

A simplified characterization of stable-like heat kernel estimates

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Abstract

We study heat kernel estimates for symmetric pure jump processes on general metric measure spaces. Building on recent progress in the local setting due to S. Eriksson-Bique, we develop a non-local version of the Whitney blending technique and use it to relate stable-like heat kernel estimates to capacity upper bounds. Under two-sided stable-like bounds on the jump kernel, we show that a capacity upper bound across annuli implies a cutoff Sobolev inequality, and we obtain a characterization of stable-like heat kernel estimates in terms of these conditions. As a consequence, we give an affirmative answer to a conjecture of A. Grigor'yan, E. Hu, and J. Hu.

1 Introduction

The behavior of the heat kernel is intimately tied to the geometry of the underlying space. A landmark result in this direction, independently due to Grigor'yan and Saloff-Coste, shows that Gaussian heat kernel estimates and the parabolic Harnack inequality for Brownian motion on a Riemannian manifold are characterized by two analytic-geometric properties: volume doubling and the Poincaré inequality [Gri, Sal]. Over the past three decades, this characterization has been extended to a variety of settings, including diffusions on local Dirichlet spaces [Stu], random walks on graphs and metric spaces [Del, MS23], diffusions on fractal-like spaces [BB, BBK, GHL15], jump processes [CKW21, CKW20a, GHH18, MS19], symmetric Markov processes with both diffusion and jumps [CKW20b]. Similar characterizations of the elliptic Harnack inequality (EHI) for diffusions have also been obtained [BM, BCM]. An important application of these characterizations is the stability of heat kernel estimates and related analytic properties such as elliptic and parabolic Harnack inequalities under perturbations of the space or the underlying Dirichlet form and also to establish such properties in examples of interest.

On fractals and many fractal-like spaces, beyond volume doubling and the Poincaré inequality, the **cutoff Sobolev inequality** and its variants constitute a fundamental analytic tool. Introduced by Barlow and Bass in the setting of random walks on graphs and subsequently extended to other settings, this inequality plays a central role in the characterization and stability of heat kernel estimates and Harnack inequalities, such as:

- (i) Parabolic Harnack inequality and two-sided sub-Gaussian heat kernel estimates for diffusions [BB, Theorem 1.5], [BBK, Theorem 2.16] [GHL15, Theorem 1.2].

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- (ii) Elliptic Harnack inequality for diffusions [BM, Theorem 5.15], [BCM, Theorem 7.9].
- (iii) Sub-Gaussian heat kernel upper bounds for diffusions [AB, Theorem 1.12].
- (iv) Two-sided stable-like heat kernel bounds for pure jump processes [CKW21, Theorem 1.13], [GHH18, Theorem 1.12], [MS19, Theorem 1.5].
- (v) Stable-like heat kernel upper bounds for pure jump processes [CKW21, Theorem 1.15], and parabolic Harnack inequality for pure jump processes [CKW20a, Theorem 1.20].
- (vi) Heat kernel bounds (two-sided and upper) and parabolic Harnack inequality for processes with both diffusions and jumps [CKW20b, Theorems 1.13, 1.14 and 1.18].

Although the cutoff Sobolev inequality and their variants are a powerful tool for establishing the stability results mentioned above, it is often difficult to verify in concrete examples. It is therefore natural to seek simpler sufficient conditions, such as an upper bound on capacity across annuli. This motivation underlies a conjecture of Grigor'yan, Hu, and Lau in setting (i) above [GHL14, Conjecture 4.15], [GHL15, p. 1493, p. 1498], as well as a related conjecture of Grigor'yan, E. Hu, and J. Hu in setting (iv) [GHH18, Conjecture 1.14]. Let us also mention that cutoff Sobolev inequality has proven to be more generally useful beyond stability results such as obtaining singularity of energy measures [KM20, Yan] and also for obtaining heat kernel estimates for reflected diffusions [Mur24] and diffusions on Laakso-type spaces [Mur26].

Recently, the conjecture of Grigor'yan, Hu, and Lau was resolved in setting (i) (and hence also in setting (ii)) by S. Eriksson-Bique in a breakthrough work introducing the technique of *Whitney blending* [Eri]. In fact, the Whitney blending technique in [Eri] is implemented in the more general framework of p -Dirichlet spaces introduced in [EM, §2.2], which allows for non-linear energies but assumes locality of the energy. The purpose of this work is to develop the Whitney blending technique in a non-local setting and to give an affirmative answer to [GHH18, Conjecture 1.14] in setting (iv). More generally, one may optimistically conjecture that the cutoff Sobolev inequality and its variants can be replaced by simpler capacity upper bounds in all of the characterizations discussed above. We leave the resolution of analogous conjectures in the remaining settings, such as (iii), (v), and (vi), as an interesting direction for future research. We refer to [Ant, Mur23, Yan] for recent partial progress on the Grigor'yan–Hu–Lau conjecture prior to its resolution in [Eri].

While Whitney covers have previously been used to obtain heat kernel bounds for diffusion processes, most notably in the analysis of reflected and killed diffusions in [GS], their use in deriving heat kernel bounds for jump processes is new. This constitutes a key methodological novelty of the present work. Moreover, the Whitney blending approach in the nonlocal setting requires substantially new ingredients not present in [Eri], including sharp two sided estimates on the jump kernel and the use of truncated Dirichlet forms.

Our main results are as follows:

- (A) Two-sided stable-like bounds on the jump kernel together with a capacity upper bound imply a cutoff Sobolev inequality (Theorem 1.6).
- (B) A characterization of stable-like heat kernel estimates in terms of the two-sided bounds on the jump kernel and the capacity upper bound above (Theorem 1.7).

The result (B) provides an affirmative answer to [GHH18, Conjecture 1.14].

1.1 Framework and results

Let (X, d) be a complete, locally compact, separable metric space, and let m be a positive Radon measure on M with full support. We always assume that X contains at least two points. Such a triple (X, d, m) is called a *metric measure space*. We set $B(x, r) := \{y \in X \mid d(x, y) < r\}$, $\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}$, and $V(x, r) = m(B(x, r))$ for $x \in X, r \in (0, \infty)$. We assume that the measure m satisfies the following *volume doubling* property (VD): there exists $C_D > 1$ such that

$$V(x, 2r) \leq C_D V(x, r), \quad \text{for any } x \in X, r \in (0, \infty). \quad (\text{VD})$$

We consider a *symmetric Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$. In other words, \mathcal{F} is a dense linear subspace of $L^2(X, m)$, $\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ is symmetric, non-negative definite, bilinear form that is *closed* (\mathcal{F} is a Hilbert space under the inner product $\mathcal{E}_1(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + \langle \cdot, \cdot \rangle_{L^2(X, m)}$) and *Markovian* (for any $f \in \mathcal{F}$, we have $\hat{f} := (0 \vee f) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\hat{f}, \hat{f}) \leq \mathcal{E}(f, f)$). We assume that $(\mathcal{E}, \mathcal{F})$ is *regular*; that is, $\mathcal{F} \cap \mathcal{C}_c(X)$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(\mathcal{C}_c(X), \|\cdot\|_{\sup})$. We assume that $(\mathcal{E}, \mathcal{F})$ is a *pure jump* type Dirichlet form; that is, there exists a symmetric positive Radon measure on $M \times M \setminus \text{diag}$ such that

$$\mathcal{E}(f, f) = \int_{M \times M \setminus \text{diag}} (f(x) - f(y))^2 J(dx, dy), \quad \text{for all } f \in \mathcal{F},$$

where $\text{diag} = \{(x, x) \mid x \in X\}$ denotes the diagonal. The Radon measure J is called the *jump measure*; cf. [FOT, Theorem 3.2.1].

We assume that the measure m satisfies the following *volume doubling* property VD: there exists $C_D > 1$ such that

$$V(x, 2r) \leq C_D V(x, r), \quad \text{for any } x \in X, r \in (0, \infty). \quad (\text{VD})$$

Setting $\alpha = \log_2 C_D$ and iterating VD, we obtain the following well-known consequence

$$V(x, s) \leq C_D^2 \left(\frac{d(x, y) + s}{r} \right)^\alpha V(y, r), \quad \text{for all } x \in X, 0 < r < s < \infty. \quad (1.1)$$

We recall the closely related reverse volume doubling property and a generalized version introduced in [Mal, Definition 1.4].

Definition 1.1. We say that a metric measure space (X, d, m) satisfies the *reverse volume doubling* property RVD if there exists $\beta \in (0, \infty), C \in (1, \infty)$ such that for all $x \in X, 0 < r \leq R < \text{diam}(X, d)$, we have

$$\frac{m(B(x, R))}{m(B(x, r))} \geq C^{-1} \frac{R^\beta}{r^\beta}. \quad (\text{RVD})$$

For any $x \in X$, define $D_x := \inf_{y \in X \setminus \{x\}} d(x, y)$. We say that (X, d, m) satisfies the *quasi reverse volume doubling* property QRVD if there exists $\beta \in (0, \infty), C \in (1, \infty)$ such that for all $x \in X, D_x < r \leq R < \text{diam}(X, d)$, we have

$$\frac{m(B(x, R))}{m(B(x, r))} \geq C^{-1} \frac{R^\beta}{r^\beta}. \quad (\text{QRVD})$$

Much of the existing work on stable-like heat kernel estimates assumes the reverse volume doubling condition **RVD**. Since many natural examples, such as graphs with polynomial volume growth, do not satisfy **RVD** but do satisfy **QRVD**, we work throughout in this more general setting.

