla u, \nabla \eta) + R\eta u) - \eta u \log u^2 - \eta u - n\eta u \right] \, dV - \left( \int \eta u \, dV \right) \left( \mu(g, \tau) - \int u^2 \, dV \right) \\
= \int \left[ \tau(4(\nabla u, \nabla \eta) + R\eta u) - \eta u \log u^2 - n\eta u - \mu(g, \tau)u \right] \, dV + \left( \int \eta u \, dV \right) - \left( \int \eta u \, dV \right) \\
= \int \left[ \tau(-4\eta \Delta u + R\eta u) - \eta u \log u^2 - n\eta u - \mu(g, \tau)u \right] \, dV.
\]

Since this holds for any test function \(\eta\), it follows (by taking \(\eta\) to be a bump function on arbitrarily small balls) that
\[
\tau(-4\Delta u + Ru) - u \log u^2 - nu = \mu(g, \tau)u.
\]
That is to say, the minimizer $u$ satisfies the Euler-Lagrange equation in the special case of $\tau = (4\pi)^{-1}$. As mentioned at the beginning, this completes the proof.

\[\square\]

We now remind the reader of a common mode of convergence for smooth manifolds called ‘Cheeger-Gromov convergence’ (or symbolically as $C^\infty_{loc}(M)$ convergence).

**Definition (Convergence on compact sets).** We say $(U_i, g_i) \to (M_\infty, g_\infty)$ on compact sets in $M$ if for any subset $K \subset M$ with compact closure, and for each $k \in \mathbb{N}$, $\|g_i - g\|_{C^\alpha(K)} := \sup_{0 \leq \alpha \leq k} \sup_{x \in K} |\partial^\alpha (g_i - g_\infty)|_g \to 0$.

**Definition (Cheeger-Gromov convergence).** A sequence $\{(M^n_i, g_i, x_i)\}$ of complete pointed manifolds converges in the Cheeger-Gromov sense to a complete manifold $(M_\infty, g_\infty, x_\infty)$ if there exists an exhaustion $\{U_i\}$ of $M_\infty$ by open sets $x_\infty \in U_i$ and a series of diffeomorphisms $\phi_i : U_i \to V_i = \phi_i(U_i) \subset M$ with $\phi_i(x_\infty) = x_i$ such that $(U_i, \phi_i^*(g_i)) \to (M_\infty, g_\infty)$ on compact sets.

The following proposition shows that AE manifolds converge to Euclidean space in the Cheeger-Gromov sense whenever the points are chosen to go to infinity. And of course, this makes a sort of sense since $(M, g)$ looks more and more like $(\mathbb{R}^n, g_E)$ as you ‘zoom out’.

**Proposition.** For an AE manifold $M^n$, we have $(M^n, g, p_k) \to (\mathbb{R}^n, g_E, p_\infty)$ for any sequence of points $p_k \to \infty$.

**Proof.** WLOG, suppose all the points $p_k$ satisfy

$$\min_{x \in E} r(x) < r(p_k)/2.$$  

Let $p_\infty = 0$ and let $U_i = B(0, r(p_k)/2)$ so that $\{U_i\}$ is an exaustion of $\mathbb{R}^n$. Let the diffeomorphisms $\phi_i : U_i \to V_i = \phi_i(U_i) \subset M$ be defined as

$$\phi_i(x) = \Phi^{-1}(x + p_i)$$

which is well defined by our assumption on the points $p_k$. Then by the AE condition, we have

$$|\partial^\alpha \phi_i^*(g) - g_E| = O(r^{-k-\sigma}) = O(r(p_i)^{-k-\sigma})$$

for any multiindex $|\alpha| = k$.

\[\square\]

The following lemma shows that the $\mu$-functional interacts nicely with Cheeger-Gromov convergence.

**Lemma 3.2.** If $(M_i, g_i, p_i) \to (M_\infty, g_\infty, p_\infty)$ in the Cheeger-Gromov sense and $\mu(g_\infty, \tau)$ is finite, then

$$\mu(g_\infty, \tau) \geq \lim_{i \to \infty} \sup \mu(g_i, \tau).$$

**Proof.** For any $\epsilon > 0$, choose some $u \in W^{1,2}_c(M_\infty)$ such that $W(g_\infty, u, \tau) \leq \mu(g_\infty, \tau) + \epsilon$ (since this quantity is the infimum of all $u \in W^{1,2}_0$ and compactly supported functions are dense in this space). For large $i$, we can find $u_i \in W^{1,2}_c$ which are the pull-back functions of $u$ and
\[
\lim_{i \to \infty} \mathcal{W}(g_i, u_i, \tau) = \mathcal{W}(g_\infty, u_\infty, \tau) \]

by the \( C^2 \) convergence on the compact set on which \( u \) is supported. Then since \( \mu_i(g_i, \tau) \) is the infimum over all such \( \mathcal{W}(g_i, u_i, \tau) \), we have

\[
\limsup_{i \to \infty} \mu(g_i, \tau) \leq \lim_{i \to \infty} \mathcal{W}(g_i, u_i, \tau) \leq \mu(g_\infty, \tau) + \epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, this completes the proof. \( \square \)

In the case of \((\mathbb{R}^n, g_E)\), it follows from log-Sobolev inequality \([12]\) that

**Theorem** (Leonard Gross 1975). \( \mathcal{W}(g_E, u, \tau) \geq 0 \) for any smooth \( u \) satisfying \( \int_M u^2 dV = (4\pi\tau)^{n/2} \) and equality holds if \( u^2 = e^{-\frac{|x|^2}{4\tau}} \).

A proof can be found in \([30]\). It is thus immediately obvious that \( \mu(g_E, \tau) = 0 \).

**Corollary.** \( \mu(g, \tau) \leq 0 \) for any AE manifold.

**Proof.** Since \((M, g, p_n) \to (\mathbb{R}^n, g_E, p_\infty)\) in the Cheeger-Gromov sense for any sequence of points \( p_n \to \infty \), we have

\[
0 = \mu(g_E, \tau) \geq \limsup_{i \to \infty} \mu(g_i, \tau) = \mu(g, \tau).
\]

\( \square \)

**Corollary.** The corresponding value of \( \mu(g, \tau) \) at infinity is 0 for any AE manifold \((M, g)\).

**Proof.** Similarly to the proof of Lemma 3.2, for any \( \epsilon > 0 \), choose some \( u \in W^{1,2}(\mathbb{R}^n) \) such that \( \mathcal{W}(g_E, u, \tau) \leq \mu(g_E, \tau) + \epsilon \). For a sequence \( r_k \to \infty \), choose \( p_k \to \infty \) in \( M \) such that if the \( u_k \) are the pull-back functions under the maps \( \phi_k(x) = \Phi^{-1}(x + p_i) \), then \( \text{supp}(u_k) \subset M \setminus B(p, r_k) \).

Then since \((M, g, p_k) \to (\mathbb{R}^n, g_E, 0)\) we have

\[
\epsilon = \mu(g_E, \tau) + \epsilon \geq \mathcal{W}(g_E, u, \tau) \geq \lim_{k \to \infty} \mathcal{W}(g, u_k, \tau) \geq \lim_{r_k \to \infty} \mu(M \setminus B(p, r_k), g, \tau).
\]

Since \( \epsilon > 0 \) was arbitrary, we see that the corresponding value of \( \mu(g, \tau) \) is 0. \( \square \)

Therefore, since \( \tau^{-1}g \) is still an AE metric, if \( \mu(\tau^{-1}g, 1) = \mu(g, \tau) < 0 \), then there is a smooth positive minimizer for the functional \( \mu(g, \tau) \). The following theorem says that if there is a solution to the Ricci flow for long enough time, starting at a (non-trivial) AE metric \( g \), then we always have \( \mu(g, \tau) < 0 \).

**Theorem 3.3.** If \((M^n, g)\) is a manifold with bounded geometry such that a solution \( g(t) \) of bounded curvature to the Ricci flow with \( g(0) = g \) exists for \( t \in [0, T) \), then for any \( \tau \in (0, T) \), \( \mu(g, \tau) < 0 \) unless \((M^n, g)\) is isometric to \((\mathbb{R}^n, g_E)\).

**Remark.** Theorem 3.3 does not assume that \((M, g)\) is an AE manifold. If it is, the proof is much shorter, as will soon be apparent.
Proof. Let $\tau(t) = \tau - t$, $y \in M$ and consider the corresponding fundamental solution

$$v(x, t) = (4\pi \tau(t))^{-n/2} e^{-f(x, t)}$$

to the adjoint heat equation

$$\partial_tv = -\Delta v + Rv$$

with $\lim_{t \to \tau} v(\cdot, t) = \delta_y$. The existence of the fundamental solutions to the adjoint heat equation on noncompact manifolds and their basic properties can be found in chapters 24-25 in [7].

Then by monotonicity of the entropy, we have

$$\mu(g, \tau) = \mu(g, \tau(0)) \leq \mathcal{W}(g(0), f(0), \tau(0)) \leq \limsup_{t \uparrow \tau} \mathcal{W}(g(t), f(t), \tau(t)) \leq 0.$$  

The last inequality above can be found in [3], Theorem 7.9. If $\mu(g, \tau) = 0$, then we must have equality throughout. Therefore

$$0 \equiv \frac{d}{dt} \mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_M |Rc + \nabla^2 f - \frac{g}{2\tau}|^2 (4\pi \tau)^{n/2} dV$$

So

$$Rc + \nabla^2 f - \frac{g}{2\tau} \equiv 0$$

for $t \in [0, \tau]$. This is precisely the equation of a shrinking solition solution $g(t)$ with singular time $\tau$. Since $g(t)$ is a solition solution, we have

$$\tau(t) \max_{x \in M} |Rm(g(t))(x)| \equiv C$$

for all $t \in [0, \tau]$. Taking $t \to \tau$ (so that $\tau(t) \to 0$), we can see that either $|Rm(g(t))| \equiv 0$ or $|Rm(g(t))| \to \infty$. But the latter is impossible since we’ve assumed $(M, g(t))$ has bounded curvature. Thus $Rm \equiv 0$ on all of $M \times [0, \tau]$. In particular, $g$ is Ricci-flat. If we were assuming $(M, g)$ is AE, we could conclude that $(M, g) \cong (\mathbb{R}^n, g_E)$ by theorem 2.4. But we’d like to be more general here. So instead, we see that

$$\nabla^2 f - \frac{g}{2\tau} \equiv 0.$$  

Set $\mathcal{J} = 4\pi \tau f$ so that $\nabla^2 \mathcal{J} = 4\pi \nabla^2 f = 2g$. Thus $\mathcal{J}$ is convex (either that it is semi-positive definite, or else that $\mathcal{J} \circ \gamma$ is a convex function for all geodesics $\gamma$).

Let $O \in M$ be a fixed point, then for any $x \in M$, we have a minimizing geodesic (by completeness) $s(t)$, $0 \leq t \leq d(x, O)$ such that $|\dot{s}(t)| \equiv 1$. Then we have

$$\frac{d^2}{dt^2} \mathcal{J}(s(t)) = \nabla^2 \mathcal{J}(\nabla d, \nabla d) = 2g(\nabla d, \nabla d) = 2$$

where $d(y) = d(y, O)$. This equation can be seen by working in coordinates with say $x^1 = s(t)$ and using the geodesic equation (note that most things cancel). Note of course that $g(\nabla d, \nabla d) = 1$ if we work in this coordinate system. Therefore by integrating, we have

$$\frac{d}{dt} \mathcal{J}(s(t)) = \langle \nabla \mathcal{J}, \dot{s} \rangle = \langle \nabla \mathcal{J}, \nabla d \rangle = 2t + \langle \nabla \mathcal{J}, \nabla d \rangle|_{t=0}$$

since $\dot{s}$ and $\nabla d$ are both unit vectors that point in the same direction (that is the direction of $x$ which is along the curve $s$). Integrating again, we have

$$\mathcal{J}(s(t)) = \mathcal{J}(s(0)) + t(\nabla \mathcal{J}, \nabla d)|_{t=0} + t^2 = \mathcal{J}(O) + t(\nabla \mathcal{J}, \nabla d)|_{t=0} + t^2.$$
Putting these things together, we have \( f = f(s(t)) = f(O_1) + t^2 = f(O_1) + d(s(t),O_1)^2 \) for \( x = s(t) \) since \( s \) is unit speed. Taking traces of the equation \( \nabla^2 f = 2g \) and using \( \text{tr}_g \nabla^2 = \Delta \) and \( \text{tr}_g g = n \), we have
\[
\Delta d^2 = \Delta(f(O_1) + d^2) = \text{tr}_g \nabla^2 f = \text{tr}_g 2g = 2n.
\]

Now let \( B_r(O_1) \) be a geodesic ball of radius \( r \). Recall that by Gauss’ Lemma, in normal coordinates, the outward unit normal to \( \partial B_r \) is
\[
\partial_r = \frac{x^i}{\sqrt{(x^1)^2 + \cdots + (x^n)^2}} \frac{\partial}{\partial x^i} = \frac{x^i}{d(x)} \frac{\partial}{\partial x^i}.
\]
So we have by Green’s theorem,
\[
2n \text{Vol}(B_r) = \int_{B_r} 2n \, dV = \int_{\partial B_r} \Delta d^2 \, dV = \int_{\partial B_r} \langle \nabla d^2, \partial_r \rangle \, d\sigma.
\]
But also note that
\[
\nabla d^2 = \partial_i[(x^1)^2 + \cdots + (x^n)^2] \frac{\partial}{\partial x^i} = 2x^i \frac{\partial}{\partial x^i} = 2d(x) \partial_r.
\]

But since \( d(x) = d(x,O_1) \) and \( B_r = B_r(O_1) \), we have that \( d(x) = r \) on \( \partial B_r \). So we have
\[
\int_{\partial B_r} \langle \nabla d^2, \partial_r \rangle \, d\sigma = \int_{\partial B_r} 2d(x) \langle \partial_r, \partial_r \rangle \, d\sigma = \int_{\partial B_r} 2r \, d\sigma = 2r \text{Vol}(S_r).
\]
Putting these things together, we have
\[
\text{Vol}(B_r) = \frac{r}{n} \text{Vol}(S_r).
\]
On the other hand, the Bishop-Gromov volume comparison theorem says that for \( s < r \),
\[
\text{Vol}(S_r) \leq \text{Vol}(S_s) \frac{r^{n-1}}{s^{n-1}}
\]
and if there is equality, then \( B_r \) is isometric to the Euclidean ball \( B_{gh}(0,r) \). So
\[
\text{Vol}(B_r) = \int_0^r \text{Vol}(S_{\rho}) \, d\rho \geq \int_0^r \text{Vol}(S_{\rho}) \frac{\rho^{n-1}}{s^{n-1}} \, d\rho = \frac{r}{n} \text{Vol}(S_s).
\]
But we just showed that this inequality must be an equality. Thus \( B_r(O_1) \) is isometric to a Euclidean ball of radius \( r \). Recall that this held for all \( r < \text{inj}(M) \). If \( \text{inj}(M) = \infty \), then we can take \( r \to \infty \) and conclude that \( (M,g) \) is isometric to \( (\mathbb{R}^n, g_E) \). So suppose \( \text{inj}(M) < \infty \). Then there is some minimum \( R > 0 \) such that the exponential map at \( O_1 \) fails to be a diffeomorphism. Either there are two distinct geodesics starting at \( O_1 \) that intersect on \( \partial B_R \) or else there is a non-zero Jacobi field that vanishes at \( O_1 \) and on \( \partial B_R \). Both of these are continuous failures, and since neither of these things occur on a Euclidean ball, we can apply the above logic to \( r \to R^- \) to get a contradiction. This completes the proof.

\[\Box\]

**Remark.** One could also use the Laplacian comparison theorem in the above proof, though I personally like this method more.

The following is really the main theorem of the paper and is definitely the most work to prove. It is used in the proof of the main theorem several times.
Theorem 3.4. If \((M^n, g)\) is an AE manifold such that \(R > 0\), then \(\lim_{\tau \to \infty} \mu(g, \tau) = 0\).

Proof. Assume for a contradiction there exists \(\tau_k \to \infty\) such that
\[
\mu_\infty := \liminf_{\tau \to \infty} \mu(g, \tau) = \lim_{k \to \infty} \mu(g, \tau_k) < 0.
\]

WLOG, we’ll assume \(\mu(g, \tau_k) < 0\) for all \(k\). Therefore, \(\mu(g, \tau_k)\) has a smooth positive minimizer \(u_k\) for each \(k\). We’ve seen that the \(\{u_k\}\) satisfy their Euler-Lagrange equation
\[
\tau_k (-4\Delta u_k + Ru_k) - u_k \log u_k^2 = \mu(g, \tau_k) u_k.
\]

We claim that the \(\{u_k\}\) are uniformly bounded. To prove this claim, we’ll first prove a lemma:

Lemma 3.5. For \(u \in W^{1, 2}(M)\), the following Sobolev inequality holds:
\[
\|u\|^2_{L^{2n/\sqrt{n-2}}} = \left( \int_M u^{2n/\sqrt{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq C \int_M (4|\nabla|^2 + Ru^2) \, dV
\]
where \(C\) depends on the dimension, the curvature bound, injective radius lower bound, AE coordinate system and the infimum of the scalar curvature \(R\) over some compact set.

Proof of Lemma 3.5. Let \(M^n = K \cup E\) be the disjoint union of a compact set \(K\) and an AE end \(E\) as given in the definition of an AE manifold. Let \(K_1\) be a compact set such that \(K \subset K_1\) and a smooth cutoff function \(1 \geq \phi_0 \geq 0\) such that \(\text{supp}(\phi_0) \subset K_1\), \(\phi_0 \equiv 1\) on \(K\). Let \(\phi_1 = 1 - \phi_0\). For any \(u \in W^{1, 2}(M)\), we have
\[
\|u\|_{2n/\sqrt{n-2}} = \|\phi_0 u + \phi_1 u\|_{2n/\sqrt{n-2}} \leq \|\phi_0 u\|_{2n/\sqrt{n-2}} + \|\phi_1 u\|_{2n/\sqrt{n-2}} =: I^2 + II^2
\]
by Minkowski inequality. We’ll look to control each of these in turn. Let’s look at \(I\) first.