We say that $\phi : [0, \infty) \rightarrow [0, \infty)$ is a *scale function* if ϕ is a homeomorphism such that there exist constants $C_\phi \geq 1$, $\beta_2 \geq \beta_1 > 0$ such that

$$C_\phi^{-1} \left(\frac{R}{r} \right)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \leq C_\phi \left(\frac{R}{r} \right)^{\beta_2} \quad \text{for all } 0 < r \leq R. \quad (1.2)$$

Definition 1.2. Let (X, d, m) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ be a pure jump Dirichlet form on $L^2(X, m)$ with jump measure J . We say that the two-sided jump kernel bounds $J(\phi)$ hold if there exist a symmetric function $J : X \times X \rightarrow [0, \infty)$ and constants $c, C > 0$ such that

$$J(dx, dy) = J(x, y) m(dx) m(dy),$$

and

$$\frac{c}{m(B(x, d(x, y))) \phi(d(x, y))} \leq J(x, y) \leq \frac{C}{m(B(x, d(x, y))) \phi(d(x, y))}, \quad J(\phi)$$

for all $x, y \in X$ with $x \neq y$. The above upper and lower bounds on J will be denoted by $J(\phi)_\leq$ and $J(\phi)_\geq$, respectively.

For the remainder of this work, we always assume that the jump measure J on $X \times X \setminus \text{diag}_X$ is of the form $J(dx, dy) = J(x, y) m(dx) m(dy)$ where $J : X \times X \rightarrow [0, \infty)$ is a non-negative, symmetric, measurable function. We define the *energy measure* $\Gamma(f, f)$ of a function $f \in \mathcal{F}$ by

$$\Gamma(f, f)(A) = \int_A \int_X (f(x) - f(y))^2 J(x, y) m(dy) m(dx). \quad (1.3)$$

We also denote by $\Gamma(f, f) : X \rightarrow \mathbb{R}$, the density of the measure $\Gamma(f, f)$ with respect to m given by

$$\Gamma(f, f)(x) := \int_X (f(x) - f(y))^2 J(x, y) m(dy).$$

By an abuse of notation, we use the same symbol for both the measure and its density with respect to m for J and Γ ; the meaning will be clear from the context.

Definition 1.3. (i) Let $U \subset V$ be open sets in X with $U \subset \bar{U} \subset V$. A **cutoff function** for $U \subset V$ is a non-negative, bounded, measurable function ϕ such that $\phi \equiv 1$ on U , $\phi \equiv 0$ on V^c , and $0 \leq \phi \leq 1$ on X .

(ii) We say that the **cutoff Sobolev inequality** $\text{CSJ}(\phi)$ holds if there exist constants $a_0 \in (0, 1]$ and $C_1, C_2 \in (0, \infty)$ such that for all $0 < r \leq R$, $x \in X$, and all $f \in \mathcal{F}$, there exists a cutoff function $\psi \in \mathcal{F}$ for $B(x, R) \subset B(x, R + r)$ such that

$$\int_{B(x, R+(1+a_0)r)} f^2 d\Gamma(\psi, \psi) \leq C_1 \int_{U \times U^*} (f(x) - f(y))^2 J(dx, dy) + \frac{C_2}{\phi(r)} \int_{B(x, R+(1+a_0)r)} f^2 dm, \quad \text{CSJ}(\phi)$$

where

$$U = B(x, R + r) \setminus B(x, R), \quad U^* = B(x, R + (1 + a_0)r) \setminus B(x, R - a_0r).$$

- (iii) We say that the **capacity upper bound** $\text{Cap}(\phi)_{\leq}$ holds if there exist constants $A_1, A_2, C_1 \in (1, \infty)$ such that for all $x \in X$ and $0 < r < \text{diam}(X, d)/A_2$, there exists a cutoff function ψ for $B(x, r) \subset B(x, A_1 r)$ satisfying

$$\mathcal{E}(\psi, \psi) \leq C_1 \frac{V(x, r)}{\phi(r)}.$$

Remark 1.4. Assuming $\text{Cap}(\phi)_{\leq}$, we may, without loss of generality, take $A_1 = 2$. Indeed, this can be achieved by covering $B(x, r)$ with a uniformly bounded number of balls of radius r/A_1 whose centers lie in $B(x, r)$, and then taking the maximum of the corresponding cutoff functions associated with these smaller balls. For such a covering argument, see, for instance, [Mur24, Lemma 6.2].

By [CKW21, Proposition 2.3-(5)], under **VD**, and $\mathbf{J}(\phi)_{\leq}$ we have the implication

$$\text{CSJ}(\phi) \implies \text{Cap}(\phi)_{\leq}. \quad (1.4)$$

Given a Dirichlet form $(\mathcal{E}, \mathcal{F})$, there is an associated **Markov semigroup** $(P_t)_{t \geq 0}$ on $L^2(X, m)$. Furthermore, by [FOT, Theorem 1.3.1 and Lemma 1.3.4] the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given in terms of the semigroup by

$$\begin{aligned} \mathcal{F} &= \left\{ f \in L^2(X, m) : \lim_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle < \infty \right\}, \\ \mathcal{E}(f, f) &= \lim_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle, \quad \text{for all } f \in \mathcal{F}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(X, m)$ inner product. The **heat kernel** associated with the Markov semigroup $\{P_t\}$ (if it exists) is a family of measurable functions $p(t, \cdot, \cdot) : X \times X \mapsto [0, \infty)$ for every $t > 0$, such that

$$\begin{aligned} P_t f(x) &= \int p(t, x, y) f(y) m(dy), \quad \text{for all } f \in L^2(X, m), t > 0 \text{ and } x \in X, \\ p(t, x, y) &= p(t, y, x), \quad \text{for all } x, y \in X \text{ and } t > 0, \\ p(t + s, x, y) &= \int p(s, x, y) p(t, y, z) m(dy), \quad \text{for all } t, s > 0 \text{ and } x, y \in X. \end{aligned}$$

Definition 1.5. We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the **stable-like heat kernel estimates** $\text{HK}(\phi)$ if there exist $C_1 \in (1, \infty)$ and a heat kernel $\{p_t\}_{t > 0}$ of the semigroup corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ such that for each $t \in (0, \infty)$,

$$p_t(x, y) \leq C_1 \left(\frac{1}{m(B(x, \phi^{-1}(t)))} \wedge \frac{t}{m(B(x, d(x, y))) \phi(d(x, y))} \right) \quad (1.5)$$

and

$$p_t(x, y) \geq C_1^{-1} \left(\frac{1}{m(B(x, \phi^{-1}(t)))} \wedge \frac{t}{m(B(x, d(x, y))) \phi(d(x, y))} \right) \quad (1.6)$$

for m -a.e. $x, y \in X$, where $\phi^{-1}(\cdot)$ denotes the inverse of the homeomorphism $\phi(\cdot) : [0, \infty) \rightarrow [0, \infty)$.

The following is the first main result of the work and provides a sufficient condition for cutoff Sobolev inequality for pure jump process.

Theorem 1.6. *Let (X, d, m) be a metric measure space that satisfies **VD**. Let $(\mathcal{E}, \mathcal{F})$ be a pure jump Dirichlet form on $L^2(X, m)$ that satisfies **J**(ϕ), and **Cap**(ϕ) $_{\leq}$. Then $(\mathcal{E}, \mathcal{F})$ satisfies the cutoff energy inequality **CSJ**(ϕ).*

Theorem 1.6 along with known characterizations of stable-like heat kernel bounds imply an affirmative answer to [GHH18, Conjecture 1.14] as we state below.

Theorem 1.7. *Let (X, d, m) be a metric measure space that satisfies **VD** and **QRVD**. Let $(\mathcal{E}, \mathcal{F})$ be a pure jump Dirichlet form on $L^2(X, m)$. Then the following are equivalent:*

- (a) *Jump kernel bound **J**(ϕ), and capacity upper bound **Cap**(ϕ) $_{\leq}$.*
- (b) *Stable-like heat kernel bound **HK**(ϕ).*

Remark 1.8. (a) Theorems 1.6 and 1.7 also hold for the more general notion of regular scale function introduced in [BM, Definition 5.4]. These extensions follow from the present formulation by a quasi-symmetric change of metric, as in [BM, Proof of Lemma 5.9].

- (b) Theorem 1.6 implies that the cutoff Sobolev inequality appearing in the characterization of stable-like heat kernel bounds in the discrete-time graph setting in [MS19, Theorem 1.5] can be replaced by capacity upper bounds analogous to those in Theorem 1.7.
- (c) We expect Theorem 1.7 to simplify the verification of stable-like heat kernel estimates in concrete examples. For instance, the proof of stable-like heat kernel bounds for the boundary trace of reflected diffusions in [KS, Theorem 5.14] relies on establishing exit time bounds together with jump kernel estimates. In this case, proving capacity upper bounds is substantially simpler than proving exit time bounds.

1.2 Outline of the proof

Since Theorem 1.7 follows readily from Theorem 1.6 together with known results, we only outline the proof of Theorem 1.6. The Whitney blending argument proceeds in three main steps:

1. We establish a cutoff Sobolev-type inequality in the case where the cutoff function is chosen as the equilibrium potential for $K \subset \Omega$, for functions that vanish either in a neighborhood of K or in a neighborhood of Ω^c (Lemma 2.6).
2. Given $f \in \mathcal{F}$, $x_0 \in X$, and $r > 0$, we construct functions $f_1, f_2 \in \mathcal{F}$ such that $|f| \leq |f_1| + |f_2|$, where f_1 vanishes in a neighborhood of $\overline{B(x_0, r)}$ and f_2 vanishes in a neighborhood of $B(x_0, 2r)^c$. This allows us to apply the cutoff Sobolev-type inequality from the previous step to f_1 and f_2 , yielding a cutoff Sobolev inequality for the equilibrium potential associated with $\overline{B(x_0, r)} \subset B(x_0, 2r)$. A key difficulty is controlling the energies of f_1 and f_2 in terms of the energy of f . This is accomplished using Whitney covers and constitutes the most delicate part of the proof (see Lemma 2.9). The construction of f_1, f_2 also relies on the explicit description of the domain of the Dirichlet form established in Proposition 2.1.
3. The previous step produces a non-local analogue of the simplified cutoff Sobolev inequality introduced in [Mur24, Definition 6.1] (see Proposition 2.10). A self-improvement argument, following [Mur24, Proof of Lemma 6.2], then yields the desired cutoff Sobolev inequality (see §2.6).

Notation. In the following, we will use the notation $A \lesssim B$ for quantities A and B to indicate the existence of an implicit constant $C \geq 1$ depending on some inessential parameters such that $A \leq CB$. We write $A \asymp B$, if $A \lesssim B$ and $B \lesssim A$.

2 Proofs

2.1 Identification of the domain of the Dirichlet form

The Whitney blending technique involves the construction of a piecewise defined function (see (2.46)). In order to verify that such a function belongs to the domain of the Dirichlet form, it is necessary to have a concrete sufficient condition ensuring membership in the domain. In the local setting, such a criterion is provided in [Eri, Lemma 4.9]. Here, we develop a non-local analogue based on an explicit description of the domain.

Let $(\mathcal{E}, \mathcal{F})$ be a pure jump Dirichlet form on $L^2(X, m)$ with jump measure J . A natural candidate for the domain \mathcal{F} is the following subspace:

$$\widehat{\mathcal{F}} := \left\{ f \in L^2(X, m) : \int_{X \times X} (f(x) - f(y))^2 J(dx, dy) < \infty \right\}. \quad (2.1)$$

While it is immediate that $\mathcal{F} \subset \widehat{\mathcal{F}}$, the converse inclusion requires proof. Several ideas used in establishing the equality $\mathcal{F} = \widehat{\mathcal{F}}$ in following proposition will later reappear in the Whitney blending argument in § 2.5. We also note that a considerably shorter proof of this result was obtained earlier under the *a priori* stronger assumption of stable-like heat kernel estimates in [KM25+, Theorem 2.40(d)].

Proposition 2.1. *Let (X, d, m) be a metric measure space that satisfies **VD**. Let $(\mathcal{E}, \mathcal{F})$ be a pure jump Dirichlet form that satisfies **J**(ϕ), and **Cap**(ϕ) $_{\leq}$. Then we have $\mathcal{F} = \widehat{\mathcal{F}}$.*

We first establish some preliminary lemmas that are needed for the proof of Proposition 2.1.

Lemma 2.2. *If the jump measure J of the pure-jump Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ satisfies **J**(ϕ), then there exists $C > 0$ such that for all $x, y \in X$ and $r_1, r_2 \in (0, d(x, y)/4)$, we have*

$$\frac{1}{V(x, r_1)V(y, r_2)} \int_{B(x, r_1)} \int_{B(y, r_2)} J(z, w) m(dz) m(dw) \leq C J(x, y),$$

and

$$\frac{1}{V(y, r_2)} \int_{B(y, r_2)} J(x, w) m(dw) \leq C J(x, y).$$

Proof. Let $x, y \in X$, $r_1, r_2 \in (0, d(x, y)/4)$. Then by the triangle inequality,

$$d(x, y)/2 \leq d(x, y) - r_1 - r_2 \leq d(z, w) \leq d(x, y) + r_1 + r_2 \leq 2d(x, y)$$

for all $z \in B(x, r_1), w \in B(y, r_2)$. Hence for all $z \in B(x, r_1), w \in B(y, r_2)$, we have

$$J(z, w) \asymp \frac{1}{V(z, d(z, w))\phi(d(z, w))} \stackrel{(1.1), (1.2)}{=} \frac{1}{V(x, d(x, y))\phi(d(x, y))} \asymp J(x, y).$$

□

We recall that an r -net N of X is a maximal r -separated subset of X ; that is $d(x, y) \in \{0\} \cup [r, \infty)$ for all $x, y \in N$ and for any $\tilde{N} \not\supseteq N$, there exists $x, y \in \tilde{N}$ such that $d(x, y) \in (0, r)$. We record the following *bounded overlap property* for balls centered at an r -net. For any $A > 0$, there exists a constant $C = C(A)$, depending only on A and the constants in the volume doubling property **VD**, such that for any $r > 0$ and any r -net N ,

$$\sum_{x \in N} \mathbf{1}_{B(x, Ar)} \leq C \mathbf{1}. \quad (2.2)$$

The following construction of partition of unity indexed by a net is standard in the setting of strongly local Dirichlet forms. However, the non-local case requires a slightly different argument, and the verification that $\psi_x \in \mathcal{F}$ is often omitted in the literature (see, for example, [Mur20, Lemma 2.5] and the references therein). For this reason, we provide a detailed proof.

Lemma 2.3. *There exist $C, c > 0$ such that for any $r > 0$ and for any r -net N , there exists a collection of functions $\{\psi_x : x \in N\}$ satisfying the following properties:*

$$(i) \sum_{x \in N} \psi_x \equiv \mathbf{1}.$$

$$(ii) \text{ For each } x \in X, \psi_x \in \mathcal{F} \cap C(X),$$

$$c \mathbf{1}_{B(x, r)} \leq \psi_x \leq \mathbf{1}_{B(x, 2r)}, \quad \text{and,} \quad \mathcal{E}(\psi_x, \psi_x) \leq C \frac{V(x, r)}{\phi(r)}. \quad (2.3)$$

Proof. Let N be any r -net, where $r > 0$. By **Cap**(ϕ) $_{\leq}$, Remark 1.4, and the regularity of the Dirichlet form, for each $x \in N$, there exists $\tilde{\psi}_x \in \mathcal{F} \cap C(X)$ be such that

$$\tilde{\psi}_x \in \mathcal{F}, \quad \mathbf{1}_{B(x, r)} \leq \tilde{\psi}_x \leq \mathbf{1}_{B(x, 2r)}, \quad \mathcal{E}(\tilde{\psi}_x, \tilde{\psi}_x) \lesssim \frac{V(x, r)}{\phi(r)}.$$

We claim that the family $\{\psi_x : x \in N\}$, defined by

$$\psi_x := \frac{\tilde{\psi}_x}{\sum_{y \in N} \tilde{\psi}_y}, \text{ for all } x \in N, \quad (2.4)$$

satisfies the desired properties. By the maximality of the net and the bounded overlap property (2.2), there exists $c \in (0, 1)$ such that

$$\mathbf{1} \leq \sum_{x \in N} \tilde{\psi}_x \leq c^{-1} \mathbf{1}. \quad (2.5)$$

Using McSchane's extension theorem [Hei, Theorem 6.2], we pick a $\sqrt{2}$ -Lipschitz function $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$T(a, b) = \begin{cases} a/b & \text{if } a \in [0, 1], b \in [1, \infty), \\ 0 & \text{if } a = 0, b = 0. \end{cases}$$