By the Sobolev embedding theorem on a compact manifold (say on \(K_1\)), we have \(W^{1, 2} \hookrightarrow L^{2n/\sqrt{n-2}}\) (continuous embedding), which is to say that
\[
\left( \int_M (\phi_0 u)^{\frac{2n}{\sqrt{n-2}}} \right)^\frac{n-2}{n} = \left( \int_{K_1} (\phi_0 u)^{\frac{2n}{\sqrt{n-2}}} \right)^\frac{n-2}{n} \leq C \int_{K_1} (|\nabla (\phi_0 u)|^2 + \phi_0^2 u^2) \, dV
\]
\[
= C \int_{K_1} (|\nabla \phi_0 u + \phi_0 \nabla u|^2 + \phi_0^2 u^2) \, dV.
\]
Now using the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) we have
\[
C \int_{K_1} (|\nabla \phi_0 u + \phi_0 \nabla u|^2 + \phi_0^2 u^2) \, dV \leq C \int_{K_1} (|\nabla \phi_0 u|^2 + |\phi_0 \nabla u|^2 + \phi_0^2 u^2) \, dV
\]
Now on \(K_1\), \(\phi_0 \leq 1\) and \(\nabla \phi_0\) is bounded. Moreover, since \(R > 0\), it achieves a (positive) minimum over the compact set \(K_1\). Thus, we have
\[
I \leq C \int_{K_1} (|\nabla \phi_0 u|^2 + |\phi_0 \nabla u|^2 + \phi_0^2 u^2) \, dV \leq C \int_{K_1} (|\nabla u|^2 + u^2) \, dV \leq C \int_{K_1} (4|\nabla u|^2 + Ru^2) \, dV
\]
where the constant \(C\) depends on the dimension (from Sobolev embedding), the AE coordinate system (from choice of \(K\) and \(K_1\)), and the infimum of \(R\) over the compact set \(K_1\).
On the AE end $E$, by enlarging $K$ and $K_1$ if necessary, we can assume the $L^2$ Sobolev inequality of the Euclidean type holds. More precisely, on $\mathbb{R}^n$, we have the following $L^2$ Gagliardo-Nirenberg-Sobolev inequality:

$$
\left( \int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} \, dV_{g_E} \right)^{\frac{n-2}{n}} \leq C \int_{\mathbb{R}^n} |\nabla_{g_E} u|^2 \, dV_{g_E}
$$

for any $u \in C^1_c(\mathbb{R}^n)$ ($C^1$ & compact support) and some $C > 0$ depending only on $n$. Note that the above also holds for any $u \in W^{1,2}_0(\mathbb{R}^n)$ since $C^1_c(\mathbb{R}^n)$ is dense in $W^{1,2}_0(\mathbb{R}^n)$. Since $E$ and an AE end (similarly as before), we can assume there is some $C > 0$ such that

$$
C^{-1} dV_{g_E} \leq dV \leq C dV_{g_E}, \quad \text{and} \quad C^{-1} |\nabla_{g_E} u|^2 \leq |\nabla g u|^2 \leq C |\nabla_{g_E} u|^2.
$$

Hence for any $u \in C^1_0(E)$, we have (by the above comparsion of volume forms and $\nabla$, as well as the GNS inequality on $\mathbb{R}^n$),

$$
\left( \int_E u^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} \leq \left( \int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} \, dV_{g_E} \right)^{\frac{n-2}{n}} \leq C \int_{\mathbb{R}^n} |\nabla_{g_E} u|^2 \, dV_{g_E} \leq C \int_{\mathbb{R}^n} |\nabla g u|^2 \, dV_{g_E} \leq C \int_E |\nabla g u|^2 \, dV_g.
$$

Then since $\text{supp}(\phi_1) \subset E$, we can apply the above to see

$$
II = \left( \int_E (\phi_1 u)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq C \int_E |\nabla (\phi_1 u)|^2 \, dV
$$

$$
= C \int_M |\nabla (\phi_1 u)|^2 \, dV \leq C \int_M |\nabla \phi_1|^2 + |\phi_1 \nabla u|^2 \, dV.
$$

Now similarly as before, $|\phi_1|, |\nabla \phi_1| \leq 1$ on $M$ and $\phi_1$ is constant outside of the compact set $K_1$ (over which $R$ achieves its positive minimum). So we have

$$
II \leq C \int_M |\nabla \phi_1|^2 + |\phi_1 \nabla u|^2 \, dV \leq C \int_{K_1} u^2 \, dV + C \int_M |\nabla u|^2 \, dV
$$

$$
\leq C \int_{K_1} Ru^2 \, dV + C \int_M |\nabla u|^2 \, dV \leq C \int_M (4|\nabla u|^2 + Ru^2) \, dV
$$

as desired. Here $C$ depends on the dimension (from the Sobolev inequality on $\mathbb{R}^n$), the asymptotic end (from comparing the volume form and gradient with those from $g_E$), and the infimum of $R$ on a compact set. This completes the proof. 

\[\Box\]

Now we can prove the claim (that the $\{u_k\}$ are uniformly bounded) using Moser’s iteration. For simplicity, we will drop the subscript $k$ and simply write $\mu$ for $\mu(g, \tau_k)$. Recall that the positive minimizer $u$ satisfies

$$
\tau (-4\Delta u + Ru) - u \log u^2 - nu = \mu u
$$

so dividing by $\tau$, we can rewrite this as

$$
4\Delta u - Ru + \frac{2}{\tau} u \log u + \frac{n + \mu}{\tau} u = 0.
$$

Now since $\mu \leq 0$, this implies

$$
4\Delta u - Ru + \frac{2}{\tau} u \log u + \frac{n}{\tau} u \geq 0.
$$

(11)
Now let $p > 1$ (to be chosen later). We have

$$\Delta u^p = \nabla (pu^{p-1}\nabla u) = p(p-1)u^{p-2}|\nabla u|^2 + pu^{p-1}\Delta u.$$  

Set $w = u^p$ and use \([11]\) and the positivity of $u$ to see

$$4\Delta w = 4\Delta u^p \geq 4p(p-1)u^{p-2}|\nabla u|^2 + 4pu^{p-1}\Delta u \geq 4pu^{p-1}\Delta u$$

$$\geq pu^{p-1}(Ru - \frac{2}{\tau}u \log \frac{n}{\tau} u) = pu^p - \frac{2p}{\tau}u^p \log \frac{np}{\tau} u = pu^p - \frac{2p}{\tau}w \log \frac{np}{\tau} w.$$  

Let $\phi \in C^\infty_c(M)$. From the above, an integration by parts, and the positivity of $u$ (and thus $w$), we have

$$4 \int \langle \nabla (\phi w^2), \nabla w \rangle \, dV = \int \phi \cdot \langle \nabla \nabla \phi w^2, \nabla w \rangle \, dV \leq \int \phi \cdot \langle \nabla \nabla \phi w^2, \nabla w \rangle \, dV$$

$$= \frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV + \frac{np}{\tau} \int w^2 \phi^2 \, dV - \int pu^2 \phi^2 \, dV.$$  

On the other hand

$$\langle \nabla (\phi w^2), \nabla w \rangle = g^{ij} \partial_i (\phi w^2) \partial_j w = \phi^2 g^{ij} \partial_i w \partial_j w + 2\phi w g^{ij} \partial_i \phi \partial_j w$$

$$= \phi^2 g^{ij} \partial_i w \partial_j w + 2\phi w g^{ij} \partial_i \phi \partial_j w + w^2 g^{ij} \partial_i \phi \partial_j \phi - w^2 g^{ij} \partial_i \phi \partial_j \phi$$

$$= g^{ij} (\phi \partial_i w + w \partial_i \phi) (\phi \partial_j w + w \partial_j \phi) - |\nabla \phi|^2 w^2 = |\nabla (\phi w^2)|^2 - |\nabla \phi|^2 w^2.$$  

Therefore, we have

$$4 \int |\nabla \phi|^2 w \, dV = 4 \int \langle \nabla (\phi w^2), \nabla w \rangle \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV$$

$$\leq \frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV + \frac{np}{\tau} \int w^2 \phi^2 \, dV - \int pu^2 \phi^2 \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV.$$  

The problem term above will be the first one (with the logarithm). So let’s try to control it now. There is some constant $c_1$ (depending on $n$) such that

$$\log u \leq u^{\frac{2}{n}} + c_1$$

since $u^{\frac{2}{n}}$ grows more quickly than $\log u$. Therefore

$$\frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV \leq \frac{2p}{\tau} \int w^2 \phi^2 u^{\frac{2}{n}} \, dV + \frac{2pc_1}{\tau} \int w^2 \phi^2 \, dV.$$  

Now applying Hölder to the first term on the RHS (with exponents $n$ and $\frac{n-1}{n}$), and using $\|u\|_2^2 = (4\pi)^{n/2}$, we have

$$\frac{2p}{\tau} \int w^2 \phi^2 u^{\frac{2}{n}} \, dV \leq \frac{2p}{\tau} \left( \int (w^2 \phi^2)^{\frac{2n}{2n-1}} \, dV \right)^{\frac{n-1}{n}} \left( \int u^2 \, dV \right)^{\frac{1}{n}} = \frac{2p\sqrt{4\pi}}{\sqrt{\tau}} \left( \int (w^2 \phi^2)^{\frac{2n}{2n-1}} \, dV \right)^{\frac{n-1}{n}}.$$  

From Hölder’s inequality $\|fh\| \leq \|f\|_p \|h\|_q$ with $f = (w^2 \phi^2)^{\frac{2n}{2n-1}}$, $p = \frac{2(n-1)}{n-2}$ and $q = \frac{2(n-1)}{n}$, we have

$$\left( \int (w^2 \phi^2)^{\frac{2n}{2n-1}} \, dV \right)^{\frac{n-1}{n}} \leq \left( \int (w^2 \phi^2)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \left( \int w^2 \phi^2 \, dV \right)^{\frac{1}{n}}$$

$$\leq \lambda \left( \int (w^2 \phi^2)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} + \frac{1}{4\lambda} \int w^2 \phi^2 \, dV.$$
where the last inequality comes from Young’s inequality (also called Peter-Paul) \( ab \leq \frac{a^2}{4} + \frac{b^2}{4} \).

Therefore using this in (13), we have

\[
\frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV \leq \frac{c_2 \lambda p}{\sqrt{\tau}} \lambda \left( \int (w \phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} + \frac{pc_2}{4\lambda \sqrt{\tau}} \int w^2 \phi^2 \, dV + \frac{2pc_1}{\tau} \int w^2 \phi^2 \, dV
\]

where \( c_2 = 2\sqrt{4\pi} \).

We still have some work to do to simplify the first term on the RHS. Recall that Lemma 3.5 gives

\[
\frac{1}{C} \left( \int (w \phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq \int 4|\nabla (w \phi)|^2 + R(w \phi)^2 \, dV.
\]

Now applying (12) to the first term (along with the facts that \( p \geq 1 \) and \( R > 0 \)) along with (14) gives

\[
\int 4|\nabla (w \phi)|^2 + R(w \phi)^2 \, dV \leq \frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV + \frac{np}{\tau} \int w^2 \phi^2 \, dV
\]

\[
- \int pRw^2 \phi^2 \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV + \int R(w \phi)^2 \, dV
\]

\[
\leq \frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV + \frac{np}{\tau} \int w^2 \phi^2 \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV
\]

\[
\leq \frac{c_2 \lambda p}{\sqrt{\tau}} \lambda \left( \int (w \phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} + \frac{pc_2}{4\lambda \sqrt{\tau}} \int w^2 \phi^2 \, dV + \frac{2pc_1}{\tau} \int w^2 \phi^2 \, dV
\]

\[
+ \frac{np}{\tau} \int w^2 \phi^2 \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV.
\]

If we let \( \lambda = \frac{\sqrt{\tau}}{2C^2p} \) so that \( \frac{c_2 \lambda p}{\sqrt{\tau}} = \frac{1}{2C} \), then the above calculation gives

\[
\frac{1}{2C} \left( \int (w \phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq \frac{p^2 c_2 C}{2\tau} \int w^2 \phi^2 \, dV + \frac{2pc_1}{\tau} \int w^2 \phi^2 \, dV
\]

\[
+ \frac{np}{\tau} \int w^2 \phi^2 \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV
\]

\[
= \left( \frac{c_2 C}{2} + 2c_1 + n \right) \frac{p^2}{\tau} \int w^2 \phi^2 \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV
\]

\[
\leq \left( \frac{c_2 C}{2} + 2c_1 + n \right) \frac{p^2}{\tau} \int w^2 \phi^2 \, dV + 4 \int |\nabla \phi|^2 w^2 \, dV.
\]

Therefore, there is some \( C_0 > 0 \) such that

\[
\left( \int (w \phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq \frac{C_0 p^2}{\tau} \int w^2 \phi^2 \, dV + C_0 \int |\nabla \phi|^2 w^2 \, dV.
\]
Now for any \( x \in M \) and any \( k \in \mathbb{N} \), choose \( 0 \leq \phi_k \leq 1 \) such that \( \text{supp}(\phi_k) \subset B(x, \sqrt{1+2^k}) \) and \( \phi_k \equiv 1 \) on \( B(x, \sqrt{1+1+2^k}) \) and \( |\nabla \phi_k| \leq \frac{C_2}{\sqrt{r}} \). We have

\[
\left( \int_{B(x, \sqrt{1+2^k+1})} w^\frac{2n}{n-2} \, dV \right)^{\frac{n-2}{n}} \leq \left( \int_{B(x, \sqrt{1+2^k+1})} (\phi_k w)^\frac{2n}{n-2} \, dV \right)^{\frac{n-2}{n}}
\]

\[
\phi = 1 \text{ on } B
\]

\[
\leq \frac{C_0 p^2}{\tau} \int w^2 \phi_k^2 \, dV + C_0 \int |\nabla \phi_k|^2 w^2 \, dV
\]

\[
\leq C_0 \int_{B(x, \sqrt{1+2^k+1})} \frac{p^2}{\tau} w^2 + |\nabla \phi_k|^2 w^2 \, dV \quad \text{supp}(\phi_k) \subset B
\]

\[
\leq C_0 \int_{B(x, \sqrt{1+2^k+1})} \frac{p^2}{\tau} w^2 + C_2^2 2^k \sqrt{\frac{p^2}{\tau}} w^2 \, dV \quad |\nabla \phi_k| \leq \frac{C_2^k}{\sqrt{r}}
\]

\[
\leq C_1 \frac{2^k p^2}{\tau} \int_{B(x, \sqrt{1+2^k+1})} w^2 \, dV.
\]

Let \( p_0 = \frac{n}{n-2} \) and set \( p = p_0^k \) (which is \( > 1 \)). Since \( w = u^p \), we have

\[
p^2 = p_0^{2k}, \quad \frac{n-2}{n} = \frac{1}{p_0}, \quad w^{\frac{2n}{n-2}} = (u^p)^{\frac{2n}{n-2}} = u^{2p_0} = u^{2p_0^k}, \quad \text{and } w^2 = (u^p)^2 = u^{2p_0^k}.
\]

So the above can be written

\[
\left( \int_{B(x, \sqrt{1+2^k+1})} u^{2p_0^k+1} \, dV \right)^{\frac{1}{p_0}} \leq C_1 \frac{(2p_0)^{2k}}{\tau} \int_{B(x, \sqrt{1+2^k})} u^{2p_0^k} \, dV.
\]

Raising both sides to the power of \( \frac{1}{p} = \frac{1}{p_0} \) gives

\[
||u^2||_{L^{2^{p_0}+1}(B(x, \sqrt{1+2^k+1}))} \leq \left( \int_{B(x, \sqrt{1+2^k+1})} u^{2p_0^k+1} \, dV \right)^{\frac{1}{p_0}}
\]

\[
\leq C_1 \frac{(2p_0)^{2k}}{\tau} \left( \int_{B(x, \sqrt{1+2^k})} u^{2p_0^k} \, dV \right)^{\frac{1}{p_0}}
\]

\[
\leq C_1 \frac{(2p_0)^{2k}}{\tau} ||u^2||_{L^{2^{p_0}}(B(x, \sqrt{1+2^k}))}.
\]

Iterating the above for all \( k = 0, 1, \ldots \) and using the fact that \( ||f||_{L^p} \to ||f||_{L^\infty} \) as \( p \to \infty \), we have

\[
\max_{B(x, 2\sqrt{r})} u^2 = \lim_{k \to \infty} \|u^2\|_{L^{2^{p_0}}(B(x, \sqrt{1+2^k}))} \leq C_1 \frac{(2p_0)^{2k}}{\tau} \sum_{k \geq 0} \frac{2k}{2^{p_0^k}} \int_{B(x, 2\sqrt{r})} u^2 \, dV.
\]

Now notice that

\[
\sum_{k \geq 0} \frac{1}{2^{p_0^k}} = \frac{1}{1 - \frac{1}{p_0}} = \frac{1}{1 - \frac{n-2}{n}} = \frac{n}{2}
\]

and that \( \sum_{k \geq 0} \frac{2k}{2^{p_0^k}} \) converges by comparison test. Therefore we have

\[
\max_{B(x, 2\sqrt{r})} u^2 \leq \frac{C_2}{\tau^2} \int_{B(x, 2\sqrt{r})} u^2 \, dV \leq \frac{C_2}{\tau^2} \int_{B(x, 2\sqrt{r})} u^2 \, dV = \frac{C_2}{\tau^2} (4\pi r)^{\frac{n}{2}} = C_2 (4\pi)^{\frac{n}{2}}.
\]

Since the point \( x \in M \) in which we centered the balls was arbitrary, and since the RHS is independent of that choice, we can see that the family \( \{u_k\} \) is uniformly bounded. This proves
We fill the work from equation (29) in [24]. Let’s drop the subscripts of \(k\) we have that \(u\) Jensen’s inequality:

We'll show that there is a limit in \(W^1\) in \(k\) \(u\) exponentially decreasing. Thus, there is a maximum point \(p_k\) in \(M\) for each \(u_k\). Hence \(\Delta u_k(p_k) \leq 0\). So when evaluated at \(p_k\), (11) becomes

\[
\tau_k R u_k - u_k \log u_k^2 - nu_k - \mu_k u_k \leq 0.
\]

We can divide through by \(u_k\) since each minimizer is strictly positive on \(M\). Then rearranging the above and using \(R \geq 0\), we have

\[
C \geq u_k(p_k) \geq \exp \left( \frac{R(p_k)\tau_k - n - \mu_k}{2} \right) \geq \exp \left( \frac{-n - \mu_k}{2} \right).
\]

Therefore it cannot be that \(\mu_k \to -\infty\). But \(\mu_k \to \mu_\infty\) and so \(\mu_\infty\) is finite.