Hence, combining (2.4) and (2.5) with the generalized contraction property of Dirichlet forms ([KS, Theorem A.2] or [BH, Exercise I.3.5, Proof of Proposition I.3.3.1]), we obtain that

$$\psi_x = T \left(\tilde{\psi}_x, \sum_{y \in N \cap B(x, 4r)} \tilde{\psi}_y \right) \in \mathcal{F},$$

and

$$\begin{aligned}
\mathcal{E}(\psi_x, \psi_x) &\leq 2 \left(\mathcal{E}(\tilde{\psi}_x, \tilde{\psi}_x) + \mathcal{E} \left(\sum_{y \in N \cap B(x, 4r)} \tilde{\psi}_y, \sum_{y \in N \cap B(x, 4r)} \tilde{\psi}_y \right) \right) \\
&\leq 2 \left(\mathcal{E}(\tilde{\psi}_x, \tilde{\psi}_x) + \#(N \cap B(x, 4r)) \sum_{y \in N \cap B(x, 4r)} \mathcal{E}(\tilde{\psi}_y, \tilde{\psi}_y) \right) \\
&\stackrel{(2.2), (2.3)}{\lesssim} \sum_{y \in N \cap B(x, 4r)} \frac{V(y, r)}{\phi(r)} \stackrel{(1.1)}{\lesssim} \frac{V(x, r)}{\phi(r)}.
\end{aligned}$$

□

Proof of Proposition 2.1. Let $x_0 \in X, r > 0$ and $f \in \hat{\mathcal{F}}$. Let N be any r -net, where $r > 0$. Let $\{\psi_x : x \in N\}$ denote the partition of unity satisfying the conclusion of Lemma 2.3. We define for each $n \in \mathbb{N}$,

$$f_{r,n}(\cdot) := \sum_{y \in B(x_0, n) \cap N} f_{B(y, r)} \psi_y(\cdot),$$

and set $f_r(\cdot) := \sum_{y \in N} f_{B(y, r)} \psi_y(\cdot)$, $F_r(\cdot) := \sum_{y \in N} |f_{B(y, r)}| \psi_y(\cdot)$.

First, we note that by the bounded overlap property (2.2) and Lemma 2.3, we have

$$\begin{aligned}
\int |F_r|^2 dm &\stackrel{(2.2)}{\lesssim} \int \sum_{y \in N} |f_{B(y, r)}|^2 \psi_y^2 dm \stackrel{(2.3)}{\lesssim} \sum_{y \in N} |f_{B(y, r)}|^2 V(y, 2r) \stackrel{\text{VD}}{\lesssim} \sum_{y \in N} |f_{B(y, r)}|^2 V(y, r) \\
&\lesssim \sum_{y \in N} \int_{B(y, r)} f^2 dm \stackrel{(2.2)}{\lesssim} \int f^2 dm < \infty. \quad (\text{by Jensen's inequality})
\end{aligned} \tag{2.6}$$

Therefore by the dominated convergence theorem $f_{r,n} \rightarrow f$ in $L^2(X, m)$. Since $f_{r,n}$ is a finite linear combination of functions in \mathcal{F} , we have that $f_{r,n} \in \mathcal{F}$ for all $f \in \hat{\mathcal{F}}$ and $n \in \mathbb{N}$.

Next, let us estimate the energy $\mathcal{E}(f_{r,n}, f_{r,n})$. By the bounded overlap property (2.2), and Cauchy-Schwarz inequality, we have for all $x, y \in X, r > 0, f \in \hat{\mathcal{F}}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
(f_{r,n}(x) - f_{r,n}(y))^2 &= \left(\sum_{w \in N \cap B(x_0, n)} f_{B(w, r)} (\psi_w(x) - \psi_w(y)) \right)^2 \\
&\lesssim \sum_{w \in N \cap B(x_0, n)} f_{B(w, r)}^2 (\psi_w(x) - \psi_w(y))^2.
\end{aligned} \tag{2.7}$$

Therefore for all $n \in \mathbb{N}, r > 0$ and $f \in \hat{\mathcal{F}}$, we have

$$\begin{aligned}
\mathcal{E}(f_{r,n}, f_{r,n}) &= \int (f_{r,n}(x) - f_{r,n}(y))^2 J(dx, dy) \\
&\leq \sum_{w \in N \cap B(x_0, n)} f_{B(w, r)}^2 \int (\psi_w(x) - \psi_w(y))^2 J(dx, dy) \leq \sum_{w \in N} f_{B(w, r)}^2 \mathcal{E}(\psi_w, \psi_w) \\
&\stackrel{(2.3)}{\lesssim} \sum_{w \in N} f_{B(w, r)}^2 \frac{V(w, r)}{\phi(r)} \stackrel{(2.6)}{\lesssim} \frac{1}{\phi(r)} \int_X f^2 dm.
\end{aligned}$$

In particular, $\liminf_{n \rightarrow \infty} \mathcal{E}(f_{r,n}, f_{r,n}) < \infty$. Combining this with $f_{r,n} \rightarrow f$ in $L^2(X, m)$ and using the fact that $(\mathcal{E}, \mathcal{F})$ is a closed form, we conclude that $f_r \in \mathcal{F}$ and

$$\mathcal{E}(f_r, f_r) \leq \liminf_{n \rightarrow \infty} \mathcal{E}(f_{r,n}, f_{r,n}) \lesssim \frac{1}{\phi(r)} \int_X f^2 dm < \infty. \quad (2.8)$$

Here, the conclusions $f_{r,n} \in \mathcal{F}$ and (2.8) follow from the standard equivalence between closedness of the quadratic form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ and lower semicontinuity on $L^2(X, m)$ of the functional

$$g \mapsto \begin{cases} \mathcal{E}(g, g), & \text{if } g \in \mathcal{F}, \\ \infty, & \text{otherwise,} \end{cases}$$

see [Mos, p. 372, §1(c)].

Similar to the argument above showing that $f_r \in \mathcal{F}$, to prove that $f \in \mathcal{F}$ it suffices to show that

$$\lim_{r \downarrow 0} \int_X |f - f_r|^2 dm = 0. \quad (2.9)$$

and

$$\liminf_{r \downarrow 0} \mathcal{E}(f_r, f_r) < \infty. \quad (2.10)$$

Next, we show (2.9). To this end, note that for any $r > 0$,

$$\begin{aligned} \int |f - f_r|^2 dm &= \int \left| f(x) - \sum_{y \in N} f_{B(y,r)} \psi_y(x) \right|^2 m(dx) = \int \left| \sum_{y \in N} (f(x) - f_{B(y,r)}) \psi_y(x) \right|^2 m(dx) \\ &\lesssim \sum_{y \in N} \int |(f(x) - f_{B(y,r)}) \psi_y(x)|^2 m(dx) \quad (\text{by (2.2), (2.3), Cauchy-Schwarz}) \\ &\lesssim \sum_{y \in N} \int_{B(y,2r)} |(f(x) - f_{B(y,r)})|^2 m(dx), \quad (\text{since } \psi_y \leq \mathbb{1}_{B(y,2r)}) \\ &\lesssim \sum_{y \in N} \frac{1}{V(y,r)} \int \int \mathbb{1}_{B(y,2r)}(x_1) \mathbb{1}_{B(y,r)}(x_2) |f(x_1) - f(x_2)|^2 m(dx_1) m(dx_2) \\ &\quad (\text{by Jensen's inequality}) \\ &\lesssim \sum_{y \in N} \int \int \mathbb{1}_{B(y,2r)}(x_1) \mathbb{1}_{B(y,r)}(x_2) \frac{|f(x_1) - f(x_2)|^2}{V(x_1, r)} m(dx_1) m(dx_2) \quad (\text{by (1.1)}) \end{aligned} \quad (2.11)$$

By the bounded overlap property (2.2), we have

$$\sum_{y \in N} \mathbb{1}_{B(y,2r)}(x_1) \mathbb{1}_{B(y,r)}(x_2) \lesssim \mathbb{1}_{B(x_2,3r)}(x_1), \quad \text{for all } x_1, x_2 \in X. \quad (2.12)$$

If $d(x_1, x_2) < 3r$, then by (1.2), $J(\phi)$ and (1.1), we have

$$\frac{1}{V(x_1, r)} \lesssim \frac{1}{V(x_1, d(x_1, x_2))} \lesssim \phi(r) \frac{1}{V(x_1, d(x_1, x_2)) \phi(d(x_1, x_2))} \lesssim \phi(r) J(x_1, x_2). \quad (2.13)$$

Combining (2.11), (2.12), and (2.13), we have

$$\begin{aligned} \int |f - f_r|^2 dm &\stackrel{(2.11), (2.12)}{\lesssim} \int \int \mathbf{1}_{B(x_1, 3r)}(x_2) \frac{|f(x_1) - f(x_2)|^2}{V(x_1, r)} m(dx_1) m(dx_2) \\ &\stackrel{(2.13)}{\lesssim} \phi(r) \int \int |(f(x_1) - f(x_2))|^2 J(x_1, x_2) m(dx_1) m(dx_2) \end{aligned} \quad (2.14)$$

By $f \in \widehat{\mathcal{F}}$, (2.14), and (1.2), we obtain (2.9).