Since the \(\{u_k\}\) are \(L^2\) normalized, we have

\[
\int_M u_k^2 \, dV = \int_K u_k^2 \, dV + \int_E u_k^2 \, dV = (4\pi \tau_k)^\frac{n}{2}.
\]

Since the \(u_k\) are uniformly bounded and \(K\) compact, the integral over \(K\) is uniformly bounded in \(k\). Since \(\tau_k \to \infty\), we can find some \(c_0 > 0\) such that for all \(k\) sufficiently large, we have

\[
\int_E u_k^2 \, dV \geq c_0 (4\pi \tau_k)^\frac{n}{2}.
\]

On the end \(E\) (which we can identify with \(\mathbb{R}^n \setminus K\) by the diffeomorphism \(K\)), we define functions \(\tilde{u}_k\) and a new metric \(\tilde{g}\) by

\[
\tilde{u}_k(x) = u_k(\tau_k x), \quad \tilde{g}_{ij}(x) = g_{ij}(\sqrt{\tau_k} x).
\]

The metric \(\tilde{g}\) on \(E\) is just \(\tau_k^{-1}g\) after the stretching diffeomorphism \(x \mapsto \sqrt{\tau_k} x\). The corresponding Laplacian \(\tilde{\Delta}_k\) and scalar curvature \(\tilde{R}(x)\) are given by

\[
\tilde{\Delta}_k(x) = \frac{1}{\tau_k} \Delta(\sqrt{\tau_k} x), \quad \tilde{R}(x) = \frac{1}{\tau_k} R(\sqrt{\tau_k} x)
\]

By the AE condition, \((E, \tilde{g}_k)\) converges in the Cheeger-Gromov sense to \((\mathbb{R}^n \setminus \{0\}, g_E)\) and the convergence is smooth away from the origin. The (new) Euler-Lagrange equation is

\[
-4\Delta_k \tilde{u}_k + \tilde{R} \tilde{u}_k - \tilde{u}_k \log \tilde{u}_k^2 - n \tilde{u}_k = \mu_k \tilde{u}_k.
\]

We’ll show that there is a limit in \(W^{1,2}(\mathbb{R}^n)\) for the sequence \(\{\tilde{u}_k\}\).

We fill the work from equation (29) in [24]. Let’s drop the subscripts of \(k\) for clarity. Note that since

\[
\int_M u^2 (4\pi)^{-n/2} \, dV = 1,
\]

we have that \(u^2 (4\pi)^{-n/2} dV\) is a probability measure. Since \(x \mapsto \log x\) is concave, we can apply Jensen’s inequality:

\[
\int u^2 \log u^2 (4\pi)^{-n/2} \, dV = \int u^2 \log u^{4\pi^{-n} (4\pi)^{-n/2}} \, dV = \frac{n - 2}{2} \int u^2 \log u^{4\pi^{-n} (4\pi)^{-n/2}} \, dV
\]

\[
\leq \frac{n - 2}{2} \log \left( \int u^2 u^{4\pi^{-n} (4\pi)^{-n/2}} \, dV \right) = \frac{n - 2}{2} \log \left( \int u^{2n} (4\pi)^{-n/2} \, dV \right).
\]
Now apply Lemma 3.5 to see
\[ \frac{n-2}{2} \log \left( \int u \frac{2n}{n-2} (4\pi)^{-n/2} dV \right) \leq \frac{n-2}{2} \log \left[ \left( C \int 4|\nabla u|^2 + Ru^2 \, dV \right)^{\frac{n}{n-2}} (4\pi)^{-n/2} \right] \]
for some constant \( C \) depending on \( g \). Now we can rearrange and use properties of the logarithm to see
\[
\frac{n-2}{2} \log \left[ \left( C \int 4|\nabla u|^2 + Ru^2 \, dV \right)^{\frac{n}{n-2}} (4\pi)^{-n/2} \right] \\
= \frac{n-2}{2} \log \left( C \int (4|\nabla u|^2 + Ru^2) (4\pi)^{-\frac{n}{2}} \, dV \right) \\
= \frac{n}{2} \log \left( \tau \int (4|\nabla u|^2 + Ru^2) (4\pi)^{-\frac{n}{2}} \, dV \right) + \log ((4\pi(C)^{\frac{n}{2}}).
\]

Now let
\[ A_k := \tau_k \int (4|\nabla u_k|^2 + Ru_k^2) (4\pi\tau_k)^{-\frac{n}{2}} \, dV. \]

Then we have
\[
0 \geq \mu_k = \int (\tau_k (4|u_k|^2 + Ru_k^2) - u_k^2 \log u_k^2 - nu_k^2) (4\pi\tau_k)^{-n/2} \, dV \\
= A_k - \int (u_k^2 \log u_k^2 + nu_k^2) (4\pi\tau_k)^{-n/2} \, dV \geq A_k - \frac{n}{2} \log A_k - n - \log ((4\pi(C)^{\frac{n}{2}})
\]
and therefore the sequence \( \{A_k\} \) is bounded. That is to say that
\[ \tau_k \int_M |\nabla u_k|^2 (4\pi\tau_k)^{-n/2} \, dV \leq C \]
for some \( C > 0 \) independent of \( k \). Then using \( d\tilde{V} = \tau_k^{-n/2}dV \) (from the determinate and square root in the volume form) and \( |\nabla \tilde{u}_k|^2 = \tau_k |\nabla u_k (\tau_k x)|^2 \), we have
\[ \int_{C_{a,A}} \tilde{u}_k^2 d\tilde{V}, \int_{C_{a,A}} |\tilde{u}_k|^2 d\tilde{V} \leq C_1 \]
where \( C_{a,A} = \{x \in \mathbb{R}^n : a < |x| < A \} \) is an open annulus and \( C_1 > 0 \) is independent of \( a, A \) and \( k \). Note that the integral bound on the \( \tilde{u}_k \) come from the \( L^2 \) normalization of the \( u_k \) and the fact that the scaling doesn’t affect the bound on the integral. In other words, the \( \tilde{u}_k \) form a bounded sequence in the Hilbert space \( W^{1,2}(C_{a,A}) \). It thus has a a subsequence that converges weakly to a function \( u_\infty \in W^{1,2}(C_{a,A}) \). The Rellich–Kondrachov theorem states that \( W^{1,2}(C_{a,A}) \) compactly embeds into \( L^q(C_{a,A}) \) for every \( 1 \leq q < \frac{2n}{n-2} \). As a consequence of the compact embedding, \( \tilde{u}_k \to u_\infty \) strongly in \( L^q \) (since limits are unique). Choosing two sequences \( a_m \to 0 \) and \( A_m \to \infty \) and iterating the above (each time replacing the sequence with the new one... this is just Cantor diagonalization), we have a function \( u_\infty \) defined on \( \mathbb{R}^n \setminus \{0\} \) such that for every compact set \( C \subset \mathbb{R}^n \setminus \{0\} \), there is some \( N > 0 \) such that \( \{\tilde{u}_k, k \geq N\} \) converges weakly to \( u_\infty \) in \( W^{1,2}(C) \) and strongly to in \( L^q \) for \( 1 \leq q < \frac{2n}{n-2} \). Of course \( N \) can be chosen large enough so that \( C \subset C_{a_m,A_m} \) for all \( m \geq N \).

By the standard \( L^p \) regularity property of the elliptic equation
\[ -4\Delta_k \tilde{u}_k + R\tilde{u}_k - \tilde{u}_k \log \tilde{u}_k^2 - nu_k^2 = \mu_k \tilde{u}_k, \]
the convergence is in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ for some $\alpha > 0$. Therefore, if we let $k \to \infty$ in the above elliptic equation, we have

$$-4\Delta_{g_E} u_\infty - u_\infty \log u_\infty - nu_\infty = \mu_\infty u_\infty.$$ 

By the standard regularity of elliptic operators and bootstrapping (see Theorem 6.17 in [11]), we know that $u_\infty \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and furthermore that either $u_\infty \equiv 0$, or else $u_\infty > 0$ by the strong maximum principle [23].

Moreover, since $\tilde{u}_k \to u_\infty$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and the metrics $\tilde{g}_k$ converge smoothly to $g_E$ away from the origin, we have the following for any $A > a > 0$:

$$\int_{B_A \setminus B_a} u_\infty^2 \, dV_{g_E} = \lim_{k \to \infty} \int_{B_A \setminus B_a} \tilde{u}_k^2 \, d\tilde{V} \leq (4\pi)^{n/2}$$

and

$$\int_{B_A \setminus B_a} |\nabla u_\infty|^2 \, dV_{g_E} = \lim_{k \to \infty} \int_{B_A \setminus B_a} |\nabla \tilde{u}_k|^2 \, d\tilde{V} \leq C.$$ 

Therefore, by taking $A \to \infty$ and $a \to 0$, we have

$$\int_{\mathbb{R}^n \setminus \{0\}} u_\infty^2 \, dV_{g_E} \leq (4\pi)^{n/2}, \quad \text{and} \quad \int_{\mathbb{R}^n \setminus \{0\}} |\nabla u_\infty|^2 \, dV_{g_E} \leq C.$$ 

Note that since $u_\infty$ is smooth (and thus $u_\infty$ and $|\nabla u_\infty|$ are measurable) and both the above integrands are positive, the respective integrals really do exist. Now we could define $u_\infty$ to be 0 at the origin, but we still could not conclude that $u_\infty \in W^{1,2}(\mathbb{R}^n)$ since in order to prove that

$$\int_{\mathbb{R}^n} u_\infty \partial_i \phi \, dV_{g_E} = -\int_{\mathbb{R}^n} \partial_i u_\infty \phi \, dV_{g_E},$$

for any test function $\phi$, we would like to use an integration by parts. But IBP requires $u_\infty$ to be differentiable on the domain, and we don’t know its behaviour at 0. So instead, we need to replace $\mathbb{R}^n$ with $\mathbb{R}^n \setminus B(0,r)$, but that introduces a boundary term which we want to go to 0 as $r \to 0$. Specifically, we need to prove the following limit

$$\lim_{r \to 0} \int_{S_r} u_\infty \phi v^i \, d\sigma = 0.$$ 

The following lemma gives a sufficient condition for making this work.

**Lemma 3.6.** For a function $f \in C^1(\mathbb{R}^n \setminus \{0\})$, if $|f(x)| \leq C|x|^{-\alpha}$ for some $\alpha < n-1$ and small $x$ and $|\nabla f|$ is integrable on the punctured ball $B_1 \setminus \{0\}$, then the function

$$\tilde{f}(x) = \begin{cases} f(x) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

has the weak derivative

$$g_i(x) = \begin{cases} \partial_i f(x) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$.

**Proof.** Let $v^i$ be the $i$th component of the inner normal vector on $S_r$ and let $\phi \in C^\infty_c(\mathbb{R}^n)$. Then we have

$$\left| \int_{S_r} f \phi v^i \, d\sigma \right| \leq |S_r| \max_{S_r} |\phi| \max_{S_r} |f| \leq C' r^{n-1} \max_{S_r} |f| \leq CC' r^{n-1-\alpha}.$$
Since $\alpha < n - 1$, the RHS goes to 0 as $r \to 0$. Therefore
\[
\lim_{r \to \infty} \left| \int_{S_r} f \phi v^i \, d\sigma \right| = 0.
\]
Note also that since $|\nabla f| \in L^1(B_1 \setminus \{0\})$ and $\phi$ has compact support $g_i \phi \in L^1(\mathbb{R}^n)$. So we have
\[
- \int_{\mathbb{R}^n} g_i \phi \, dV_{g_E} + \lim_{r \to \infty} \int_{S_r} f \phi v^i \, d\sigma = - \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B_r} \partial_i f \phi \, dV_{g_E} + \lim_{r \to \infty} \int_{S_r} f \phi v^i \, d\sigma.
\]
Then an IBP yields
\[
- \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B_r} \partial_i f \phi \, dV_{g_E} + \lim_{r \to \infty} \int_{S_r} f \phi v^i \, d\sigma = \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B_r} f \partial_i \phi \, dV_{g_E}.
\]
Since $f$ is integrable on the punctured ball $B_1(0) \setminus \{0\}$, $\tilde{f}$ is integrable on $B_1(0)$ Therefore
\[
\lim_{r \to 0} \int_{\mathbb{R}^n \setminus B_r} f \partial_i \phi \, dV_{g_E} = 0 \implies \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B_r} f \partial_i \phi \, dV_{g_E} = \int_{\mathbb{R}^n} \tilde{f} \partial_i \phi \, dV_{g_E}
\]
Since $\phi \in C^\infty_c (\mathbb{R}^n)$ was arbitrary, we see that $g_i$ is the weak derivative of $\tilde{f}$. □

Applying Moser’s iteration to
(16) 
\[-4 \Delta_{g_E} u_\infty - u_\infty \log u_\infty^2 - nu_\infty = \mu_\infty u_\infty\]
we have for any $0 < r \leq 1$ and $|p| = r$,
\[
 u_\infty(x)^2 \leq \max_{B(p, r/4)} u_\infty^2 \leq \frac{C}{r^n} \int_{B(p, r/2)} u_\infty^2 \, dV_{g_E} \leq \frac{C}{r^n} \int_{\mathbb{R}^n \setminus \{0\}} u_\infty^2 \, dV_{g_E} \leq \frac{C(4\pi)^{n/2}}{r^n}.
\]
Hence we have
\[
 u_\infty(x) \leq \frac{C}{|x|^{n/2}}
\]
for $|x| \leq 1$. Since $n \geq 3$, we indeed have $n/2 < n - 1$. Therefore, applying the above lemma with $\alpha = n/2$, we can conclude that $u_\infty \in W^{1,2}(\mathbb{R}^n)$ as discussed right before the statement of the lemma.

Now recall that we had two possible cases for what $u_\infty$ looked like on $\mathbb{R}^n \setminus \{0\}$ (either $u_\infty \equiv 0$ or $u_\infty > 0$). Let’s investigate this further.

**Case 1**: $u_\infty > 0$. In this case we have
\[
0 < \int_{\mathbb{R}^n} u_\infty^2 \, dV_{g_E} \leq (4\pi)^{n/2} \implies 0 < \int_{\mathbb{R}^n} u_\infty^2 (4\pi)^{-n/2} \, dV_{g_E} = c_1^2 \leq 1.
\]
Define $\tilde{u}_\infty = u_\infty/c_1$. Then from (16) and the fact that $\mu_{\mathbb{R}^n}(g_E, 1) = 0$, we have
\[
0 \leq \int_{\mathbb{R}^n} (4|\nabla \tilde{u}_\infty|^2 - \tilde{u}_\infty^2 \log \tilde{u}_\infty^2 - n\tilde{u}_\infty^2)(4\pi)^{-n/2} \, dV_{g_E}
\]
\[
= \int_{\mathbb{R}^n} \left(4 \left| \nabla u_\infty \right|^2 - \frac{u_\infty^2}{c_1^2} \log \left( \frac{u_\infty}{c_1} \right)^2 - n \left( \frac{u_\infty}{c_1} \right)^2 \right)(4\pi)^{-n/2} \, dV_{g_E}
\]
\[
= \frac{1}{c_1^2} \int_{\mathbb{R}^n} (4\nabla u_\infty|^2 - u_\infty^2 \log u_\infty^2 + u_\infty^2 \log c_1^2 - n u_\infty^2)(4\pi)^{-n/2} \, dV_{g_E}
\]
\[
= \frac{1}{c_1^2} \int_{\mathbb{R}^n} (\mu_\infty u_\infty^2 + u_\infty^2 \log c_1^2)(4\pi)^{-n/2} \, dV_{g_E} = \mu_\infty + \log c_1^2 \leq \mu_\infty
\]
since $c \leq 1$. But this is a contradiction since $\mu_{\infty} < 0$. So we must be in case 2.

**Case 2: $u_{\infty} \equiv 0$.** In this case, it means that $\tilde{u}_k(x) = u_k(\tau_k x)$ converges uniformly to 0 on any compact set of the asymptotic end $E$. This fact is due to the elliptic regularity (the convergence $\tilde{u}_k \to u_{\infty}$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$). We claim the stronger statement that

$$\limsup_{k \to \infty} \max_{\mathbb{R}^n \setminus \mathcal{B}(0,a)} \tilde{u}_k(x) = 0.$$  

To prove the claim, assume to the contrary that the above quantity is positive. Then there exists a sequence of points $\{p_k\}$ such that $\tilde{u}_k(p_k) > 0$ (possibly up to a subsequence). By the fact that $\tilde{u}_k$ converges uniformly to 0 on compact sets, we can assume $p_k \to \infty$. On the other hand, $(M, \tilde{g}_k, p_k)$ converges smoothly in the Cheeger-Gromov sense to $(\mathbb{R}^n, g_E, p_{\infty})$ and hence $\tilde{u}_k$ converges to $u_{\infty}$ which is not identically zero since $u'_{\infty}(p_{\infty}) = \lim_{k \to \infty} \tilde{u}_k(p_k) > 0$. This is a contradiction to our assumption that the limiting function is identically 0. Thus our claim of (17) holds.

Recall that there was some $c_0 > 0$ such that

$$\int_E u_k^2 \, dV \geq c_0 (4\pi \tau_k)^{n/2}.$$  

Since the $u_k$ are uniformly bounded we see that

$$\int_{\mathcal{B}(0,\sqrt{\tau_k} a)} u_k^2 \, dV \leq C(a \sqrt{\tau_k})^n.$$  

Thus we can choose a small $a > 0$ such that

$$C(a \sqrt{\tau_k})^n \leq \frac{c_0}{2} (4\pi \tau_k)^{n/2}$$

and so

$$\int_{E \setminus \mathcal{B}(0,2a \sqrt{\tau_k})} u_k^2 \, dV \geq \frac{c_0}{2} (4\pi \tau_k)^{n/2}.$$  

Choose a function smooth function $\phi$ such that $\text{supp} \phi \subset \subset \mathbb{R}^n \setminus B(0, a)$ and $\phi \equiv 1$ on $\mathbb{R}^n \setminus B(0, 2a)$. Then again using the identity

$$\langle \nabla(w \phi^2), \nabla w \rangle = |\nabla(w \phi)|^2 - |\nabla \phi|^2 \phi^2,$$

we have

$$\int \left( 4|\tilde{\nabla}(\phi \tilde{u}_k)|^2 + (\tilde{R} - n)(\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log \tilde{u}_k \right) (4\pi)^{-n/2} d\tilde{V}$$

$$= \int \left( 4\tilde{\nabla}(\phi^2 \tilde{u}_k), \tilde{\nabla} \tilde{u}_k \right) + |\nabla \phi|^2 \tilde{u}_k^2 + (\tilde{R} - n)(\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log \tilde{u}_k \right) (4\pi)^{-n/2} d\tilde{V}$$

$$= \int \left( -4\phi \tilde{u}_k \tilde{\nabla} \tilde{u}_k + |\tilde{\nabla} \phi|^2 \tilde{u}_k^2 + (\tilde{R} - n)(\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log \tilde{u}_k \right) (4\pi)^{-n/2} d\tilde{V}$$

$$= \mu_k(\phi \tilde{u}_k)^2 (4\pi)^{-n/2} d\tilde{V} + \int_{C_{a,2a}} |\nabla \phi|^2 \tilde{u}_k^2 (4\pi)^{-n/2} d\tilde{V}$$

$$\leq \mu_k(\phi \tilde{u}_k)^2 (4\pi)^{-n/2} d\tilde{V} + C \int_{C_{a,2a}} \tilde{u}_k^2 (4\pi)^{-n/2} d\tilde{V}$$

where $C$ depends only on $a$ (since we can choose $\phi$ to have derivative bounded only based on $a$). But $\tilde{u}_k$ converges uniformly to 0 on $C_{a,2a}$, so there is some sequence $\epsilon_k \to 0$ such that

$$\int \left( 4|\tilde{\nabla}(\phi \tilde{u}_k)|^2 + (\tilde{R} - n)(\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log \tilde{u}_k \right) (4\pi)^{-n/2} d\tilde{V} \leq \epsilon_k + \mu_k(\phi \tilde{u}_k)^2 (4\pi)^{-n/2} d\tilde{V}.$$  

(18)
On the other hand, we can use the fact that $0 \leq \phi \leq 1$ to see

$$(4\pi)^{n/2} \geq \int \tilde{u}_k^2 d\tilde{V} \geq \int (\phi \tilde{u}_k)^2 d\tilde{V} \geq \int_{\mathbb{R}^n \setminus B_{2a}} \tilde{u}_k^2 d\tilde{V}.$$ 

But recall that $a$ was chosen so that

$$\int_{E \setminus B(0, 2a \sqrt{\tau_k})} u_k^2 dV \geq \frac{c_0}{2} (4\pi \tau_k)^{n/2}$$

and also we know that $d\tilde{V} = \tau_k^{-n/2} dV$ so that

$$\int_{E \setminus B(0, 2a \sqrt{\tau_k})} \tilde{u}_k^2 d\tilde{V} = \int_{E \setminus B(0, 2a \sqrt{\tau_k})} u_k^2 dV \geq \frac{c_0}{2} (4\pi)^{n/2}.$$ 

Thus we have

$$\frac{c_0}{2} (4\pi)^{n/2} \leq \int (\phi \tilde{u}_k)^2 d\tilde{V} \leq (4\pi)^{n/2}$$

So if we set $\eta_k$ such that

$$\int (\phi \tilde{u}_k)^2 d\tilde{V} = \frac{\eta_k^2}{2} (4\pi)^{n/2}$$

then $\eta_k^2 \in [c_0/2, 1]$. Now let $\psi_k = \frac{\phi \tilde{u}_k}{\eta_k}$ so that

$$\int \psi_k^2 (4\pi)^{-n/2} dV = \int \frac{(\phi \tilde{u}_k)^2}{\eta_k^2} (4\pi)^{-n/2} dV = \frac{\eta_k^2}{2} (4\pi)^{-n/2} \frac{\eta_k}{\eta_k^2} = 1$$

Now dividing (18) by $\eta_k^2$ gives

$$\int \left( 4|\tilde{\nabla} \psi_k|^2 + (\tilde{R} - n) \psi_k^2 - \psi_k^2 \log \psi_k^2 \right) (4\pi)^{-n/2} d\tilde{V} > 0 \text{ for large } k$$

$$= \int \frac{1}{\eta_k^2} \left( 4|\tilde{\nabla} \tilde{u}_k|^2 + (\tilde{R} - n) (\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log \tilde{u}_k^2 + (\phi \tilde{u}_k)^2 \log \eta_k^2 \right) (4\pi)^{-n/2} d\tilde{V}$$

$$\leq \eta_k^{-2} \epsilon_k + \mu \eta_k^{-2} \int (\phi \tilde{u}_k)^2 (4\pi)^{-n/2} d\tilde{V} + \eta_k^{-2} \log \eta_k \int (\phi \tilde{u}_k)^2 (4\pi)^{-n/2} d\tilde{V}$$

$$= \eta_k^{-2} \epsilon_k + \mu \eta_k^{-2} \log \eta_k.$$ 

Now we use the bounds on $\eta_k^2$. Namely $\eta_k^2 \leq 1$ (so $\log \eta_k^2 \leq 0$) and $\eta_k^2 \geq c_0/2$ (so $\eta_k^{-2} \geq 2c_0^{-1}$) which gives

$$(19) \int \left( 4|\tilde{\nabla} \psi_k|^2 + (\tilde{R} - n) \psi_k^2 - \psi_k^2 \log \psi_k^2 \right) (4\pi)^{-n/2} d\tilde{V} \leq 2c_0^{-1} \epsilon_k + \mu \eta_k.$$ 

Recall that $\mu \eta_k \to \mu_\infty < 0$ and $\epsilon_k \to 0$ and so for $k$ large enough, the RHS above is negative. But since $\psi_k = \frac{\phi \tilde{u}_k}{\eta_k}$ and the $\eta_k$ are uniformly bounded from below by $\sqrt{\tau_k} > 0$ and by the properties of $\phi$, we have

$$\sup_{x \in \mathbb{R}^n} |\psi_k(x)| = \sup_{x \in \mathbb{R}^n} \left| \frac{\phi(x) \tilde{u}_k(x)}{\eta_k} \right| \leq \sqrt{\frac{2}{c_0}} \sup_{x \in \mathbb{R}^n} |\phi(x) \tilde{u}_k(x)| \leq \sqrt{\frac{2}{c_0}} \sup_{x \in \mathbb{R}^n \setminus B_a} |\tilde{u}_k(x)| \to 0.$$ 

But this last quantity goes to zero as $k \to \infty$ by (17). Therefore, $\psi_k$ converges uniformly to zero on $\mathbb{R}^n$.

But $n + \log x^2 < 0$ for $x$ small enough, therefore since $\psi_k \to 0$ uniformly, we have

$$4|\tilde{\nabla} \psi_k|^2 + (\tilde{R} - n) \psi_k^2 - \psi_k^2 \log \psi_k^2 \geq -\psi_k^2 (n + \log \psi_k^2) > 0$$
for $k$ large enough on all of $\mathbb{R}^n$. Integrating the above quantity contradicts (19). This completes the proof of Theorem 3.4.

As mentioned earlier, we’ll use Theorem 3.4 several times in the proof of the main theorem. But one of the places that it will be used is in establishing the behaviour of $|\text{Rm}(t)|$ as $t \to \infty$. This will, in turn, allow us to prove that the metrics $g(t)$ converge smoothly to some limit metric $g(\infty)$ (which we will in turn show is actually just the Euclidean metric). But in order to do any of this, we will be required to prove the following:

Claim: $(M, g_i(t), p_i)$ converges smoothly in the Cheeger-Gromov sense to a complete eternal Ricci flow solution $(M_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, \infty)$ where $g_i(t) = Q_i g(t_i + Q_i^{-1} t)$ and $Q_i = |\text{Rm}(x_i, t_i)|$ for some specially chosen $(x_i, t_i) \in M \times [0, \infty)$.

To prove this claim (several times), we’ll use Hamilton’s Compactness Theorem which assumes bounded curvature as well as some lower bound on injectivity radius at the points $p_k$.

**Theorem** (Hamilton’s compactness of Ricci flow, cf. Theorem 8.2 in [25]). Let $(M_k, g_k(t), p_k)$ $(k \in \mathbb{N}$ and $t \in (a, b)$) be a sequence of pointed Ricci flows. If there exists $C_0 > 0$ and $\iota_0 > 0$ such that for all $k$,

1. (curvature bound)
   $$|\text{Rm}_{g_k}| g_k \leq C_0 \text{ on } M_k \times (a, b),$$

2. (injectivity radius estimate at $t = 0$)
   $$\text{inj}_{g_k(0)}(p_k) \geq \iota_0,$$

then there exists a subsequence $j_k$ such that $(M_{j_k}, g_{j_k}, p_{j_k})$ converges to a complete pointed Ricci flow $(M_\infty, g_\infty(t), p_\infty)$ $(t \in (a, b))$.

Assumption (1) (bounded curvature) will be satisfied based on our choice of points $(p_i, t_i)$. However, showing condition (2) is much harder. Indeed, in order to deal with this, we have to introduce the following concept of $\kappa$-noncollapsedness.

**Definition.** A Riemannian manifold is $\kappa$-noncollapsed on all scales if for any metric ball $B(x, r)$ satisfying $|\text{Rm}(y)| \leq r^{-2}$ for all $y \in B(x, r)$, then we have

$$\frac{\text{Vol} B(x, r)}{r^n} \geq \kappa.$$

We’ll use the following fact proved by Cheeger-Gromov-Taylor (see [5]).

**Theorem** (Cheeger-Gromov-Taylor). For any $c > 0$, $r_0 > 0$ and $n \in \mathbb{N}$, there exists $\iota_0 = \iota(n, c) > 0$ such that for any complete Riemannian manifold $(M^n, g)$ with the sectional curvature bounded by $\pm 1$, if $p \in M$ satisfies

$$\frac{\text{Vol} B(p, r_0)}{r_0^n} \geq c,$$

then $\text{inj}(p) \geq \iota_0$. 

$$\square$$
Lemma \((\kappa\text{-noncollapsed implies injectivity radius bound})\). Let \((M^n, g)\) be a complete Riemannian manifold and let \(\rho > 0\). Suppose that \(g\) is \(\kappa\)-noncollapsed below the scale \(\rho\). That is to say, for any \(B(x, r) \subset M\) with \(r < \rho\) such that \(|Rm| \leq r^{-2}\) on \(B(x, r)\), then
\[
\frac{\text{Vol}(B(p, r_0))}{r_0^n} \geq \kappa.
\]
Then there exists \(\delta = \delta(n, \kappa)\) such that for any \(x \in M\) and \(r \leq \rho\), if \(|Rm| \leq r^{-2}\) on \(B(x, r)\), then \(\text{inj}(x) \geq \delta r\).

Proof. Suppose that \(|Rm| \leq \frac{1}{n} r^{-2}\) on \(B(x, r)\) for some \(r \leq \rho\). Then consider the metric \(g_r := \frac{1}{r^2} g\) so that \(B(x, r) = B_g(x, r) = B_{g_r}(x, 1)\). Then since \(|Rm_{g_{r}}|_{g_{r}} \leq 1\), the \(\kappa\)-noncollapsed condition implies that
\[
\text{Vol}_{g_{r}} B_{g_{r}}(x, 1) = \frac{\text{Vol}_{g}(B(x, r))}{r^n} \geq \kappa.
\]
Then by the previous theorem, there exists some \(\delta = \delta(n, \kappa)\) such that \(\text{inj}_{g_{r}}(x) \geq \delta\) and thus \(\text{inj}_{g_{r}}(x) \geq \delta r\).

Thus if we are able to pick our points \((p_i, t_i)\) in such a way that \(|Rm_{g_{r}}|_{g_{r}} \leq C\) on all of \(M\) and uniformly in \(i \in \mathbb{N}\) then in particular,
\[
|Rm_{g_{r}}|_{g_{r}} \leq r^{-2} \quad \text{for all} \quad y \in B_r(x) \quad \text{whenever} \quad r \geq \sqrt{\frac{1}{C} =: \rho}.
\]
If we could show that the original Ricci flow solution \(g(t), t \in [0, \infty)\) is \(\kappa\)-noncollapsed on all scales, then we would have
\[
\frac{\text{Vol}_{g_{i}(0)} B_{g_{i}(0)}(p_i, r)}{r^n} = \frac{\text{Vol}_{g_{Q_i}(t_i)} B_{g_{Q_i}(t_i)}(p_i, r)}{r^n} = \frac{Q_i^{n/2} \text{Vol}_{g_{Q_i}(t_i)} B_{g_{Q_i}(t_i)}(p_i, r/\sqrt{Q_i})}{r^n} \geq \kappa.
\]
Therefore, the metrics \(g_i\) would be (uniformly) \(\kappa\)-noncollapsed below the scale \(\rho\) and thus, by the previous lemma, we would have \(\text{inj}_{g_{i}(0)}(p_i) \geq \delta \rho\). This is precisely condition (2) in Hamilton’s Compactness Theorem. We therefore have

Lemma. If the flow \(g(t), t \in [0, \infty)\) is \(\kappa\)-noncollapsed on all scales, and if \((p_i, t_i) \in M \times [0, \infty)\) are chosen so that the metrics \(\{g_i\}\) have uniformly bounded curvature, then condition (2) holds. That is to say, there is some \(t_0 > 0\) such that
\[
\text{inj}_{g_{i}(0)}(p_i) \geq t_0 \quad \text{for all} \quad i \in \mathbb{N}.
\]
And therefore, by Hamilton’s Compactness, \((M_k, g_k(t), p_k)\) converges (up to a subsequence) to a complete pointed Ricci flow \((M_\infty, g_\infty(t), p_\infty)\).

All this is to say that it suffices to show that the flow \(g(t), t \in [0, \infty)\) is \(\kappa\)-noncollapsed on all scales. We will do this now.
Theorem 3.9. Let \( g(t), t \in [0, \infty), \) be the Ricci flow solution on an AE manifold \( M^n \) with \( R(t) > 0 \) for all \( t \). Then there exists a \( \kappa > 0 \) such that \( g(t) \) is \( \kappa \)-noncollapsed on all scales.

Proof. We break into two cases. First consider \( t \in [0, 1] \). Then by Theorem 2.2, the Ricci flow preserves the AE coordinate system and inequalities so it suffices to show that \( (M, g(0)) \) is \( \kappa \)-noncollapsed on this interval. Since \( (M, g) \) is AE, we can find some compact \( K \) such that for all \( x \not\in K \) and \( r > 0 \) we have

\[
\frac{\text{Vol}(B(x, r))}{r^n} \geq \delta.
\]

Then the function

\[
F : K \times [0, \infty) \to \mathbb{R}, \quad (x, r) \mapsto \frac{\text{Vol}(B(x, r))}{r^n}
\]

is continuous and positive. Moreover since \( F(x, \cdot) \to \alpha_n \) as \( r \to \infty \) (say by Lemma 2.5), we can use a compactness argument to see that there is some \( r_0 > 0 \) such that whenever \( r \geq r_0 \) and \( x \in K \), then \( F(x, r) \geq \alpha_n/2 \). Finally, \( K \times [0, r_0] \) is compact and \( F \) is positive so it achieves its (positive) minimum. Thus, there is some \( \kappa_1 > 0 \) such that

\[
\frac{\text{Vol}(B(x, r))}{r^n} \geq \kappa_1
\]

for all \( x \in M, r \geq 0 \).

For \( t \in [1, \infty), r > 0 \) and \( x \in M \) such that \( |\text{Rm}(g)| \leq r^{-2} \) for all \( y \in B_{g(t)}(x, r) \) we have the following inequality which is Proposition 5.37 in [9]:

\[
\mu(g(t), r^2) \leq \log \left( \frac{\text{Vol}(B_{g(t)}(x, r))}{r^n} \right) + C(n).
\]

Since \( \mu \) is continuous and \( \lim_{r \to \infty} \mu(g, r) = 0 \) from Theorem 3.4, we can conclude that \( \mu(g, \cdot) \) is bounded (by say \(-C\)). So using this, the above inequality and the monotonicity of \( \mu \), we have

\[
-C \leq \mu(g(0), r^2 + t) \leq \mu(g(t), r^2) \leq \log \left( \frac{\text{Vol}(B_{g(t)}(x, r))}{r^n} \right) + C(n).
\]

Therefore

\[
\frac{\text{Vol}(B_{g(t)}(x, r))}{r^n} \geq \exp(-C - C(n)) =: \kappa_2.
\]

Letting \( \kappa = \min\{\kappa_1, \kappa_2\} \) gives the result.

\( \square \)

CHAPTER 4: ANALYSIS OF SINGULARITY AT TIME INFINITY

For the Ricci flow \( (M, g(t)), t \in [0, \infty) \), there are two different types of singularity at infinity classified by Hamilton [13].

Case 1 (Type IIb): \( \sup_{t \in [0, \infty)} t |\text{Rm}(x, t)| = \infty. \)

In this case, we take any sequence of times \( T_i \to \infty \) and choose \( p_i = (x_i, t_i) \in M^n \times [0, T_i] \) such that

\[
t_i(T_i - t_i)|\text{Rm}(x_i, t_i)| = \sup_{M^n \times [0, T_i]} t(T_i - t)|\text{Rm}(x, t)|.
\]

This is of course possible since \( [0, T_i] \) is compact and each \( (M, g(t)) \) is AE with the same coordinate system as \( (M, g(0)) \) so \( |\text{Rm}| \to 0 \) uniformly in \( t \) as \( d(x) \to \infty \).
Claim: \( t_i \to \infty \).

Proof: Otherwise, there is a subsequence (still calling it \( t_i \)) that is bounded.

Find \( L_i, y_i \) such that \( \lim_{i \to \infty} L_i |\text{Rm}(y_i, L_i)| \to \infty \) and \( L_i \leq T_i/2 \).

We claim that \( t_i \to \infty \). Otherwise, there is a subsequence (still calling it \( t_i \)) that is bounded. Since we’re assuming we’re in type IIb, we can find \( L_i \to \infty \) and \( y_i \in M \) satisfying \( \lim_{i \to \infty} L_i |\text{Rm}(y_i, L_i)| \to \infty \) and furthermore that \( L_i \leq T_i/2 \) (which is fine since \( T_i \to \infty \) as well). Then

\[
t_i(T_i-t_i)|\text{Rm}(x_i, t_i)| = \sup_{M^n \times [0, T_i]} t(T_i-t)|\text{Rm}(x, t)| \geq L_i(T_i-L_i)|\text{Rm}(y_i, L_i)| \geq \frac{1}{2} T_i L_i |\text{Rm}(y_i, L_i)|. 
\]

Therefore we get

\[
t_i|\text{Rm}(x_i, t_i)| - \frac{t_i^2}{T_i} |\text{Rm}(x_i, t_i)| \geq \frac{1}{2} L_i |\text{Rm}(y_i, L_i)| 
\]

Since the \( t_i \) remain bounded, so does \( |\text{Rm}(x_i, t_i)| \) (this quantity is bounded by \( \sup_{M^n \times [0, S]} |\text{Rm}| \) for some \( S \geq t_i \)). So as \( i \to \infty \), the RHS \( \to \infty \) and the LHS goes to \( t_i |\text{Rm}(x_i, t_i)| \). Since we have the inequality, this quantity must be going to infinity as well, and therefore \( t_i \to \infty \) which contradicts our assumption of bounded \( t_i \). This proves the claim.

If we set \( Q_i = |\text{Rm}(x_i, t_i)| \), then we claim that \((M, g_i(t), p_i)\) converges smoothly in the Cheeger-Gromov sense to a complete eternal Ricci flow solution \((M_\infty, g_\infty(t), p_\infty)\), \( t \in (-\infty, \infty) \) where \( g_i(t) = Q_i g(t_i + Q_i^{-1} t). \) By Hamilton’s Compactness of Ricci flow and the \( \kappa \)-noncollapsiveness, it suffices to show that \( |\text{Rm}_{g_i}|_{g_i} \) is uniformly bounded for \( i \) large enough. Furthermore, it suffices to prove this on the interval \((-a, a)\) for some \( a > 0 \) as we can take \( a \to \infty \) after to conclude the result on all of \( \mathbb{R} \).