Since the bound (2.8) is not enough to obtain (2.10), we need an improved bound on

$$\mathcal{E}(f_r, f_r) = \int (f_r(x) - f_r(y))^2 J(dx, dy).$$

We estimate the above integral by considering two cases, according to whether $d(x, y) < 18r$. To this end, we need two estimates on $(f_r(x) - f_r(y))^2$. The first estimate is obtained by the same argument as in (2.7), yielding the following estimate for all $x, y \in X$, $r > 0$, $f \in \tilde{\mathcal{F}}$, and $K \in \mathbb{R}$:

$$\begin{aligned} (f_r(x) - f_r(y))^2 &= \left(\sum_{w \in N} (f_{B(w, r)} - K)(\psi_w(x) - \psi_w(y)) \right)^2 \\ &\lesssim \sum_{w \in N} (f_{B(w, r)} - K)^2 (\psi_w(x) - \psi_w(y))^2. \end{aligned} \quad (2.15)$$

The second estimate for $(f_r(x) - f_r(y))^2$ follows from the Cauchy–Schwarz inequality and (2.3), yielding

$$\begin{aligned} (f_r(x) - f_r(y))^2 &= \left(\sum_{w_1, w_2 \in N} (f_{B(w_1, r)} - f_{B(w_2, r)}) \psi_{w_1}(x) \psi_{w_2}(y) \right)^2 \\ &\leq \#(N \cap B(x, 2r)) \cdot \#(N \cap B(y, 2r)) \sum_{\substack{w_1 \in N \cap B(x, 2r), \\ w_2 \in N \cap B(y, 2r)}} (f_{B(w_1, r)} - f_{B(w_2, r)})^2 \\ &\stackrel{(2.2)}{\lesssim} \sum_{\substack{w_1 \in N \cap B(x, 2r), \\ w_2 \in N \cap B(y, 2r)}} (f_{B(w_1, r)} - f_{B(w_2, r)})^2 \\ &\lesssim \int_{B(x, 3r)} \int_{B(y, 3r)} \frac{(f(z_1) - f(z_2))^2}{V(z_1, r)V(z_2, r)} m(dz_1) m(dz_2), \end{aligned} \quad (2.16)$$

where the last line follows from by (1.1), Jensen's inequality and $B(w_1, r) \subset B(x, 3r)$, $B(w_2, r) \subset B(y, 3r)$ for all $w_1 \in N \cap B(x, 2r)$, $w_2 \in N \cap B(y, 2r)$. Note that, if $d(x, y) \geq 18r$, then for any $z_1 \in B(x, 3r)$, $z_2 \in B(y, 3r)$, we have

$$\frac{1}{4}d(z_1, z_2) \geq \frac{1}{4}(d(x, y) - 6r) \geq 3r,$$

and hence by Lemma 2.2 and (1.1), we have

$$\frac{1}{V(z_1, r)V(z_2, r)} \int_{B(z_1, 3r)} \int_{B(z_2, 3r)} J(w_1, w_2) m(dw_1) m(dw_2) \lesssim C J(z_1, z_2). \quad (2.17)$$

Using the estimate (2.16) and interchanging the order of integration via Fubini's theorem, we obtain

$$\begin{aligned}
& \int \int (f_r(x) - f_r(y))^2 \mathbb{1}_{\{d(x,y) \geq 18r\}} J(x,y) m(dx) m(dy) \\
& \stackrel{(2.16)}{\lesssim} \int \int \int_{B(z_2, 3r)} \int_{B(z_1, 3r)} \mathbb{1}_{\{d(x,y) \geq 18r\}} \frac{(f(z_1) - f(z_2))^2}{V(z_1, r)V(z_2, r)} J(x,y) m(dx) m(dy) m(dz_1) m(dz_2) \\
& \stackrel{(2.17)}{\lesssim} \int \int (f(z_1) - f(z_2))^2 J(z_1, z_2) m(dz_1) m(dz_2). \tag{2.18}
\end{aligned}$$

To estimate the integral under $d(x, y) < 18r$, we use that

$$\mathbb{1}_{B(x, 18r)}(y) \leq \sum_{w \in N} \mathbb{1}_{B(w, r)}(x) \mathbb{1}_{B(w, 19r)}(y) \leq \sum_{w \in N} \mathbb{1}_{B(w, 19r)}(x) \mathbb{1}_{B(w, 19r)}(y),$$

for all $x, y \in X$ such that $d(x, y) < 18r$. Furthermore, for such $x, y \in X$, using (2.3), we observe that

$$\psi_w(x) - \psi_w(y) = 0, \quad \text{for all } w \in N \cap B(x, 20r)^c.$$

Combining the two displays above, and choosing $K = f_{B(x, r)}$ in (2.15), we obtain that for all $x, y \in X$ with $d(x, y) < 18r$,

$$(f_r(x) - f_r(y))^2 \leq \sum_{w \in N \cap B(x, 20r) \cap B(y, 20r)} (f_{B(w, r)} - f_{B(x, r)})^2 (\psi_w(x) - \psi_w(y))^2. \tag{2.19}$$

By Jensen's inequality and (1.1), for any $w \in N$ and any $x \in B(w, 20r)$, we have

$$\begin{aligned}
& |f_{B(w, r)} - f_{B(x, r)}|^2 \\
& \lesssim \frac{1}{m(B(w, r))^2} \int_{B(w, 21r)} \int_{B(w, 21r)} (f(z_1) - f(z_2))^2 m(dz_1) m(dz_2) \\
& \stackrel{(1.2)}{\lesssim} \frac{\phi(r)}{m(B(w, r))} \int_{B(w, 21r)} \int_{B(w, 21r)} \frac{(f(z_1) - f(z_2))^2}{V(z_1, d(z_1, z_2))\phi(d(z_1, z_2))} m(dz_1) m(dz_2) \\
& \lesssim \frac{\phi(r)}{m(B(w, r))} \int_{B(w, 21r)} \int_{B(w, 21r)} (f(z_1) - f(z_2))^2 J(z_1, z_2) m(dz_1) m(dz_2) \tag{2.20}
\end{aligned}$$

Hence by (2.19), (2.20), we obtain

$$\begin{aligned}
& \int \int (f_r(x) - f_r(y))^2 \mathbb{1}_{\{d(x,y) < 18r\}} J(x,y) m(dx) m(dy) \\
& \leq \sum_{w \in N} \frac{\phi(r)}{m(B(w, r))} \int_{B(w, 21r)} \int_{B(w, 21r)} (f(z_1) - f(z_2))^2 J(z_1, z_2) m(dz_1) m(dz_2) \mathcal{E}(\psi_w, \psi_w) \\
& \stackrel{(2.3)}{\lesssim} \sum_{w \in N} \int_{B(w, 21r)} \int_{B(w, 21r)} (f(z_1) - f(z_2))^2 J(z_1, z_2) m(dz_1) m(dz_2) \\
& \lesssim \int_X \int_X (f(z_1) - f(z_2))^2 J(z_1, z_2) m(dz_1) m(dz_2) \quad (\text{by (2.2)}). \tag{2.21}
\end{aligned}$$

Combining (2.18) and (2.21), we obtain (2.10) which concludes the proof that $f \in \mathcal{F}$ and hence $\mathcal{F} = \widehat{\mathcal{F}}$. \square

2.2 Truncated Dirichlet form

A useful and well-known technique for handling difficulties arising from non-local Dirichlet forms is to truncate the jump kernel so that it behaves more like a local Dirichlet form. For a probabilistic interpretation of the resulting decomposition of the jump kernel in terms of the Meyer decomposition, we refer the reader to [BGK, § 3].

Let $(\mathcal{E}, \mathcal{F})$ be a pure jump Dirichlet form with jump measure $J(dx, dy) = J(x, y) m(dx) m(dy)$. For $\rho > 0$, we define the ρ -truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ by

$$\mathcal{E}^{(\rho)}(f, g) := \int (f(x) - f(y))(g(x) - g(y)) J^{(\rho)}(dx, dy), \quad f, g \in \mathcal{F}, \quad (2.22)$$

where

$$J^{(\rho)}(x, y) := \mathbf{1}_{\{d(x, y) \leq \rho\}} J(x, y), \quad J^{(\rho)}(dx, dy) := \mathbf{1}_{\{d(x, y) \leq \rho\}} J(dx, dy). \quad (2.23)$$

By [CKW21, Lemma 2.1 and (2.3)] and $\mathbf{J}(\phi)_{\leq}$, there exists a constant $C > 0$ such that for all $\rho > 0$ and all $f \in \mathcal{F}$,

$$0 \leq \mathcal{E}(f, f) - \mathcal{E}^{(\rho)}(f, f) \leq \frac{C}{\phi(\rho)} \int_X f^2 dm. \quad (2.24)$$

Denote the corresponding energy measure by $\Gamma^{(\rho)}(f, f)(dx) = \Gamma^{(\rho)}(f, f)(x) m(dx)$, where

$$\Gamma^{(\rho)}(f, f)(x) := \int_X (f(y) - f(x))^2 J^{(\rho)}(x, y) m(dy).$$

By [CKW21, Lemma 2.1], there exists $C > 0$ such that for any $f \in \mathcal{F} \cap L^\infty(X, m)$,

$$0 \leq \Gamma(f, f)(x) - \Gamma^{(\rho)}(f, f)(x) \leq \frac{C}{\phi(\rho)} \|f\|_\infty^2, \quad \text{for } m\text{-a.e. } x \in X. \quad (2.25)$$