Let \( \Omega_i = (T_i - t_i)Q_i \) and \( A_i = t_i Q_i \) so that the flow \( g_i \) is defined for all \( t \in (-A_i, \Omega_i) \). Note that

\[
\frac{1}{1/\Omega_i + 1/A_i} = \frac{A_i \Omega_i}{A_i + \Omega_i} = \frac{t_i(T_i - t_i)Q_i^2}{(T_i - t_i)Q_i + t_i Q_i} = \frac{t_i(T_i - t_i)}{T_i} Q_i
\]

which then immediately implies that \( A_i \Omega_i \to \infty \) as \( i \to \infty \). In particular, we may assume that both \( A_i, \Omega_i \) are uniformly large for all \( i \) (and \( A_i, \Omega_i \) at the times before the dilation and \( t \) the times after dilation. That is to say that \( s = t_i + tQ_i^{-1} \) and \( t = (s - t_i)Q_i \). Then by the choice of \( p_i, t_i \), we have

\[
(t + A_i)(\Omega_i - t)|\text{Rm}_{g_i}(x, t)|_{g_i} = (t + A_i)(\Omega_i - t)Q_i^{-1}|\text{Rm}_{g}(x, t + tQ_i^{-1})|_{g} = (s - t_i)Q_i + tQ_i |(T_i - t_i)Q_i - (s - t_i)Q_i|Q_i^{-1}|\text{Rm}_{g_i}(x, t + (s - t_i)Q_i Q_i^{-1})|_{g}
\]

\[
= Q_i s(T_i - s)|\text{Rm}_{g}(x, s)|_{g} \leq Q_i t_i(Q_i - t_i)|\text{Rm}(p_i, t_i)| \leq t_i(Q_i - t_i)Q_i^2 = A_i \Omega_i 
\]

for any \( t \in [-A_i, \Omega_i] \). In particular, if we restrict to \( t \in (-a, a) \), then

\[
t + A_i \geq A_i - a, \quad \text{and} \quad \Omega_i - t \geq \Omega_i - a
\]

and so we have

\[
\sup_{x \in M, t \in (-a, a)} |\text{Rm}_{g_i}(x, t)|_{g_i} \leq \frac{A_i \Omega_i}{(A_i - s)(\Omega_i - s)},
\]
The RHS is uniformly bounded in $i$ because $A_i, \Omega_i \gg s$ and the RHS $\to 1$ as $i \to \infty$. In other words, there is some constant $C_0 > 0$ such that

$$\sup_{x \in M, t \in (-\infty, a), i \in \mathbb{N}} |\text{Rm}_{g_i}(x, t)|_{g_i} \leq C_0.$$  

Thus, $(M, g_i(t), p_i)$ converges smoothly in the Cheeger-Gromov sense to $(M_\infty, g_\infty(t), p_\infty)$.

Then for any $\tau > 0$,

$$\mu(g_\infty(0), \tau) \geq \limsup_{i \to \infty} \mu(Q_i g(t_i), \tau)$$

Lemma 3.2

$$= \limsup_{i \to \infty} \mu(g(t_i), \frac{\tau}{Q_i})$$

scaling of $\mu$

$$\geq \limsup_{i \to \infty} \mu(g(0), \frac{\tau}{Q_i} + t_i)$$

monotonicity of $\mu$

$$= 0$$

Theorem 3.4

By theorem 3.3, $(M^n, g_\infty(0)) \cong (\mathbb{R}^n, g_E)$. But this is impossible since

$$|\text{Rm}|_{g_\infty(0)}(x_\infty) = \lim_{i \to \infty} |\text{Rm}|_{g_i(0)}(x_i) = \lim_{i \to \infty} Q_i |\text{Rm}|(t_i, x_i) = \lim_{i \to \infty} Q_i Q_i^{-1} = 1.$$  

Case 2 (Type III): $A := \sup_{M \times [0, \infty)} t |\text{Rm}| < \infty$.

If $\limsup_{t \to \infty} t \sup_{x \in M} |\text{Rm}| > 0$, then we can find a sequence of points and times $p_i = (x_i, t_i)$ with $t_i \to \infty$ such that

$$t_i |\text{Rm}|(x_i, t_i) = t_i \sup_{x \in M} |\text{Rm}|(x, t_i) \geq c > 0.$$  

One part of this is just using the definition of the limsup being positive, while the other part is using the fact that each $(M, g(t))$ is AE and so $|\text{Rm}|(x, t) \to 0$ as $d(x) \to \infty$ so the supremum of $|\text{Rm}|(\cdot, t)$ over $M$ is achieved for each $t \geq 0$. If we set $Q_i = |\text{Rm}(x_i, t_i)|$, and $g_i(t) = Q_i g(t_i + Q_i^{-1} t)$, then

$$\sup_{x \in M, t \in [0, \infty)} |\text{Rm}_{g_i}(x, t)|_{g_i} = |\text{Rm}(x_i, t_i)|^{-1} \sup_{x \in M, t \in [0, \infty)} |\text{Rm}(x, t_i + Q_i^{-1} t)|$$

$$\leq \frac{1}{c} \sup_{x \in M, t \in [0, \infty)} (t_i + Q_i^{-1} t) |\text{Rm}(x, t_i + Q_i^{-1} t)| \leq \frac{A}{c}.$$  

Therefore, by Hamilton’s Compactness, $(M, g_i(t), p_i)$ converges smoothly in the Cheeger-Gromov sense to a Ricci flow solution $(M_\infty, g_\infty(t), P_\infty), t \in [0, \infty)$. Then for any $\tau > 0$,

$$\mu(g_\infty(0), \tau) \geq \limsup_{i \to \infty} \mu(Q_i g(t_i), \tau)$$

Lemma 3.2

$$= \limsup_{i \to \infty} \mu(g(t_i), \frac{\tau}{Q_i})$$

scaling of $\mu$

$$\geq \limsup_{i \to \infty} \mu(g(0), \frac{\tau}{Q_i} + t_i)$$

monotonicity of $\mu$

$$= 0$$

Theorem 3.4

By theorem 3.3, $(M^n, g_\infty(0))$ is isometric to $(\mathbb{R}^n, g_E)$ but this is impossible since

$$|\text{Rm}|_{g_\infty(0)}(x_\infty) = \lim_{i \to \infty} |\text{Rm}|_{g_i(0)}(x_i) = \lim_{i \to \infty} Q_i |\text{Rm}|(t_i, x_i) = \lim_{i \to \infty} Q_i Q_i^{-1} = 1.$$  

So it must be the case that $\limsup_{t \to \infty} t \sup_{x \in M} |\text{Rm}| = 0$.  

Choose $\epsilon \in (0, 1/2)$ to be determined later. By possibly translating in time, we can assume that
\[
\sup |Rm| \leq \frac{\epsilon}{1+t} \quad \text{for all } t \geq 0.
\]
We will see later that $\epsilon$ only needs to satisfy
\[
(20) \quad \epsilon < \frac{p}{2p+1} - \frac{1}{8}
\]
where $p \in (\frac{n}{2+\sigma}, \frac{n}{\sigma})$. In particular, the quantity on the RHS is positive. So we could take $\epsilon$ to be half of the RHS and not have to worry about the order of quantifiers.

Next, we prove a gradient estimate and Harnack inequality for the solution of heat equation under the condition of (4.5). The proof is a long time version of the Li-Yau estimates, see [18].

Set $u_0 = r^{-2-\sigma}$ where $r$ is the function defined on $M$ by the definition of an AE manifold ($r$ is a positive smooth function such that $r(x) = |\Phi(x)|$ when $x \in E$ where $\Phi : E \to \mathbb{R}^n \setminus B$ is the AE diffeomorphism; $\sigma$ is the order of the AE end $E$ which satisfies $(n-2)/2 < \sigma \leq n-2$).

Consider the solution to the heat equation
\[
\begin{align*}
&u_t = \Delta u, \quad u(0) = u_0
\end{align*}
\]
which is necessarily positive for all $t \geq 0$, $x \in M$ by the maximum principle.

**Lemma.** For any $T > 0$, there exists $c_1(T), c_2(T)$ such that for all $x$ outside a compact set, we have
\[
\begin{align*}
c_1(T)r^{-2-\sigma} &\leq u(t) \leq c_2(T)r^{-2-\sigma}, \\
c_1(T)r^{-3-\sigma} &\leq |\nabla u(t)| \leq c_2(T)r^{-3-\sigma}
\end{align*}
\]

**Proof.** First, let $h = r^{2+\sigma}$ and $w = hu$ so that (just like in the proof of Theorem 2.2), we have
\[
(\partial_t - \Delta)w \leq Bw - 2\nabla \log h \nabla w
\]
where $B = \frac{2|\nabla h|^2 - h\Delta h}{h^2}$ which is uniformly bounded just like before as
\[
|B| = \frac{2|\nabla h|^2 - h\Delta h}{h^2} \leq C \left| \frac{|\nabla g(0)|^2 - h\Delta g(0)}{h^2} \right| \leq C \frac{r^{2+2\sigma} + r^{2+\sigma}r^{4-\sigma}}{r^{2(2+\sigma)}} = Cr^{-2} \leq C
\]
as $r$ achieves a positive minimum. Since $u(0) = r^{-2-\sigma}$, we conclude by Theorem 2.1 that there is some constant (depending on $T$) such that for all $t \in [0, T]$ we have $u(t) \leq Cr^{-2-\sigma}$.

Similarly, $u^{-1}(0) = r^{2+\sigma}$ and $u^{-1}$ satisfies
\[
(\partial_t - \Delta)u^{-1} = -u^{-2}u_t + u^{-2}\Delta u - 2u^{-3}\nabla u|^2 = -u^{-2}\Delta u + u^{-2}\Delta u - 2u^{-3}\nabla u|^2 \leq 0.
\]
so letting $h = r^{-2-\sigma}$ and $w = hu$ gives
\[
(\partial_t - \Delta)w \leq Bw - 2\nabla \log h \nabla w
\]
where $B = \frac{2|\nabla h|^2 - h\Delta h}{h^2}$ is uniformly bounded because
\[
|B| = \frac{2|\nabla h|^2 - h\Delta h}{h^2} \leq C \left| \frac{|\nabla g(0)|^2 - h\Delta g(0)}{h^2} \right| \leq C \frac{r^{-2-\sigma} + r^{-2-\sigma}r^{-4-\sigma}}{r^{2(2-2-\sigma)}} = Cr^{-2} \leq C.
\]

Then by Theorem 2.1, $w \leq C(T)$ on $[0, T]$. In other words
\[
r^{-2-\sigma}u^{-1} = w \leq C(T) \implies u^{-1} \leq r^{2+\sigma}C(T) \implies C(T)^{-1}r^{-2-\sigma} \leq u.
\]
Obtaining the same bounds on $|\nabla u|$ are just obtained by replacing $u$ with $|\nabla u|$ above and $h$ with either $r^{1+\sigma}$ or $r^{-1-\sigma}$ respectively. Note that $|\nabla u|(0) \leq Cr^{1+\sigma}$ and $|\nabla u|^{-1}(0) \leq Cr^{-1-\sigma}$ at least outside of a compact set. There is a small problem in that $|\nabla u|$ might be arbitrarily close to 0 outside of the AE end $E$, but this is only a compact set and what we really care about is the behaviour of $u$ and $\nabla u$ at infinity. □

Let $f = \log u$. Then $f$ satisfies
\[
 f_t = \frac{u_t}{u} = \frac{\Delta u}{u^2} - \frac{|\nabla u|^2}{u^2} + \frac{\nabla u^2}{u^2} = \nabla (\frac{\nabla u}{u}) + \frac{\nabla u^2}{u^2} = \Delta f + |\nabla f|^2.
\]
If we set $H(x, t) = t(|\nabla f|^2 - 2f_t)$, then we have the following lemma:

**Lemma.** Under the condition that $\sup_M |Rm|(x, t) \leq \frac{r}{1+t}$, we have
\[
 \Delta H - H_t \geq -2|\nabla f| \cdot \nabla H + \frac{t}{n}((|\nabla f|^2 - f_t)^2 - (|\nabla f|^2 - 2f_t) - 3|\nabla f|^2 - \frac{4e^2}{1+t}.
\]

**Proof.** We have
\[
 \Delta H = t\Delta(|\nabla f|^2 - 2f_t).
\]
Using Bochner’s formula, we have
\[
 \Delta |\nabla f|^2 = 2|\nabla f|^2 + 2\text{Re}(\nabla f \cdot \nabla f) + 2(\nabla \Delta f, \nabla f)
\]
\[
 = 2|\nabla f|^2 + 2\text{Re}(\nabla f \cdot \nabla f) - 2(|\nabla f|^2 - f_t, \nabla f)
\]
\[
 \geq 2|\nabla f|^2 - \frac{2}{1+t}|\nabla f|^2 - 2(|\nabla f|^2 - f_t), \nabla f)
\]
where the last inequality follows from a computation in normal coordinates at $p$ (so that $g_{ij} = \delta_{ij}$) and that $\sup_M |Rm|(x, t) \leq \frac{r}{1+t}$.

I claim that under the Ricci flow, we have the following identity:
\[
 (21) \quad \Delta f_t = (\Delta f)_t - 2\text{Rc}_{ij}f_{ij}.
\]
To prove this identity, we can use coordinates such that $g_{ij} = \delta_{ij}$ and $\partial_i g_{ab} = 0$. We can do this because both sides are tensorial (note $\text{Rc}_{ij}f_{ij} = \text{tr}_g \text{Rc} \nabla^2 f$). So we have
\[
 (\Delta f)_t = \partial_t \left( \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j f) \right) = \partial_t \left( \frac{1}{\sqrt{|g|}} \sqrt{|g|} g^{ij} f_{ij} \right) = 2\text{Rc}_{ij}f_{ij} + \Delta f_t.
\]
Then by Young’s inequality, we have
\[
 \Delta f_t = (\Delta f)_t - 2\text{Rc}_{ij}f_{ij} \leq (\Delta f)_t + 2|\text{Rc}|^2 + \frac{1}{2}|\nabla^2 f|^2.
\]
So we have
\[
 \Delta H \geq t \left( 2|\nabla f|^2 - \frac{2}{1+t}|\nabla f|^2 - 2(\nabla(|\nabla f^2 - f_t), \nabla f) - 2((\Delta f)_t + 2|\text{Rc}|^2 + \frac{1}{2}|\nabla^2 f|^2) \right)
\]
\[
 = t \left( |\nabla f|^2 - \frac{2}{1+t}|\nabla f|^2 - 2(\nabla(|\nabla f^2 - f_t), \nabla f) - 2((\Delta f)_t + 2|\text{Rc}|^2 \right)
\]
\[
 \geq t(|\nabla f|^2 - 2|\nabla f|^2 - 2t(\nabla(|\nabla f^2 - f_t), \nabla f) - 2(|\nabla f|^2 - f_t)_t - \frac{4e^2}{1+t}
\]
where in the last inequality we used [21], the curvature bound, and $\frac{1}{1+t} \leq 1$. The first term is still trouble. But I claim that
\[
 |\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2.
\]
We can see this by working in normal coordinates so that \( g_{ij} = \delta_{ij} \) and letting \( A_{ij} = (\nabla^2 f)_{ij} \) we have
\[
|\nabla^2 f|^2 = g^{ij}g^{kl}A_{ik}A_{jl} = \sum_{ij}A_{ij}^2 \geq \sum_i A_i^2 \geq^* \frac{1}{n}(\text{tr}A)^2 = \frac{1}{n}(\text{tr}gA)^2 = \frac{1}{n}(\Delta f)^2.
\]
The marked inequality is true by the following:
\[
(23)
\]
Thus to show \( (24) \), it suffices to show
\[
H_t = |\nabla f|^2 - 2f_t + t(|\nabla f|^2 - 2f_t)_t.
\]
So combining these two, we have
\[
\Delta H - H_t \geq \frac{t}{n}((\nabla f)^2 - f_t)^2 - 2(|\nabla f|^2 - 2f_t)_t - 2(|\nabla f|^2 - f_t)_t^2 - \frac{4\epsilon^2}{1+t}
\]
\[
= -2t(\langle \nabla f^2 - f_t, \nabla f \rangle) + 2t(|\nabla f|^2 - f_t)_t - t(|\nabla f|^2 - 2f_t)_t + |\nabla f|^2
\]
\[
+ \frac{t}{n}(|\nabla f|^2 - f_t)^2 - (\langle \nabla f^2 - 2f_t, \nabla f \rangle - 3|\nabla f|^2 - \frac{4\epsilon^2}{1+t}.
\]
So to finish the proof of the lemma, it suffices to show that
\[
(23) \quad -2t(\langle \nabla f^2 - f_t, \nabla f \rangle + 2t(|\nabla f|^2 - f_t)_t - t(|\nabla f|^2 - 2f_t)_t + |\nabla f|^2 \geq -2\langle \nabla f, \nabla H \rangle.
\]
But the RHS of \( (23) \) can be written as
\[
-2\langle \nabla f, \nabla H \rangle = -2\langle \nabla f, \nabla (t(|\nabla f|^2 - 2f_t)) \rangle = -2t(\langle \nabla f^2 - f_t, \nabla f \rangle + 2t(\langle \nabla f, \nabla f \rangle.
\]
Moreover, a portion of the LHS \( (23) \) can be simplified:
\[
+2t(|\nabla f|^2 - f_t)_t - t(|\nabla f|^2 - 2f_t)_t = 2t|\nabla f|^2.
\]
So to prove \( (23) \), it suffices to prove
\[
(24) \quad t|\nabla f|^2 + |\nabla f|^2 \geq 2t|\nabla f, \nabla f_t|.
\]
We have
\[
|\nabla f|^2 = (g^{ij}f_if_j)_t = g^{ij}f_if_j + g^{ij}(f_it)_j + g^{ij}(f_j)_i_t
\]
\[
= 2\text{Rc}_{ij}g^{ik}g^{lj}f_if_l + 2\langle \nabla f, \nabla f_t \rangle = 2\text{Rc}(\nabla f, \nabla f) + 2|\nabla f, \nabla f_t|.
\]
Thus to show \( (24) \), it suffices to show
\[
2t\text{Rc}(\nabla f, \nabla f) + |\nabla f|^2 \geq 0.
\]
For this, we use our curvature bound:
\[
|\text{Rm}| \leq \frac{\epsilon}{1+t} \implies 2t\text{Rc}(\nabla f, \nabla f) + |\nabla f|^2 \geq -2t \left( \frac{\epsilon}{1+t} \right) |\nabla f|^2 + |\nabla f|^2 \geq |\nabla f|^2(-2\epsilon + 1) \geq 0.
\]
since $\epsilon \in (0, 1/2]$. This completes the proof of the lemma. $\square$

From this inequality, we can derive the following Li-Yau type inequality by following the same technique in Theorem 4.2 of [29]:

$$\frac{\nabla u}{u^2} - 2\frac{u_t}{u} = |\nabla f|^2 - 2f_t = \frac{H}{t} \leq \frac{c_1}{t}.$$  \hfill (25)

We now prove the following Harnack inequality for $u$.