Similarly, by [CKW21, Lemma 2.1], there exists $C > 0$ such that for any $x \in X$ and $r > 0$,

$$\int_{B(x, r)^c} J(x, y) m(dy) \leq \frac{C}{\phi(r)}. \quad (2.26)$$

2.3 Estimates on the equilibrium potential

In this subsection, we establish a cutoff Sobolev-type inequality for functions with zero boundary values; see Lemma 2.6. The cutoff function is chosen to be an equilibrium potential.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$, where (X, d, m) is a metric measure space. The **1-capacity** of a set $A \subset X$ is defined by

$$\text{Cap}_1(A) := \inf \{ \mathcal{E}_1(f, f) \mid f \in \mathcal{F}, f \geq 1 \text{ } m\text{-a.e. on a neighborhood of } A \}, \quad (2.27)$$

where $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(X, m)}$ as defined before. A subset N of X is said to be \mathcal{E} -polar if $\text{Cap}_1(N) = 0$. If $\text{Cap}_1(A) > 0$, then $A \subset X$ is said to be non- \mathcal{E} -polar or *non-polar*. For $A \subset X$ and a statement $\mathcal{S}(x)$ on $x \in A$, we say that \mathcal{S} holds \mathcal{E} -quasi-everywhere on A (\mathcal{E} -q.e. on A for short), or $\mathcal{S}(x)$ holds for \mathcal{E} -quasi-every $x \in A$ (\mathcal{E} -q.e. $x \in A$ for short), if $\mathcal{S}(x)$ holds for any $x \in A \setminus N$ for some \mathcal{E} -polar $N \subset X$. When $A = X$, we often write just “ \mathcal{E} -q.e.” instead of

“ \mathcal{E} -q.e. on X ”. A non-decreasing sequence $\{F_k\}_{k \in \mathbb{N}}$ of closed subsets of X is called an \mathcal{E} -nest if $\lim_{k \rightarrow \infty} \text{Cap}_1(K \setminus F_k) = 0$ for any compact subset K of X . A function $f: D \setminus N \rightarrow [-\infty, \infty]$, defined \mathcal{E} -q.e. on an open subset D of X for some \mathcal{E} -polar $N \subset X$, is said to be \mathcal{E} -quasi-continuous on D if there exists an \mathcal{E} -nest $\{F_k\}_{k \in \mathbb{N}}$ such that $F_k \cap N = \emptyset$ and $f|_{D \cap F_k}$ is an \mathbb{R} -valued continuous function on $D \cap F_k$ for any $k \in \mathbb{N}$ (again, when $D = X$, we often omit “on X ”). For each $f \in \mathcal{F}_e$, an \mathcal{E} -quasi-continuous m -version \tilde{f} of f exists by [FOT, Theorem 2.1.7] and is unique \mathcal{E} -q.e. by [FOT, Lemma 2.1.4].

Let $A \Subset \Omega \subset X$, where Ω is a precompact open subset such that Ω^c is non-polar. By [FOT, the 0-order version of Theorem 2.1.5-(i),(ii)], we have

$$\text{Cap}(A, \Omega^c) = \inf \{ \mathcal{E}(f, f) \mid f \in \mathcal{F}, \tilde{f} \geq 1 \text{ } \mathcal{E}\text{-q.e. on } A, \tilde{f} = 0 \text{ } \mathcal{E}\text{-q.e. on } \Omega^c \} \quad (2.28)$$

and there exists a unique function $e_{A, \Omega} \in \mathcal{F}$, called the **equilibrium potential** of A in Ω , that attains the infimum in (2.28) and satisfies

$$\tilde{e}_{A, \Omega} \equiv 1 \text{ } \mathcal{E}\text{-q.e. on } A \quad \text{and} \quad \tilde{e}_{A, \Omega} \equiv 0 \text{ } \mathcal{E}\text{-q.e. on } \Omega^c. \quad (2.29)$$

The following Lemma is a non-local analogue of [Eri, Lemma 3.1] and provides estimates of energy measure of equilibrium potential. The proof of the analogous lemma in [Eri] relies on strong locality.

Lemma 2.4. *Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump Dirichlet form on $L^2(X, m)$ with jump measure $J(dx, dy) = J(x, y) m(dx) m(dy)$. Let ψ be the equilibrium potential for $K \subset \Omega$, where Ω is a non-polar, precompact open subset and K is compact. Let $g \in \mathcal{F} \cap C(X)$ be such that $0 \leq g \leq 1$ and $g \equiv 1$ for some $A \subset X$. Suppose if one of the following conditions hold:*

- (a) $g \equiv 0$ on some neighborhood of Ω^c , or
- (b) $g \equiv 0$ on some neighborhood of K .

Then $\Gamma(\psi, \psi)(A) \leq 2\Gamma(g, g)(A)$.

Proof. Let us first assume condition (a). By a quasi-continuous modification if necessary, we assume that ψ is quasi-continuous. Define $u := \psi \vee g$. Then by [FOT, Theorem 1.4.2-(i)] and (2.29), $u \in \mathcal{F}$, $u \equiv 0$ q.e. on Ω^c and $u \equiv 1$ q.e. on K . Hence by the energy minimizing property of ψ , we have

$$\int_{X \times X} (u(x) - u(y))^2 J(dx, dy) = \mathcal{E}(u, u) \geq \mathcal{E}(\psi, \psi) = \int_{X \times X} (\psi(x) - \psi(y))^2 J(dx, dy). \quad (2.30)$$

Note that

$$\begin{aligned} (u(x) - u(y))^2 &= 0, & \text{for all } x, y \in A; \\ (u(x) - u(y))^2 &\leq (g(x) - g(y))^2 & \text{for all } (x, y) \in (A \times A^c) \cup (A^c \times A); \\ (u(x) - u(y))^2 &\leq (\psi(x) - \psi(y))^2 + (g(x) - g(y))^2 & \text{for all } x, y \in A^c. \end{aligned} \quad (2.31)$$

Combining (2.30) and (2.31), and using the symmetry of J , we have

$$\int_{X \times X} (\psi(x) - \psi(y))^2 J(dx, dy)$$

$$\leq 2 \int_{A \times A^c} (g(x) - g(y))^2 J(dx, dy) + \int_{A^c \times A^c} ((\psi(x) - \psi(y))^2 + (g(x) - g(y))^2) J(dx, dy)$$

which can be rearranged to

$$\begin{aligned} & 2 \int_{A \times A^c} (\psi(x) - \psi(y))^2 J(dx, dy) + \int_{A^c \times A^c} (\psi(x) - \psi(y))^2 J(dx, dy) \\ & \leq 2 \int_{A \times A^c} (g(x) - g(y))^2 J(dx, dy) + \int_{A^c \times A^c} (g(x) - g(y))^2 J(dx, dy) \end{aligned}$$

Therefore

$$\begin{aligned} \int_A \Gamma(\psi, \psi)(dx) &= \int_{A \times (A \cup A^c)} (\psi(x) - \psi(y))^2 J(dx, dy) \\ &\leq 2 \int_{A \times A^c} (g(x) - g(y))^2 J(dx, dy) + \int_{A^c \times A^c} (g(x) - g(y))^2 J(dx, dy) \\ &\leq 2 \int_A \Gamma(g, g)(dx). \end{aligned}$$

The proof in case (b) is similar, replacing u with $v := \psi \wedge (1 - g)$ in the above argument. Note that $v = T(\psi, g)$ where $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the 1-Lipschitz function defined by $T(a, b) := \min(a, 1 - b)$. Since $T(0, 0) = 0$ by the generalized contraction property [KS, Theorem A.2], [BH, Exercise I.3.5, Proof of Proposition I.3.3.1], we have $v \in \mathcal{F}$. Hence the above argument in case (a) applies, once we observe that

$$\begin{aligned} (v(x) - v(y))^2 &= 0, & \text{for all } x, y \in A; \\ (v(x) - v(y))^2 &\leq (g(x) - g(y))^2 & \text{for all } (x, y) \in (A \times A^c) \cup (A^c \times A); \\ (v(x) - v(y))^2 &\leq (\psi(x) - \psi(y))^2 + (g(x) - g(y))^2 & \text{for all } x, y \in A^c. \end{aligned}$$

□

The following lemma is a variant of [BCLS, Lemma 6.1].

Lemma 2.5. *Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump Dirichlet form on $L^2(X, m)$ with jump measure $J(dx, dy) = J(x, y) m(dx) m(dy)$. Let $f \in \mathcal{F}$ be a non-negative function. For any $k \in \mathbb{Z}$, define*

$$f_k := (0 \vee (f - 2^k)) \wedge 2^k. \quad (2.32)$$

Then

$$\sum_{k \in \mathbb{Z}} \Gamma(f_k, f_k)(x) \leq \Gamma(f, f)(x). \quad (2.33)$$

Proof. Since $f(x), f(y) \geq 0$, we note that for any $x, y \in X$,

$$\sum_{k \in \mathbb{Z}} |f_k(x) - f_k(y)| = |f(x) - f(y)|.$$

The estimate between ℓ_1 and ℓ_2 norms is

$$\sum_{k \in \mathbb{Z}} (f_k(x) - f_k(y))^2 \leq (f(x) - f(y))^2,$$

and therefore

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \Gamma(f_k, f_k)(x) &= \int \sum_{k \in \mathbb{Z}} (f_k(x) - f_k(y))^2 J(x, y) m(dy) \\ &\leq \int (f(x) - f(y))^2 J(x, y) m(dy) = \Gamma(f, f)(x). \end{aligned}$$

□

The following result is a version of cutoff Sobolev inequality for functions with somewhat restrictive boundary conditions. This restriction on the boundary condition will be later removed by the Whitney blending argument developed in Lemma 2.9.

Lemma 2.6. *Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump Dirichlet form on $L^2(X, m)$ with jump measure $J(dx, dy) = J(x, y) m(dx) m(dy)$. Let ψ be the equilibrium potential for $K \subset \Omega$, where Ω is a non-polar, precompact open subset and K is compact. Let $f \in \mathcal{F} \cap C(X)$ be one of the following conditions hold:*

- (a) $f \equiv 0$ on some neighborhood of Ω^c , or
- (b) $f \equiv 0$ on some neighborhood of K .

Then for any $U \subset X$,

$$\int_U f^2 d\Gamma(\psi, \psi) \leq 48\Gamma(f, f)(U). \quad (2.34)$$

Proof. By replacing f by $|f| \in \mathcal{F}$ and noting that $\Gamma(|f|, |f|) \leq \Gamma(f, f)$, we assume that f is non-negative without loss of generality [FOT, Theorem 1.4.1(e)].

For each $k \in \mathbb{Z}$, we define f_k as given in (2.32), and set $g_k := 2^{-k}f_k$, so that

$$g_k \equiv 1 \quad \text{on } A_k := \{x \in U : f(x) \geq 2^{k+1}\}. \quad (2.35)$$

Note that

$$f(x)^2 \geq \sum_{k \in \mathbb{Z}} 4^k \mathbb{1}_{A_k}(x) \geq f(x)^2/12, \quad \text{for all } x \in U,$$

and hence

$$\int_U f^2 d\Gamma(\psi, \psi) \leq 12 \cdot 4^k \sum_{k \in \mathbb{Z}} \Gamma(\psi, \psi)(A_k). \quad (2.36)$$

By applying Lemma 2.4 to $g = g_k$ and $A = A_k$ using (2.35), we obtain

$$\Gamma(\psi, \psi)(A_k) \leq 2\Gamma(g_k, g_k)(A_k) \leq 2 \cdot 4^{-k} \Gamma(f_k, f_k)(A_k) \leq 2 \cdot 4^{-k} \Gamma(f_k, f_k)(U) \quad (2.37)$$

Hence by (2.36), (2.37), and Lemma 2.5, we obtain

$$\int_U f^2 d\Gamma(\psi, \psi) \leq 48 \sum_{k \in \mathbb{Z}} \Gamma(f_k, f_k)(U) \leq 48\Gamma(f, f)(U).$$

□

2.4 Whitney cover

We recall the notion of a ϵ -Whitney cover from [GS, Definition 3.16]. For a proper open set $U \subsetneq X$, we denote by

$$\delta_U(x) := \text{dist}(x, U^c) = \inf_{y \in U^c} d(x, y).$$

Definition 2.7. Let $\epsilon \in (0, 1/2)$ and $U \subsetneq X$ be open. We say a collection of balls $\mathfrak{R} := \{B(x_i, r_i) : x_i \in U, r_i > 0, i \in I\}$ is an ϵ -Whitney cover if it satisfies the following properties:

- (i) The collection of sets $\{B(x_i, r_i), i \in I\}$ are pairwise disjoint.
- (ii) The radii r_i satisfy

$$r_i = \frac{\epsilon}{1 + \epsilon} \delta_U(x_i), \quad \text{for all } i \in I.$$

- (iii) $\bigcup_{i \in I} B(x_i, K_\epsilon r_i) = U$, where $K_\epsilon = 2(1 + \epsilon) \in (2, 3)$.

Note that since $K_\epsilon < 3$ by (iii), we have

$$\bigcup_{i \in I} B_U(x_i, 3r_i) = U.$$

The existence of a Whitney cover follows by applying Zorn's lemma to select a maximal collection of balls satisfying properties (i) and (ii) in Definition 2.7. By [Mur24, Proposition 3.2-(d)], we have for any open set $U \subsetneq X$, and for any $\epsilon \in (0, 1/2)$, there exists $C > 0$ depending only on ϵ and the constants in VD such that for any ϵ -Whitney cover $\{B(x_i, r_i) : i \in I\}$, we have

$$\sum_{i \in I} \mathbf{1}_{B(x_i, r_i/\epsilon)} \leq C \mathbf{1}_U. \quad (2.38)$$

We also record the following consequence of triangle inequality for future use. Let $\{B(x_i, r_i) : i \in I\}$ be an ϵ -Whitney cover of an open set $U \subsetneq X$, where $\epsilon \in (0, 1/2)$. Then for any $\lambda > 0$, for all $i \in I$ and for all $z \in B(x_i, \lambda r_i)$, we have

$$\left(1 - \frac{\lambda\epsilon}{1 + \epsilon}\right) \delta_U(x_i) < \delta_U(z) < \left(1 + \frac{\lambda\epsilon}{1 + \epsilon}\right) \delta_U(x_i). \quad (2.39)$$

The proof for the case $\lambda = 3$ in [Mur24, Proposition 3.2-(b)] readily generalizes to yield (2.39).

The following existence of partition of unity follows from the same argument as the proof of Lemma 2.3, where the desired bounded overlap property follows from (2.38).

Lemma 2.8. *Let (X, d, m) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form that satisfies $\text{Cap}(\phi)_{\leq}$ and let m be a doubling measure. There exists $\epsilon_0 \in (0, 1/6)$, $C_1, c_1 > 0$ such that for any open set $U \subsetneq X$ be open and for any ϵ -Whitney cover $\{B_i = B(x_i, r_i) : i \in I\}$ of U where $0 < \epsilon < \epsilon_0$, there exists a partition on unity $\{\psi_{B_i} : i \in I\}$ of V such that the following properties hold:*

(a) (partition of unity) $\sum_{i \in I} \psi_{B_i} \equiv \mathbf{1}_U$.

(b) (controlled energy) For each $i \in I$, we have $\psi_{B_i} \in \mathcal{F} \cap C(X)$ and

$$c_1 \mathbf{1}_{B(x_i, 3r_i)} \leq \psi_{B_i} \leq \mathbf{1}_{B(x_i, 6r_i)}, \quad \mathcal{E}(\psi_{B_i}, \psi_{B_i}) \leq C_1 \frac{V(x_i, r_i)}{\phi(r_i)}. \quad (2.40)$$

2.5 Whitney blending

The following *Whitney blending* result is a non-local analogue of [Eri, Lemma 1.6] and is the key ingredient behind our main result.

Lemma 2.9. *For any $\eta \in (0, 1)$, there exists C depending only on the constants in $\mathbf{J}(\phi)$, \mathbf{VD} , and η such that, for any $x_0 \in X$, $0 < r < \text{diam}(X, d)/4$ and for any $f, g \in \mathcal{F} \cap C(X)$ and for any $A \in [2, \infty)$, there exists $h \in \mathcal{F} \cap C(X)$ such that $h \equiv f$ on $B(x_0, r)$, $h \equiv g$ on $B(x_0, (1 + \eta)r)^c$ and*

$$\begin{aligned} & \int_{B(x_0, Ar) \times B(x_0, Ar)} (h(x) - h(y))^2 J(dx, dy) \\ & \leq C \int_{B(x_0, Ar) \times B(x_0, Ar)} ((f(x) - f(y))^2 + (g(x) - g(y))^2) J(dx, dy) + \frac{C}{\phi(r)} \int_{B(x_0, Ar)} (f - g)^2 dm. \end{aligned} \quad (2.41)$$

Proof. As explained in [Eri, Proof of Lemma 1.6], it suffices to consider the case $g = 0$ by replacing f with $\tilde{f} = f - g$ and g with $\tilde{g} = 0$. Then by considering the function \tilde{h} with $\tilde{h} \equiv \tilde{f}$ on $B(x_0, r)$, $\tilde{h} \equiv 0$ on $B(x_0, (1 + \eta)r)^c$ with satisfying the estimate

$$\begin{aligned} & \int_{B(x_0, Ar) \times B(x_0, Ar)} (\tilde{h}(x) - \tilde{h}(y))^2 J(dx, dy) \\ & \lesssim \int_{B(x_0, Ar) \times B(x_0, Ar)} (\tilde{f}(x) - \tilde{f}(y))^2 J(dx, dy) + \frac{1}{\phi(r)} \int_{B(x_0, Ar)} (f - g)^2 dm, \end{aligned}$$

and using the estimate

$$\begin{aligned} & (\tilde{f}(x) - \tilde{f}(y))^2 \leq 2((f(x) - f(y))^2 + (g(x) - g(y))^2), \\ & ((\tilde{h} + g)(x) - (\tilde{h} + g)(y))^2 \leq 2((\tilde{h}(x) - \tilde{h}(y))^2 + (g(x) - g(y))^2), \quad \text{for all } x, y \in X, \end{aligned}$$

we obtain the desired conclusion for the function $h = \tilde{h} + g$. Thus we assume without loss of generality that $g = 0$.