**Theorem 4.2.** For any $x, y \in M$ and $0 < t_1 < t_2$,

$$(t_2/t_1)^{-c_1/2} \exp\left(-\frac{d_2(t_1)(x,y)^2}{2(t_2-t_1)}(t + t_2 - t_1)^{2\epsilon}\right).$$

Proof. Since $(M, g(t_1))$ is complete, the Hopf-Rinow theorem says that there exists a geodesic $\gamma : [t_1, t_2] \to M$ with respect to the metric $g(t_1)$ such that

$$|\mathring{\gamma}(t)| = \frac{d_2(t_1)(x,y)}{t_2 - t_1}, \quad \gamma(t_1) = x, \quad \gamma(t_2) = y.$$  \hfill (26)

Then by FTC and chain rule we have

$$\log \frac{u(y,t_2)}{u(x,t_1)} = \int_{t_1}^{t_2} \frac{d}{dt} (\log u(\gamma(t), t)) dt = \int_{t_1}^{t_2} \frac{\partial}{\partial t} \log u + \nabla \log u \cdot \frac{\partial \mathring{\gamma}}{\partial t} dt = \int_{t_1}^{t_2} f_t + \nabla f \cdot \frac{\partial \mathring{\gamma}}{\partial t} dt.$$

Now using our gradient estimate (25), we have

$$\int_{t_1}^{t_2} f_t + \nabla f \cdot \frac{\partial \mathring{\gamma}}{\partial t} dt \geq \int_{t_1}^{t_2} \frac{\nabla f}{2} - \frac{c_1}{2t} + \nabla f \cdot \frac{\partial \mathring{\gamma}}{\partial t} dt = \int_{t_1}^{t_2} -\frac{c_1}{2t} + \frac{1}{2} \left| \nabla f + \frac{\partial \mathring{\gamma}}{\partial t} \right|^2 - \frac{1}{2} \left| \frac{\partial \mathring{\gamma}}{\partial t} \right|^2 dt$$

$$\geq \int_{t_1}^{t_2} -\frac{c_1}{2t} - \frac{1}{2} \left| \frac{\partial \mathring{\gamma}}{\partial t} \right|^2_{g(t)} dt = -\frac{c_1}{2} \log \left(\frac{t_2}{t_1}\right) - \frac{1}{2} \int_{t_1}^{t_2} \left| \frac{\partial \mathring{\gamma}}{\partial t} \right|^2_{g(t)} dt.$$

Fix some $s \in [t_1, t_2]$ and for brevity, write $V = V(s) = \frac{\partial}{\partial s}(s)$. We’re interested in how the quantity $|V|_{g(t)}$ compares with $|V|_{g(t_1)}$. Firstly, by the evolution of $| \cdot |^2_{g(t)}$ along the flow, we have

$$\frac{d}{dt} |V|_{g(t)}^2 = \frac{\partial}{\partial t} (g(t)_{ij} V^i V^j) = -2Rc_{ij} V^i V^j = -2Rc(V, V).$$

Using this and the FTC and the curvature bound $|Rm| \leq \frac{c}{t^2}$, we have

$$\log \frac{|V|_{g(t)}^2}{|V|_{g(t_1)}^2} = \left| \int_{t_1}^{t} \frac{d}{dt} \left( \log \frac{|V|_{g(t)}^2}{|V|_{g(t_1)}^2} \right) dt \right| = \left| \int_{t_1}^{t} -2Rc(V, V) \left| V|_{g(t)}^2 \right| dt \right|$$

$$\leq \int_{t_1}^{t} 2 \left| Rc(V, V) \right| \frac{d}{dt} |V|_{g(t)}^2 dt \leq 2 \int_{t_1}^{t} \frac{2\epsilon}{1 + \tau} d\tau = 2\epsilon \log \frac{1 + t}{1 + t_1} \leq 2\epsilon \log \frac{1 + t_2}{1 + t_1}.$$

But since $t_1 > 0$ we have

$$\frac{1 + t_1}{1 + t_1} + \frac{t_2 - t_1}{1 + t_1} + \frac{t_2 - t_1}{1 + t_1} < 1 + t_2 - t_1$$

so that

$$\log \frac{|V|_{g(t)}^2}{|V|_{g(t_1)}^2} \leq 2\epsilon \log (1 + t_2 - t_1) \implies |V|_{g(t)}^2 \leq (1 + t_2 - t_1)^{2\epsilon} |V|_{g(t_1)}^2.$$  \hfill (26)

But recall from (26) that $\gamma$ is a geodesic in $(M, g(t_1))$ satisfying

$$|V| = |\mathring{\gamma}(s)| = \frac{d_2(t_1)(x,y)^2}{(t_2 - t_1)^2}.$$
So we have
\[ |\tilde{\gamma}(s)|^2_{\tilde{g}(t)} \leq (1 + t_2 - t_1)^2 |V|_{\tilde{g}(t)}^2 = (1 + t_2 - t_1)^2 \frac{d_{\tilde{g}(t)}(x,y)^2}{t_2 - t_1} \]
for all \( t \in [t_1, t_2] \). In particular, this holds at \( t = s \). Then since \( s \) was also arbitrary, we can integrate over \([t_1, t_2]\) to see that
\[ \int_{t_1}^{t_2} \left| \frac{\partial \tilde{\gamma}}{\partial t} \right|^2_{\tilde{g}(t)} \, dt \leq (1 + t_2 - t_1)^2 \frac{d_{\tilde{g}(t)}(x,y)^2}{(t_2 - t_1)^2} (t_2 - t_1) = (1 + t_2 - t_1)^2 \frac{d_{\tilde{g}(t)}(x,y)^2}{t_2 - t_1}. \]

So plugging this back into our earlier work, we have
\[ \log \frac{u(y, t_2)}{u(x, t_1)} \geq -\frac{c_1}{2} \int_{t_1}^{t_2} \left| \frac{\partial \tilde{\gamma}}{\partial t} \right|^2_{\tilde{g}(t)} \, dt \geq -\frac{c_1}{2} \log \left( \frac{t_2}{t_1} \right) - \frac{1}{2} (1 + t_2 - t_1) \frac{d_{\tilde{g}(t)}(x,y)^2}{t_2 - t_1}. \]
Exponentiating both sides gives the result.

\[ \square \]

Remark 4.3 We note that the proof of the above estimates does not depend on the order of decaying for the initial condition \( u_0 \).

**Theorem 4.4.** There exists \( C, \delta > 0 \) such that \( u(x, t) \leq \frac{C}{(1 + t)^{1+\delta}} \).

**Proof.** First recall that each \( u(\cdot, t) \) decays at the same rate as \( u(\cdot, 0) \). That is to say that
\[ (27) \quad c_1(T)r^{-2-\sigma} \leq u(x, t) \leq c_2(T)r^{-2-\sigma}. \]
Therefore, if we take \( T = 1 \), we can find some compact \( K \subset M \) such that \( u \leq c_2(1) \) on \((M \setminus K) \times [0,1]\). Since \( K \times [0,1] \) is compact, \( u \) is bounded (say by \( C \geq c_2 \)) on this set as well. Thus for any \( t \in [0,1] \) and any \( \delta \leq 1 \), we have
\[ u(x, t) \leq C \leq \frac{4C}{(1 + t)^{1+\delta}}. \]
Thus the claim is trivially true for \( t \in [0,1] \). So from now on, we’ll focus on the case \( t \geq 1 \).

Fix some constant \( p \in (\frac{n}{2+\sigma}, \frac{n}{2}) \) where \( \sigma \) is the order of the AE manifold \((M, g)\). Since \( p > \frac{n}{2+\sigma} \), we can use \((27)\) to see that
\[ (u(x, t))^p \leq c_2 r^{(-2-\sigma)p} \ll c_2 (t)^{p(-2-\sigma) \frac{n}{2+\sigma}} = c_2 r^{-n}. \]
That is, \( u^p(\cdot, t) \) is integrable for all \( t \). Thus by the DCT, evolution of the volume form \((\partial_t dV = -RdV)\), integrability, \( R > 0 \), and an integration by parts, gives
\[
\frac{d}{dt} \left( \int u^p \, dV \right) = \int \frac{d}{dt} (u^p \, dV) \\
= \int (pu^{p-1}u_t - Ru^p) \, dV \\
\leq \lim_{s \to \infty} \int_{r(x) \leq s} (pu^{p-1}u_t - Ru^p) \, dV \\
\leq \lim_{s \to \infty} \int_{r(x) \leq s} pu^{p-1} \Delta u \, dV \\
\leq \lim_{s \to \infty} \left( \int_{r(x) = s} pu^{p-1} \nabla u \cdot \nabla r \, d\sigma - \int_{r(x) \leq s} (p(p-1)u^{p-2} |\nabla u|^2 \, dV) \right).
\]
Note that in the integration by parts, we have also used the fact that \( p > \frac{n}{2+\sigma} \) which is \( \geq 1 \) by our assumption that \( \sigma \leq n-2 \). But by the decay of \( u \), we have

\[
\text{Vol}(\{ r(x) = r \}) |\nabla u| u^{p-1} \leq Cr^{(n-1)+(-3-\sigma)+(p-1)(-2-\sigma)} \leq Cr^{n-2-p(2+\sigma)} < Cr^{-2}.
\]

Here we have used the fact that \((M, g(t))\) is AE with the same AE coordinates as \((M, g(0))\) so that in particular,

\[
\lim_{r \to \infty} \frac{\text{Vol}(\{ r(x) = r \})}{nu_m r^{n-1}} = 1.
\]

So

\[
\lim_{s \to \infty} \left| \int_{r(x)=s} pu^{p-1} (\nabla u, \nabla r) \, ds \right| \leq \lim_{s \to \infty} Cs^{-2} = 0.
\]

Also recall that \( u \) was taken to be positive. So \( p(p-1)u^{p-2}|\nabla u|^2 \geq 0 \) on all of \( M \) and thus we have

\[
\frac{d}{dt} \left( \int u^p \, dV \right) \leq 0.
\]

Therefore for any \( t \geq 0 \), we have

\[
0 \leq \int u^p(x,t) \, dV \leq \int u^p(x,0) \, dV =: c_2 < \infty.
\]

Now fix \( x \in M \) and \( t \geq 1 \). Using Theorem 4.2 (Harnack inequality) with \( t_2 = 2t \), \( t_1 = t \), and \( y \in B_{g(t)}(x, (1+t)^{\frac{1}{2}} \epsilon) \), we have

\[
u^p(y,2t) \geq \left( \frac{2t}{t} \right)^{-pc_1/2} \exp \left( -p \frac{d_{g(t)}(y,x)^2}{2(2t-t)} (1+2t-t)^{2\epsilon} \right) u^p(x,t)
\]

\[
geq 2^{-pc_1/2} \exp \left( -p \frac{(1+t)^{1-2\epsilon}}{2t} (1+t)^{2\epsilon} \right) u^p(x,t) \quad y \in B_{g(t)}(x, (1+t)^{\frac{1}{2}} \epsilon)
\]

\[
= 2^{-pc_1/2} \exp \left( -p \frac{1+t}{2t} \right) u^p(x,t).
\]

Since \( y \in B_{g(t)}(x, (1+t)^{\frac{1}{2}} \epsilon) \) was arbitrary and \( u \) is positive, we have

\[
c_2 \geq \int_M u^p(y,2t) \, dV_{g(2t)}(y) \geq \int_{B_{g(t)}(x,(1+t)^{\frac{1}{2}} \epsilon)} u^p(y,2t) \, dV_{g(2t)}(y)
\]

\[
\geq c_2 2^{-pc_1/2} \exp \left( -p \frac{1+t}{2t} \right) u^p(x,t) \text{Vol}_{g(2t)}(B_{g(t)}(x, (1+t)^{\frac{1}{2}} \epsilon))
\]

\[
\geq c_3 u^p(x,t) \text{Vol}_{g(2t)}(B_{g(t)}(x, (1+t)^{\frac{1}{2}} \epsilon))
\]

where \( c_3 = c_2^{-pc_1/2} c_2^{-p} \leq c_2^{-pc_1/2} \exp \left( -p \frac{1+t}{2t} \right) \) which holds for any \( t \geq 1 \).

The evolution equation for the volume of a compact set \( K \subset M \) is

\[
\frac{d}{dt} \text{Vol}_{g(t)}(K) = \frac{d}{dt} \left( \int_K dV \right) = \int_K \partial_t dV = \int_K RdV \geq \frac{-\epsilon}{1+t} \text{Vol}_{g(t)}(K).
\]

So we have by a similar argument we’ve done before,

\[
\log \frac{V_{g(t)}(K)}{V_{g(0)}(K)} = \int_0^t \frac{d}{ds} \left( \log \frac{V_{g(s)}(K)}{V_{g(0)}(K)} \right) ds = \int_0^t \frac{d}{ds} \frac{dV_{g(s)}(K)}{V_{g(s)}(K)} ds \geq \int_0^t -\frac{\epsilon}{1+t} ds = -\epsilon \log(1+t).
\]

Exponentiating both sides and rearranging gives

\[
V_{g(t)}(K) \geq (1+t)^{-\epsilon} V_{g(0)}(K).
\]

\[
\text{Vol}(\{ r(x) = r \}) |\nabla u| u^{p-1} \leq Cr^{(n-1)+(-3-\sigma)+(p-1)(-2-\sigma)} \leq Cr^{n-2-p(2+\sigma)} < Cr^{-2}.
\]
for all $t \geq 1$. Therefore, we have
\[c_2 \geq c_3 u^p(x, t) \text{Vol}_{g(t)}(B_{g(t)}(x, (1 + t)^{1/2 - \epsilon})) \geq (1 + 2t)^{-\epsilon} c_3 u^p(x, t) \text{Vol}_{g(0)}(B_{g(0)}(x, (1 + t)^{1/2 - \epsilon}))\]
We would also like to replace the ball wrt the metric $g(t)$ with the one wrt the metric $g(0)$. For that, we recall from the proof of Theorem 4.2 that
\[|V|_{g(t)} \leq (1 + t)^\epsilon |V|_{g(0)}\]
for any $V \in T_x M$. Then for any $x, y \in M$, choose $\gamma : [0, 1] \to M$ to be a geodesic wrt the metric $g(0)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then
\[d_{g(t)}(x, y) \leq \int_0^1 |\gamma'(s)|_{g(t)} ds \leq \int_0^1 (1 + t)^\epsilon |\gamma'(s)|_{g(0)} ds = (1 + t)^\epsilon d_{g(0)}(x, y).\]
Therefore
\[B_{g(t)}(x, r) = \{ y \in M : d_{g(t)}(x, y) < r \} \subset \{ y \in M : (1 + t)^\epsilon d_{g(0)}(x, y) < r \} = B_{g(0)}(x, r(1 + t)^{-\epsilon}).\]
So we have
\[c_2 \geq (1 + 2t)^{-\epsilon} c_3 u^p(x, t) \text{Vol}_{g(t)}(B_{g(t)}(x, (1 + t)^{1/2 - \epsilon})) \geq (1 + 2t)^{-\epsilon} c_3 u^p(x, t) \text{Vol}_{g(0)}(B_{g(0)}(x, (1 + t)^{1/2 - 2\epsilon})).\]
Now we can use the AE condition of $g(0)$ to see that
\[c_2 \geq (1 + 2t)^{-\epsilon} c_3 u^p(x, t) \text{Vol}_{g(0)}(B_{g(0)}(x, (1 + t)^{1/2 - 2\epsilon})) \geq c_4 (1 + 2t)^{-\epsilon} (1 + t^{1/2 - 2\epsilon}) u^p(x, t).\]
Finally notice that
\[(1 + 2t)^{-\epsilon} \geq 2^{-\epsilon} \geq 2^{-1} \implies (1 + 2t)^{-\epsilon} \geq \frac{1}{2}(1 + t)^{-\epsilon}.\]
So we have
\[c_2 \geq c_4 (1 + 2t)^{-\epsilon} (1 + t^{(1/2 - 2\epsilon)n}) u^p(x, t) \geq \frac{1}{2} c_4 (1 + t)^{-\epsilon + (1/2 - 2\epsilon)n} u^p(x, t).\]
Taking $p'$th roots and rearranging this gives
\[u(x, t) \leq \left(\frac{2c_2}{c_4}\right)^{1/p} (1 + t)^{n \epsilon - (1/2 - 2\epsilon)n} =: C(1 + t)^{\frac{n \epsilon - (1/2 - 2\epsilon)n}{p}}.\]
Recall that $p < n/2$ so that $n/2 - p > 0$. So if we choose $\epsilon > 0$ satisfying (20), then we have
\[\epsilon < \frac{n}{2p + 1} \implies 8\epsilon + \frac{\epsilon - (1/2 - 2\epsilon)n}{p} < -1 \implies \delta := -1 - \frac{\epsilon - (1/2 - 2\epsilon)n}{p} > 8\epsilon > 0.\]
This proves the theorem. \hfill \Box

**Remark.** The reasoning for having the $8\epsilon$ term will make itself apparent in the proof of Theorem 4.5. It is also worth emphasizing that our constants $c_2, c_3, c_4$ depended on $p$ but never on $\epsilon$. Moreover, note that by taking $p$ arbitrarily close to $\frac{n}{2 + \sigma}$, then
\[\delta = -1 - \frac{\epsilon - (1/2 - 2\epsilon)n}{p} = -\frac{(2n + 1)\epsilon}{p} - 1 + \frac{n}{2p}\]
can be any number smaller than
\[-\frac{(2n + 1)\epsilon}{p} - 1 + \frac{n}{2 + \sigma}\]
which (by taking $\epsilon$ as small as we like) can be made arbitrarily close to (but strictly smaller than) $\sigma/2$. To conclude, Theorem 4.4 is true for any $0 < \delta < \frac{\sigma}{2}.$
With Theorem 4.4, we will prove the following estimate for the curvature tensor:

**Theorem 4.5.** There exists $C_0, \delta_0 > 0$ such that $|Rm| \leq \frac{C_0}{(1+t)^{1+\delta_0}}$.

Before proving, we need a lemma.

The original paper (even the published version) has the version of Lemma 4.6 (which is false by taking $u \equiv 1$ which is clearly a positive solution to $\partial_t = \Delta u$). The author is aware of this error.

**Lemma 4.6 (incorrect).** Let $T$ be a time dependent tensor on $M$ (such as $Rm$) and $u$ is a positive solution of $\partial_t u = \Delta u$. Then

$$(\partial_t - \Delta) |T|^2 - \frac{2}{u} \nabla u \cdot \nabla |T|^2 - \frac{2}{u^2} |\nabla T - \nabla u T|^2 + \frac{(\partial_t - \Delta) |T|^2}{u^2}.$$ 

If one computes what the LHS in the above actually is, you’d get the following formula.

**Lemma 4.6 (correct version, but not useful).** Let $T$ be a time dependent tensor on $M$ (such as $Rm$) and $u$ is a positive solution of $\partial_t u = \Delta u$. Then

$$(\partial_t - \Delta) |T|^2 - \frac{4}{u} \nabla u \cdot \nabla |T|^2 - \frac{2}{u^2} |\nabla T|^2 + \frac{2 \Delta u |T|^2}{u^2} + \frac{(\partial_t - \Delta) |T|^2}{u^2}.$$ 

The problem with this formula is that we have a positive quantity as a reaction term (we want either no reaction term or else something non-negative to apply Theorem 2.1). The formula we actually would like to use is the following (which is ironically much simpler).

**Lemma 4.6 (correct version).** Let $T$ be a time dependent tensor on $M$ (such as $Rm$) and $u$ is a positive solution of $\partial_t u = \Delta u$. Then

$$(\partial_t - \Delta) |T| - \frac{2}{u} \nabla u \cdot \nabla |T| - \frac{2 \Delta u |T|^2}{u^2} + \frac{(\partial_t - \Delta) |T|}{u}.$$ 

**Proof.** By direct computation, we have

$$(\partial_t - \Delta) |T| = \frac{u \Delta |T| - u} {u^2} - \nabla u \nabla |T| - |T| \nabla u = \frac{\partial_t |T| - |T| \Delta u}{u^2} - \nabla u \nabla |T| + \nabla |T| \nabla u = \frac{\partial_t |T| - |T| \Delta u}{u^2} - \frac{u \Delta |T|^2 - \nabla |T| \cdot \nabla u}{u^2} + \frac{u^2 \nabla |T| \cdot \nabla u + u^2 |T| \Delta u - 2u |T| \nabla u}{u^4} = \frac{(\partial_t - \Delta) |T|}{u} + \frac{2}{u} \nabla u \cdot \left( \frac{u \nabla |T| - |T| \nabla u}{u^2} \right) = \frac{2}{u} \nabla u \cdot \nabla |T| - \frac{2 \Delta u |T|^2}{u^2} + \frac{(\partial_t - \Delta) |T|}{u^2}.$$ 

Now we continue with the proof of Theorem 4.5.
Proof of Theorem 4.5. Let \( W = \frac{|Rm|}{u} \), then from the previous lemma and the well known fact that \( (\partial_t - \Delta)|Rm| \leq 8|Rm|^2 \), we have

\[
\partial_t W = \Delta W + \frac{2}{u} \nabla u \cdot \nabla W + \frac{(\partial_t - \Delta)|Rm|}{u} \leq \Delta W + \frac{2}{u} \nabla u \cdot \nabla W + \frac{8|Rm|^2}{u}.
\]

\[
= \Delta W + \frac{2}{u} \nabla u \cdot \nabla W + 8|Rm|W \leq \Delta W + \frac{2}{u} \nabla u \cdot \nabla W + \frac{\epsilon}{1+t} W.
\]

Note that since the \( u \) and \( \nabla u \) decay at the same rates as \( u(\cdot, 0) \) and \( \nabla u(\cdot, 0) \), we have

\[
2\frac{\nabla u}{u} \leq C(T) r^{2+\sigma} r^{-3-\sigma} = C(T) r^{-1}
\]

which is bounded (here we’re fixing some \( T > 0 \) and looking at \( M \times [0, T] \)). Now the solution to the ODE

\[
\frac{dU}{dt} = \frac{8\epsilon}{1+t} U, \quad U(0) = c
\]

is \( U(t) = c(1 + t)^{8\epsilon} \) by separation of variables. If we thus take \( c = \sup_{x \in M} \frac{|Rm||x, 0\rangle}{u(x, 0)} < \infty \), we have

\[
W = \frac{|Rm|}{u} \leq c(1 + t)^{8\epsilon}.
\]

Thus from Theorem 4.4, we have

\[
|Rm| \leq C u(1 + t)^{8\epsilon} \leq \frac{C_0}{(1 + t)^{1+\delta-8\epsilon}}.
\]

But recall that we chose \( \epsilon \) in such a way that \( \delta > 8\epsilon \), so \( \delta_0 := \delta - 8\epsilon > 0 \). This completes the proof. \( \square \)

We have the following version of Shi’s estimate. See also [20].

**Theorem 4.7.** For any \( k = 0, 1, \ldots \), \( |\nabla^k Rm| \leq C_k t^{-1-\delta_0-k/2} \).

**Proof.** From Theorem 4.5, the result is true for \( k = 0 \). Now assume by induction that the result holds for all \( 0 \leq l < k \). For any fixed \( s \geq 1 \), we let \( F : M \times [s, \infty) \to \mathbb{R} \) be defined as

\[
F(x, t) = (t-s)^k |\nabla^k Rm|^2 + a_1 (t-s)^{k-1} |\nabla^{k-1} Rm|^2 + \cdots + a_k |Rm|^2.
\]

We will choose the \( \{a_l\} \) later but for now, they’re just positive constants. From the evolution equation of \( |\nabla^k Rm|^2 \) under Ricci flow (standard formula but you can find it on page 153 of [8]), we have

\[
\partial_t |\nabla^k Rm|^2 = \Delta |\nabla^k Rm|^2 - 2 |\nabla^{k+1} Rm|^2 + \sum_{l=0}^{k} \nabla^l Rm \ast \nabla^{k-l} Rm \ast \nabla^k Rm
\]

\[
\leq \Delta |\nabla^k Rm|^2 - 2 |\nabla^{k+1} Rm|^2 + C \sum_{l=0}^{k} |\nabla^l Rm| |\nabla^{k-l} Rm| |\nabla^k Rm|.
\]
Let’s use the inductive hypothesis in two cases: Case 1 is for $0 < l < k$. In this case, both $l$ and $k - l$ are less than $k$, and so we have

$$(t-s)^k|\nabla^l Rm||\nabla^{k-l} Rm||\nabla^k Rm| \leq C t^{-1-\delta_0/2} t^{-1-\delta_0/2} (t-s)^k |\nabla^k Rm| \quad \text{inductive hypothesis}$$

$$= C t^{-2-2\delta_0} (t-s)^{k/2} |\nabla^k Rm| \quad \text{algebra}$$

$$= C t^{-2-2\delta_0} (t-s)^{k/2} |\nabla^k Rm| \quad t-s \leq t$$

$$\leq C t^{-2-2\delta_0} F^{1/2} \quad \text{defn of } F.$$  

For case 2, assume $l = 0$ or $l = k$. Then either $l$ or $k - l$ is equal to $k$ (and the other is equal to 0) and so

$$(t-s)^k|\nabla^l Rm||\nabla^{k-l} Rm||\nabla^k Rm| = (t-s)^k |\nabla^k Rm| |\nabla^k Rm|^2 \quad \text{simplification}$$

$$\leq C (t-s)^k t^{-1-\delta_0} |\nabla^k Rm|^2 \quad \text{base case/Theorem 4.5}$$

$$\leq C t^{-1-\delta_0} F \quad \text{defn of } F.$$  

Note that in both cases above, $C$ depends only on $k$ and not on $s$. Using these (with $a_0 := 1$) we have

$$\langle \partial_t - \Delta \rangle F = \langle \partial_t - \Delta \rangle \sum_{i=0}^k a_i (t-s)^{k-i} |\nabla^{k-i} Rm|^2$$

$$= \sum_{i=0}^{k-1} a_i (k-l) (t-s)^{k-l-1} |\nabla^{k-l} Rm|^2 + \sum_{i=0}^k a_i (t-s)^{k-l} \partial_t |\nabla^{k-l} Rm|^2$$

$$- \sum_{i=0}^k a_i (t-s)^{k-l} \Delta |\nabla^{k-l} Rm|^2$$

$$\leq \sum_{i=1}^k a_{i-1} (k-l+1) (t-s)^{k-l} |\nabla^{k-l+1} Rm|^2$$

$$- 2 \sum_{i=0}^k a_i (t-s)^{k-l} |\nabla^{k-l+1} Rm|^2 + C t^{-1-\delta_0} F + t^{-2-2\delta_0} F^{1/2}$$

$$= -2 (t-s)^{k} |\nabla^{k+1} Rm|^2 + \sum_{i=1}^k (t-s)^{k-l} |\nabla^{k-l+1} Rm|^2 \left[ a_{i-1} (k-l+1) - 2a_i \right]$$

$$+ C t^{-2-2\delta_0} (t^{1+\delta_0} F + F^{1/2})$$

$$\leq \sum_{i=1}^k (t-s)^{k-l} |\nabla^{k-l+1} Rm|^2 \left[ a_{i-1} (k-l+1) - 2a_i \right] + C t^{-2-2\delta_0} (t^{1+\delta_0} F + F^{1/2}).$$

Note that $C$ has possibly changed line by line, but still never depends on $s$ (which is crucial later). We inductively define the $\{a_i\}$ so that

$$a_1 = \frac{k a_0}{2} = \frac{k}{2}, \quad a_2 = \frac{(k-1) a_1}{2}, \quad \ldots \quad a_{k-1} = \frac{2a_{k-2}}{2}, \quad a_k = \frac{a_{k-1}}{2}.$$  

Actually, $a_k = k!/2^k$ but that’s not important. This gives,

$$\sum_{i=1}^k (t-s)^{k-l} |\nabla^{k-l+1} Rm|^2 \left[ a_{i-1} (k-l+1) - 2a_i \right] \equiv 0.$$
Thus, we have

$$(\partial_t - \Delta)F \leq Ct^{-2-2\delta_0}(t^{1+\delta_0}F + F^{1/2}).$$

Consider the following ODE on $[s, \infty)$

$$\frac{d\phi}{dt} = Ct^{-2-2\delta_0}(t^{1+\delta_0} \phi + \phi^{1/2}), \quad \phi(s) = a_k C_0^2 s^{-2-2\delta_0}$$

where $C_0$ is the bound from theorem 4.5. That is to say, $C_0$ is the constant such that

$$|\text{Rm}| \leq C_0 t^{-1-\delta_0}.$$

Note that

$$\sup_{x \in M} F(x, s) = a_k \sup_{x \in M} |\text{Rm}|^2(x, s) \leq a_k C_0^2 s^{-2-2\delta_0} = \phi(s).$$

Therefore, by Theorem 2.1, $F(x, t) \leq \phi(t)$ for any $t \geq s$. The problem is that we don’t really have a good understanding of how $\phi$ behaves. Let’s figure it out.

Note that $\phi$ is increasing (since $\frac{d\phi}{dt} > 0$), and so for any $t \geq s$,

$$\phi(t) \geq \phi(s) = a_k C_0^2 s^{-2-2\delta_0} \geq a_k C_0^2 t^{-2-2\delta_0} \implies \frac{\phi(t)^{1/2} t^{1+\delta_0}}{\sqrt{a_k C_0}} \geq 1.$$

We therefore have

$$\frac{d\phi}{dt} = Ct^{-2-2\delta_0}(t^{1+\delta_0} \phi + \phi^{1/2}) \leq Ct^{-2-2\delta_0}(t^{1+\delta_0} \phi + \phi^{1/2} \frac{\phi(t)^{1/2} t^{1+\delta_0}}{\sqrt{a_k C_0}})$$

$$= C \left(1 + \frac{1}{\sqrt{a_k C_0}}\right) t^{-1-\delta_0} \phi =: \tilde{C} t^{-1-\delta_0} \phi.$$

The ODE

$$\frac{d\psi}{dt} = \tilde{C} t^{-1-\delta_0} \psi, \quad \psi(s) = a_k C_0^2 s^{-2-2\delta_0}$$

has solution

$$\psi(t) = a_k C_0^2 s^{-2-2\delta_0} e^{\tilde{C} t^{-1-\delta_0}} e^{-C t^{-\delta_0}}.$$

And since $\psi(s) = \phi(s)$ and $\psi'(t) \leq \psi(t)$ for all $t$ and all $x \in M$, we have

$$F(x, t) \leq \phi(t) \leq \psi(t) \leq \psi(\infty) = a_k C_0^2 e^\tilde{C} s^{-2-2\delta_0} \leq a_k C_0^2 e^\tilde{C} s^{-2-2\delta_0}.$$

In particular, we can take $t = 2s$ to see that

$$s^k |\nabla^k \text{Rm}|^2(2s) = (2s-s)^k |\nabla^k \text{Rm}|^2(2s) \leq F(x, 2s) \leq a_k C_0^2 e^\tilde{C} s^{-2-2\delta_0}$$

which is equivalent to

$$|\nabla^k \text{Rm}|(2s) \leq \left(\sqrt{a_k C_0 e^{\tilde{C}/2} 2^{1+\delta_0+k/2}}\right) (2s)^{-1-\delta_0-k/2} =: C_k(2s)^{-1-\delta_0-k/2}.$$

Note that $C_k$ is independent of $s$ since $\tilde{C}$ doesn’t depend on $s$, $C_0$ is the constant from theorem 4.5 (which doesn’t depend on $s$), and $a_k = k!/2^k$. Then since $s \geq 1$ was arbitrary, this completes the proof.

We claim that from theorem 4.7 it follows that there exists a metric $g_\infty$ such that $g(t) \to g_\infty$ smoothly as $t \to \infty$. The $C^0$ convergence is clear by the flow:

$$|g_{\infty} - g(t)| = \left| \int_t^\infty \partial_s g(s) \, ds \right| \leq \int_t^\infty 2|R_c|(s) \, ds \leq \int_t^\infty 2Cs^{-1-\delta_0} \, ds = \frac{2C}{\delta_0} t^{-\delta_0} \to 0.$$

From the evolution equation of the Christoffel symbols

$$\partial_t \Gamma_{ij}^k = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij})$$
and theorem 4.7 and the $C^0$ convergence of $g(t)$, we see that
\[
|\Gamma^k_{ij}(\infty) - \Gamma^k_{ij}(t)| = \left| \int_t^\infty \partial_s \Gamma^k_{ij}(s) \, ds \right| \leq C \int_t^\infty s^{-3/2 - \delta_0} \, ds = C t^{-\delta_0 - 1/2}.
\]
Thus from the relation
\[(28) \quad \partial_i R_{jk} = \nabla_i R_{jk} + \Gamma^l_{ij} R_{lk} + \Gamma^l_{ik} R_{lj} \]
we have
\[
|\partial_i g(\infty)_{jk} - \partial_i g(t)_{jk}| = \left| \int_t^\infty \partial_s \partial_j g(s) \, ds \right| \leq C \int_t^\infty |\nabla_i R_{jk} + \Gamma^l_{ij} R_{lk} + \Gamma^l_{ik} R_{lj}| \, ds \leq C \int_t^\infty t^{-1 - \delta_0} \, ds \leq Ct^{-\delta_0}.
\]
Repeated use of the flow along with Theorem 4.7 and (28) again to see that \(\{\partial^\alpha (g(t) - g(\infty))\}\) decay like \(t^{-\delta_0}\) for any multi-index \(\alpha\).

Thus \(g(t) \to g(\infty)\) smoothly. Moreover, we have by Lemma 3.2, monotonicity of \(\mu\) and Theorem 3.4, we have (for any \(\tau > 0\))
\[
\mu(g(\infty), \tau) \geq \limsup_{t \to \infty} \mu(g(t), \tau) \geq \limsup_{t \to \infty} \mu(g(0), \tau + t) = 0.
\]
Therefore from theorem 3.3, \((M, g(\infty)) \cong (\mathbb{R}^n, g_E)\). In particular, \(M^n\) is diffeomorphic to \(\mathbb{R}^n\).

CHAPTER 5: PROOF OF THEOREM 1.2

In this section, we’ll prove Theorem 1.2 which is restated below:

**Theorem 1.2.** If there exists a solution \(g(t)\) \((0 \leq t < \infty)\) of the Ricci flow with \(g(0) = g\), then \(m(g) \geq 0\) with equality if and only if \((M^n, g) \cong (\mathbb{R}^n, g_E)\).

**Definition.** Let \((M, g)\) be an AE manifold with the AE end \(E\), the weighted space \(C^k_\beta(E)\) consists of all \(C^k\) functions \(u\) for which the norm
\[
\|u\|_{C^k_\beta} = \sum_{i=0}^k \sup_M r^{-\beta + i} |\nabla^i u|
\]
is finite.

Then we have the following convergence result in the weighted space.

**Theorem 5.1.** For any \(\sigma' \in (\frac{\sigma - 2}{2}, \sigma)\), we have \(g_{ij}(t)\) converges to \(g_{ij}(\infty)\) in \(C^\infty_\sigma\), (that is to say it converges in \(C^k_{\sigma', \sigma}\) for any \(k \in \mathbb{N}\)).

**Remark.** Theorem 5.1 implies that \((E, g_{ij}(\infty))\) is an AE coordinate system of order \(\sigma'\). This is because \((E, g(t)_{ij})\) is an AE coordinate system of order \(\sigma'\) (since \(\sigma' < \sigma\)) and thus
\[
g_{ij}(\infty) = g_{ij}(t) + g_{ij}(\infty) - g_{ij} = \delta_{ij} + O(r^{-\sigma'}) + O(r^{-\sigma'})
\]
and

$$|\partial^\alpha g_{ij}(\infty)| \leq |\partial^\alpha g_{ij}(t) + \partial^\alpha (g_{ij}(\infty) - g_{ij}(t))| = O(r^{-k-\sigma'})$$

for any $|\alpha| = k$.

Before we prove Theorem 5.1, we need a lemma.

**Lemma 5.2.** For each $k = 0, 1, \ldots$, There exists $C_k, \eta_k > 0$ such that

$$|\nabla Rm| \leq C_k t^{-1-\eta_k} r^{-k-\sigma'}.$$

**Proof.** Recall that since $\delta_0 = \delta - 8\epsilon$ and $\delta$ could be chosen to be any number smaller than $\sigma/2$ and $\epsilon$ all of our work so far only requires an upper bound on $\epsilon$, we can take $\delta_0$ to be any number smaller than $\sigma/2$. Therefore, we can take any $\sigma_0, \sigma_1$ satisfying

$$\sigma' < \sigma_1 < \sigma_0 < \sigma,$$

and $\delta_0 = \sigma_0/2$. Consider the domain $D_k = \{(x, t) \in M \times [0, \infty) \text{ s.t. } r(x) \geq t^{a_k}\}$ in the spacetime where $a_k > 1/2$ is a constant to be determined later.

For $(x, t) \notin D_k$, it is true that

$$t^{-1-\eta_k-a_k(k+\sigma')} < t^{-1-\eta_k} r^{-k-\sigma'}.$$

Thus, by taking $a_k$ close enough to $1/2$, we have

$$\sigma_0/2 + k/2 - a_k(k + \sigma') > 0$$

since $\sigma' < \sigma_0$. Thus, taking $\eta_k$ to be less than or equal to this above positive quantity, we have

$$t^{-1-\sigma_0/2-k/2} \leq t^{-1-\eta_k-a_k(k+\sigma')}.$$

So by Theorem 4.7, we have

$$|\nabla^k Rm| \leq C_k t^{-1-\sigma_0/2-k/2} \leq C_k t^{-1-\eta_k} r^{-k-\sigma'}.$$

To cover the case of $(x, t) \in D_k$, we will prove the following claim.

**Claim:** $|\nabla^k Rm| \leq C r^{-2-\sigma_1-k}$ on $D_k$.

**Proof of the claim:** Let $h_k = r^{4+2\alpha_1+2k}$ and $w_k = h_k|\nabla^k Rm|^2$ just as we have done before. Just as in the proof of theorem 2.2, we have

$$(\partial_t - \Delta) w_k \leq B_k w_k - 2\nabla \log h_k \nabla w_k + C \sum_{l=0}^{k} h_k |\nabla^l Rm| |\nabla^{k-l} Rm| |\nabla^l Rm|$$

where $B_k = \frac{2|\nabla h_k|^2 - h_k \Delta h_k}{h_k^2}$ is uniformly bounded by $C r^{-2} \leq t^{-2a_k}$ since $(x, t) \in D_k$.

For $k = 0$, we have

$$(\partial_t - \Delta) w_0 \leq B_0 w_0 - 2\nabla \log h_0 \nabla w_0 + C h_0 |Rm|^3$$

$$\leq C r^{-2} w_0 - 2\nabla \log h_0 \nabla w_0 + |Rm| w_0$$

$$\leq C t^{-2a_0} w_0 - 2\nabla \log h_0 \nabla w_0 + C t^{-1-\sigma_0/2} w_0$$

$$\leq -2\nabla \log h_0 \nabla w_0 + C t^{-1-\delta_0} w_0$$
where $\delta_0' = \min\{2a_0 - 1, \sigma_0/2\} > 0$. Moreover, on $\partial D_0$, we have
\[
|\text{Rm}| \leq Ct^{-1-\delta_0} = Cr^{-(1+\sigma_0/2)/a_0} \leq Cr^{-2-\sigma},
\]
for $a_0$ sufficiently close to 1/2 since $\sigma_1 < \sigma_0$. Therefore $w_0$ is bounded on $\partial D_0$ and also for $t = 0$. Now we apply theorem 2.1 on $D_k$ to conclude that the claim holds for $k = 0$. Note that even though in theorem 2.1 there is no boundary in spacetime for $t > 0$, if we go through the proof (see [6] Theorem 12.14), the contradiction is derive at an interior point as long as the conclusion holds also on the boundary.

Now assume the claim holds for all $0 \leq l < k$. Then by the inductive hypothesis, we have
\[
h_k|\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| = h_k|\nabla^k \text{Rm}|^2 |\text{Rm}| \leq Ct^{-1-\sigma/2} w_k
\]
for $l = 0$ or $l = k$ and
\[
h_k|\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| \leq Cr^{4+2\sigma+2k} t^{-2-\sigma_1-l} t^{-2-\sigma_1-(k-l)} |\nabla^k \text{Rm}|
\]
\[
= Cr^k |\nabla^k \text{Rm}| = Cr^{-\sigma_1-2} |\nabla^k \text{Rm}|
\]
\[
= Cr^{-\sigma_1-2} w_k^{1/2} \leq Ct^{-a_k \sigma_1-2a_k} w_k^{1/2}
\]
for $0 < l < k$. Therefore, we have
\[
(\partial_t - \Delta) w_k \leq B_k w_k - 2\nabla \log h_k \nabla w_k + C \sum_{l=0}^{k} h_k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}|
\]
\[
\leq -2\nabla \log h_k \nabla w_k + Ct^{-1-\sigma_0/2} w_k + Ct^{-a_k \sigma_1-2a_k} w_k^{1/2}
\]
\[
\leq -2\nabla \log h_k \nabla w_k + Ct^{-2a_k} w_k + Ct^{-1-\sigma_0/2} w_k + Ct^{-a_k \sigma_1-2a_k} w_k^{1/2}
\]
\[
\leq -2\nabla \log h_k \nabla w_k + Ct^{-1-\delta'(w_k + w_k^{1/2})}
\]
for $\delta' = \min\{2a_k - 1, \sigma_0/2, a_k (\sigma_1 + 2) - 1\} > 0$. Now we’re in a position to apply the modified maximum principle (Theorem 2.1 but with boundary condition) but we need to first check that the conclusion holds on $\partial D_k$. Indeed, by theorem 4.7 for $(x, t) \in \partial D_k$, we have
\[
|\nabla^k \text{Rm}| \leq C k t^{-1-\sigma_0/2-k/2} = C k t^{-(1-\sigma_0/2-k/2)} a_k \leq C k t^{-2-k-\sigma}
\]
for $a_k$ sufficiently close to 1/2 since $\sigma_1 < \sigma_0$. That is to say that $w_k$ is bounded on $\partial D_k t$ and $t = 0$. By Theorem 2.1, it is uniformly bounded in $t$. In other words
\[
w_k \leq C \implies |\nabla^k \text{Rm}| \leq C t^{-2-k-\sigma} \text{ on } D_k.
\]
This concludes the proof of the claim.

Now since $\sigma_1 > \sigma'$, we can choose $a_k$ close enough to 1/2 so that $a_k (2 + \sigma_1 - \sigma') > 1$. Taking $\eta_k > 0$ satisfying
\[
a_k (2 + \sigma_1 - \sigma') - 1 \geq \eta_k,
\]
we get
\[
|\nabla^k \text{Rm}| \leq C k t^{-2-k-\sigma_1} = C k t^{-2-\sigma_1+a'+k-\sigma'}
\]
\[
\leq C k t^{-a_k (2+\sigma_1-\sigma')} t^{-k-\sigma'} \leq C k t^{-1-\eta_k-\sigma} t^{-k-\sigma'}.
\]
This completes the proof of the Lemma. \qed
Remark. On the surface, it looks like the claim in the above lemma follows immediately from the previous work we’ve done in Theorem 2.2. But the problem is that in that theorem, the constant on the RHS depended on an upper bound $T$ on the time interval. In this claim, we sacrifice an arbitrarily small amount on the exponent (working with $\sigma_1 < \sigma$) in exchange for having a bound with a constant independent of $T$.

Proof of Theorem 5.1. With the same argument in Theorem 2.2, we conclude that $g_{ij}(t)$ converges to $g_{ij}(\infty)$ in $C^\infty$ because the term $t^{-1-\eta}$ guarantees that $|\nabla^k Rm|$ is integrable with respect to time at infinity. In other words, $g_{ij}(\infty)$ is an AE coordinate system on $E$ with a smaller order $\sigma' < \sigma$.

Now we are in a position to prove Theorem 1.2. We let $\eta$ be a smooth function such that $\eta = 0$ outside of the AE end $E$ and $\eta = 1$ outside of some compact set (ie whenever $r$ is large enough). Let $\chi(t) = (\partial_i g_{ij}(t) - \partial_j g_{ii}(t))\partial_j$ be a vector field on the AE end $E$. By the definition of the ADM mass and the fact that $(E, g_{ij}(t))$ is an AE end (so the mass is well-defined), we have

$$m(g(t)) = \lim_{r \to \infty} \int_{S^r} \chi(t) \cdot dV_{g_E} = \lim_{r \to \infty} \int_{S^r} \eta \chi(t) \cdot dV_{g_E}.$$  

Now we can use Stokes theorem ($\int_{\partial \Omega} \omega = \int_{\Omega} d\omega$), the definition of divergence ($d(\chi \cdot dV) = \text{div}(\chi) dV$) and product rule for divergence operator ($\text{div}((X \cdot dV) = f \text{div}(X) + \langle X, \nabla f \rangle$) to see that

$$\int_{S^r} \eta \chi(t) \cdot dV_{g_E} = \int_{|x| \leq r} d(\eta \chi(t) \cdot dV_{g_E})$$

$$= \int_{|x| \leq r} \text{div}(\eta \chi(t)) dV_{g_E}$$

$$= \int_{|x| \leq r} \eta \text{div}(\chi(t)) + \langle \chi(t), \nabla \eta \rangle V_{g_E}.$$  

Let’s try to get a better handle on this integrand. First recall (one representation) of the coordinate expression for the scalar curvature:

$$(29) \quad R = g^{ik}(\partial_i \Gamma^j_{jk} - \partial_k \Gamma^i_{ij} + \Gamma^i_{il} \Gamma^l_{jk} - \Gamma^i_{kl} \Gamma^l_{ij})$$

and the formula for the Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2} g^{mk}(\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij}).$$  

Consider the rate of decay of each term in the product expansion of (29): any term that is the product of two Christoffel symbols is $O(r^{-2-2\sigma'})$ since each of these terms is of the form $g^{-1} \partial_i g \partial_j g$. Also, any term in which the derivative lands on $g^{-1}$ is also $O(r^{-2-2\sigma'})$ since this will be of the form $g^{-1} \partial g^{-1} \partial g$. The only terms left are those involving the second derivatives of $g$ which are of the form $g^{ab} \partial^c a \partial_c g^{bk}$. Now even for these terms, we can assume that $g^{ab} = \delta^{ab}$ since the other contribution is from the $O(r^{-\sigma})$ portion which, when multiplied by the second
derivative of $g$, will be also $O(r^{-2-2\sigma'})$. All this is to say that

$$R = g^{jk}(\partial_t \Gamma^i_{jk} - \partial_k \Gamma^i_{tj} + \Gamma^i_{dtj} - \Gamma^i_{kjt})$$

$$= \frac{1}{2} \delta^{i} (\partial_t \partial_j g_{kl} + \partial_l \partial_k g_{j} - \partial_l \partial_k g_{jl} - \partial_k \partial_j g_{il} + \partial_k \partial_l g_{ij}) + E(g)$$

$$= \frac{1}{2} (\partial_t \partial_i g_{ij} + \partial_j \partial_i g_{ij} - \partial_j \partial_i g_{ji} - \partial_j \partial_i g_{ii} + \partial_j \partial_i g_{ij}) + E(g)$$

$$= \frac{1}{2} (\partial_j \partial_i g_{ij} + \partial_i \partial_j g_{ij} - \partial_j \partial_i g_{ii} - \partial_j \partial_i g_{ii} + E(g)) = \partial_j (\partial_i g_{ij} - \partial_i g_{ii}) + (E(g)$$

where $E(g)$ is some polynomial in $g$, $\partial g$ and $\partial^2 g$ such that $E(g) = O(r^{-2-2\sigma'})$. Moreover, we have

$$|E(g(t)) - E(g(\infty))| \leq C\|g(t) - g(\infty)\|_{C^2_{\sigma', \rho(\sigma(x)=r)}} r^{-2-2\sigma'} \leq C\|g(t) - g(\infty)\|_{C_{\sigma, \rho}(E)} r^{-2-2\sigma'}.$$

By taking the difference of equations of $R(t)$ and $R(\infty) = 0$ (since $(M^n, g(\infty)) \cong (\mathbb{R}^n, g_E)$), we have

$$R(t) = R(t) - R(\infty) = \partial_t (\partial_t g_{ij}(t) - \partial_t g_{ii}(t)) = \partial_t (\partial_t g_{ij}(\infty) - \partial_t g_{ii}(\infty)) + E(g(t)) - E(g(\infty))$$

and therefore

$$|\text{div} \chi(t) - \text{div} \chi(\infty) - R(t)| = |E(g(t)) - E(g(\infty))| \leq C\|g(t) - g(\infty)\|_{C^2_{\sigma', \rho(\sigma(x)=r)}} r^{-2-2\sigma'}.$$

We'll use this to show $m(g(0)) \geq 0$. First recall that the mass is unchanged along the flow (Theorem 2.2 since $\sigma' > \frac{n-2}{2}$ and $R$ is integrable). Also since $(M, g(\infty)) \cong (\mathbb{R}^n, g_E)$ and the mass is independent of the AE coordinate system that is used, we can use the coordinates from this isometry to conclude that

$$m(g(\infty)) = m(g_E) = 0.$$

In particular, we have

$$m(g(0)) = \lim_{t \to \infty} m(g(t)) = \lim_{t \to \infty} m(g(t)) - m(g(\infty))$$

$$= \lim_{t \to \infty} \lim_{r \to \infty} \int_{|x| \leq r} \eta (\text{div} \chi(t) - \text{div} \chi(\infty)) + \langle \chi(t) - \chi(\infty), \nabla \eta \rangle V_{g_E}$$

$$\geq \lim_{t \to \infty} \lim_{r \to \infty} \int_{|x| \leq r} \eta R(t) - \eta C\|g(t) - g(\infty)\|_{C^2_{\sigma', \rho}(E)} r^{-2-2\sigma'} + \langle \chi(t) - \chi(\infty), \nabla \eta \rangle V_{g_E}.$$

Note that

$$\sigma > \frac{n-2}{2} \implies 2\sigma' + 2 > n$$

and since $r$ is a positive smooth function which stays away from 0, $\eta^{r-2-2\sigma'}$ is integrable. Thus

$$\int_{|x| \leq r} \eta C\|g(t) - g(\infty)\|_{C^2_{\sigma', \rho}(E)} r^{-2-2\sigma'} dV_{g_E} \leq C\|g(t) - g(\infty)\|_{C^2_{\sigma', \rho}(E)}$$

which goes to 0 as $t \to \infty$ by Theorem 5.1. Finally

$$\lim_{t \to \infty} \lim_{r \to \infty} \int_{|x| \leq r} \langle \chi(t) - \chi(\infty), \nabla \eta \rangle V_{g_E} = \lim_{t \to \infty} \int_{\text{supp}(\nabla \eta)} \langle \chi(t) - \chi(\infty), \nabla \eta \rangle V_{g_E}.$$

And for $x \in \text{supp}(\nabla \eta) \subset E$, we have

$$\sup_{x \in \text{supp}(\nabla \eta)} |\chi(t) - \chi(\infty)| \leq C\|g(t) - g(\infty)\|_{C^1_{\sigma', \rho(\sigma(x)=r)}} \sup_{x \in E} r^{-2-2\sigma'}$$

$$\leq C\|g(t) - g(\infty)\|_{C^1_{\sigma', \rho}(E)} \sup_{x \in E} r^{-2-2\sigma'} \leq C\|g(t) - g(\infty)\|_{C^1_{\sigma', \rho}(E)} \to 0.$$
where we have used the fact that $r \gg 0$ on $E$ and Theorem 5.1. Therefore, we’ve proven that
\[
m(g(0)) \geq \lim_{t \to \infty} \lim_{r \to \infty} \int_{|x| \leq r} \eta R(t) \, dv_{g_E} = \lim_{t \to \infty} \int \eta R(t) \, dv_{g_E}.
\]
Using the other side of the inequality coming from the absolute values in (30) shows that
\[
m(g(0)) \leq \lim_{t \to \infty} \int \eta R(t) \, dv_{g_E} \implies m(g(0)) = \lim_{t \to \infty} \int \eta R(t) \, dv_{g_E}.
\]
Since $g(t) \to g(\infty)$ in $C^\infty_{\sigma'}$, and $\inf r > 0$, we can see that $g(t) \to g(\infty) = g_E$ in $C^k(K)$ for any $k \in \mathbb{N}$ and any compact $K \subset M$. In particular, taking $K = \text{supp}(1 - \eta)$ and $k = 2$, we have that
\[
\lim_{t \to \infty} \int (1 - \eta) R(t) \, dv_{g_E} = 0.
\]
In particular, this gives
\[
\lim_{t \to \infty} \int \Delta R \, dv_{g_E} = \lim_{t \to \infty} \int \eta R(t) \, dv_{g_E} + \lim_{t \to \infty} \int (1 - \eta) R(t) \, dv_{g_E} = m(g(0)) = 0.
\]
Also since $g(t) \to g(\infty)$ in $C^\infty_{\sigma'}$, $dv_{g(t)}$ is uniformly comparable to $dv_{g_E}$ on all of $M$ for all $t$ large enough. That is to say, there is some $C(t)$ which goes to 1 as $t \to \infty$ such that $C(t)^{-1} dv_{g_E} \leq dv_{g(t)} \leq C(t) dv_{g_E}$ on all of $M$. So
\[
m(g(0)) = \lim_{t \to \infty} \int R(t) \, dv_{g_E} = \lim_{t \to \infty} \int R(t) \, dv_{g(t)}.
\]
On the other hand, using the DCT and the evolution of the scalar curvature $(\partial_t R = \Delta R + 2|Rc|^2)$ and the evolution of the volume form $(\partial_t dv = -R \, dv)$, we have
\[
\frac{d}{dt} \left( \int R \, dv \right) = \int \frac{\partial}{\partial t} (R \, dv) = \int \Delta R + 2|Rc|^2 - R^2 \, dv = \int 2|Rc|^2 - R^2 \, dv.
\]
The last equality uses
\[
\int \Delta R \, dv = \lim_{r \to \infty} \int_{r(x) \leq r} \Delta R \, dv = \lim_{r \to \infty} \int_{S_r} \nabla^i R \, dv^j
\]
and the Lemma 11 in [19] which says
\[
\lim_{r \to \infty} \int_{S_r} |\nabla R| \, dA = 0
\]
as we have used earlier in the paper. Now we claim that $n|Rc|^2 \geq R^2$. To see this, work in normal coordinates (so that $g_{ij} = \delta_{ij}$). Then it suffices to show
\[
n \sum_{ij} Rc_{ij}^2 = n \delta^{ik} \delta^{jl} Rc_{ij} Rc_{kl} = ng^{ik} g^{jl} Rc_{ij} Rc_{kl} = n|Rc|^2 \geq R^2 = (g^{ij} Rc_{ij})^2 = (\delta^{ij} Rc_{ij})^2 = \left( \sum_i Rc_{ii} \right)^2.
\]
Letting $A_{ij} = Rc_{ij}$. To show the above inequality, it suffices to show that
\[ n \sum_{ij} A_{ij}^2 \geq n \sum_i A_{ii}^2 \geq \left( \sum_i A_{ii} \right)^2 = (\text{tr} A)^2 \]
but we did this already in (22). Therefore, we have
\[ \frac{d}{dt} \left( \int R \, dV \right) = \int 2|Rc|^2 - R^2 \, dV \geq \int \frac{2}{n} R^2 - R^2 \, dV = -\frac{n-2}{n} \int R^2 \, dV. \]
Now use the curvature bound $R \leq C (1 + t)^{\frac{1}{1+\delta}}$ to get
\[ \frac{d}{dt} \left( \int R \, dV \right) \geq -\frac{n-2}{n} \int R^2 \, dV \geq -\frac{C}{(1 + t)^{1+\delta}} \int R \, dV. \]
Let $f(t) = \int R(t) \, dV_{g(t)}$ and let $h : [0, \infty) \to \mathbb{R}$ be defined as
\[ h(t) = Ae^{\frac{C}{(1+t)^{1-\delta}}}, \quad A = f(0)e^{-\frac{C}{\delta}} \]
so that $h(0) = f(0)$ and
\[ h'(t) = A \frac{C}{\delta} e^{\frac{C}{(1+t)^{1-\delta}}} (-\delta)(1+t)^{-1-\delta} \]
\[ = -CAe^{\frac{C}{(1+t)^{1-\delta}}} (1+t)^{-1-\delta} \]
\[ = -C(1+t)^{-1-\delta} h(t). \]
Then $f(t) \geq h(t)$ for all $t \geq 0$. But this gives
\[ m(g(0)) = \lim_{t \to \infty} \int R \, dV = \lim_{t \to \infty} f(t) \geq \lim_{t \to \infty} h(t) = \lim_{t \to \infty} Ae^{\frac{C}{(1+t)^{-\delta}}} \]
\[ = A = f(0)e^{-\frac{C}{\delta}} = \int R(0) \, dV_{g(0)}e^{-\frac{C}{\delta}}. \]
This final quantity is manifestly positive by our assumption that $R > 0$. This completes the proof.

REFERENCES


NOTES ON ‘RICCI FLOW ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS’