Let $U = B(x_0, (1 + \eta)r) \setminus \overline{B(x_0, r)}$ and let $\{B_i = B(x_i, r_i) : i \in I\}$ denote a ϵ -Whitney cover of U , where ϵ is such that

$$0 < \epsilon < \frac{\eta}{36}, \quad (2.42)$$

and such that

$$A_1 := \frac{7\epsilon}{1 - 5\epsilon} < \frac{1}{8}, \quad A_2 := \frac{A_1}{1 - A_1} < \frac{1}{4}, \quad A_3 := \frac{6A_2}{(1 - 4A_2)(1 - A_2)} < 1, \quad (2.43)$$

and

$$A_4 := \frac{1 + 4\epsilon}{1 + \epsilon} A_3 + \frac{3\epsilon}{1 + \epsilon}, \quad A_5 := A_4 + (1 + A_4) \frac{6\epsilon(1 - 5\epsilon)}{(1 + \epsilon)^2} \leq \frac{1}{1 + \epsilon}. \quad (2.44)$$

For instance, $\epsilon = \eta/100$ satisfies all of the above requirements.

Define

$$\tilde{I} = \{i \in I : B(x_i, 6r_i) \cap B(x_0, (1 + (\eta/2))r) \neq \emptyset\}.$$

Let $\{\psi_{B_i} : i \in I\}$ denote the partition of unity as given in Lemma 2.8.

Note that the choice of ϵ ensures that if $i \in \tilde{I}$, then $B(x_i, 6r_i) \subset B(x, (1 + (\eta/2)r + 6r_i)) \subset B(x_0, (1 + (2\eta/3))r)$, so that

$$\mathbb{1}_{B(x_0, (1 + (\eta/2))r) \setminus \overline{B(x_0, r)}} \leq \sum_{i \in \tilde{I}} \psi_{B_i} \leq \mathbb{1}_{B(x_0, (1 + (2\eta/3))r) \setminus \overline{B(x_0, r)}}. \quad (2.45)$$

The desired function h is defined by

$$h(x) := \begin{cases} f(x) & \text{if } x \in \overline{B(x_0, r)}, \\ \sum_{i \in \tilde{I}} f_{B_i} \psi_{B_i}(x) & \text{if } x \notin \overline{B(x_0, r)}. \end{cases} \quad (2.46)$$

satisfies $h = f$ on $B(x_0, r)$ and $h = 0$ on $B(x, (1 + (2\eta/3))r)^c$.

Next, we verify that $h \in \mathcal{F}$ using Proposition 2.1. To this end, let us first check that $h \in L^2(X, m)$. Using the bounded overlap property (2.38) and Jensen's inequality, we have

$$\begin{aligned} \int h^2 dm &\lesssim \int_{\overline{B(x_0, r)}} f^2 dm + \sum_{i \in \tilde{I}} \int |f_{B_i}|^2 \mathbb{1}_{B(x_i, 6r_i)} dm \quad (\text{by (2.38), (2.40), Cauchy-Schwarz}) \\ &\lesssim \int_{\overline{B(x_0, r)}} f^2 dm + \sum_{i \in \tilde{I}} \int |f_{B_i}|^2 m(B_i) \\ &\lesssim \int_{\overline{B(x_0, r)}} f^2 dm + \sum_{i \in \tilde{I}} \int_{B_i} f^2 dm \leq \int_{B(x_0, (1 + \eta)r)} f^2 dm < \infty \quad (\text{by Jensen's inequality}). \end{aligned} \quad (2.47)$$

Next we show that

$$\int \int (h(x) - h(y))^2 J(x, y) m(dx) m(dy) < \infty. \quad (2.48)$$

Let $K = \overline{B(x_0, r)}$, $F = B(x_0, (1 + \eta)r)^c$. Using the symmetry of J , in order to estimate the integral (2.48), we divide into five cases based depending on if x, y belong to K, U , or F .

Case 1: $x, y \in K$. In this case, since $h = f$ on K , we have

$$\int_K \int_K (h(x) - h(y))^2 J(x, y) m(dx) m(dy) = \int_K \int_K (f(x) - f(y))^2 J(x, y) m(dx) m(dy). \quad (2.49)$$

Case 2: $x, y \in F$. Since $h = 0$ in F , we have

$$\int_F \int_F (h(x) - h(y))^2 J(x, y) m(dx) m(dy) = 0. \quad (2.50)$$

Case 3: $x, y \in U$. This case is similar to the proof of Proposition 2.1 (in particular, the argument to establish (2.10)) and is divided into two subcases depending on whether or not $d(x, y) \geq A_3(\delta_U(x) \wedge \delta_U(y))$, where A_3 is given by (2.43).

For any $x \in U$, we denote

$$\tilde{I}(x) := \{i \in \tilde{I} : \psi_{B_i}(x) \neq 0\}.$$

By (2.38) and (2.40), $\sup_{x \in U} \tilde{I}(x) \lesssim 1$. Using this, (2.40) and the Cauchy-Schwarz inequality, we have

$$(h(x) - h(y))^2 = \left(\sum_{i \in \tilde{I}(x), j \in \tilde{I}(y)} (f_{B_i} - f_{B_j}) \psi_{B_i}(x) \psi_{B_j}(y) \right)^2$$

$$\lesssim \sum_{i \in \tilde{I}(x), j \in \tilde{I}(y)} (f_{B_i} - f_{B_j})^2 \mathbb{1}_{B(x_i, 6r_i)}(x) \mathbb{1}_{B(x_j, 6r_j)}(y). \quad (2.51)$$

Note that by (2.39), for any $x \in U, i \in \tilde{I}(x)$, we have

$$B_i \subset B(x, 7r_i) = B\left(x, \frac{7\epsilon\delta_U(x_i)}{1+\epsilon}\right) \subset B(x, A_1\delta_U(x)), \quad \text{where } A_1 \text{ is as given in (2.43).} \quad (2.52)$$

Hence by Jensen's inequality, (2.51) and (2.52), we have for all $x, y \in U$

$$(h(x) - h(y))^2 \lesssim \int_{B(y, A_1\delta_U(y))} \int_{B(x, A_1\delta_U(x))} \frac{(f(z) - (f(w))^2}{V(z, A_2\delta_U(z))V(w, A_2\delta_U(w))} m(dz) m(dw). \quad (2.53)$$

Note that for any $x \in U$ and for any $z \in B(x, A_1\delta_U(x))$, by triangle inequality

$$(1 - A_1)\delta_U(x) \leq \delta_U(z) \leq (1 + A_1)\delta_U(x), \quad \mathbb{1}_{B(x, A_1\delta_U(x))}(z) \leq \mathbb{1}_{B(z, A_2\delta_U(z))}(x) \quad (2.54)$$

If $d(z, w) < 4A_2(\delta_U(z) \vee \delta_U(w))$, $x \in B(z, A_2\delta_U(z))$, $w \in B(z, A_2\delta_U(z))$, then by triangle inequality

$$d(x, y) < 6A_2(\delta_U(z) \vee \delta_U(w)) \leq \frac{6A_2}{1 - 4A_2}(\delta_U(z) \wedge \delta_U(w)) \leq \frac{6A_2}{(1 - 4A_2)(1 - A_2)}(\delta_U(x) \wedge \delta_U(y)).$$

Hence by the choice of A_3 in (2.43), for all $x, y \in U, z \in B(x, A_1\delta_U(x)), w \in B(y, A_1\delta_U(y))$ such that $d(x, y) \geq A_3(\delta_U(x) \wedge \delta_U(y))$, we have

$$A_2(\delta_U(z) \vee \delta_U(w)) \leq d(z, w)/4. \quad (2.55)$$

Hence for any $x, y \in U$ such that Thus by (2.53) and Lemma 2.2, we have

$$\begin{aligned} & \int_U \int_U (h(x) - h(y))^2 \mathbb{1}_{\{d(x, y) \geq A_3(\delta_U(x) \wedge \delta_U(y))\}} J(x, y) m(dx) m(dy) \\ & \stackrel{(2.53), (2.54), (2.55)}{\lesssim} \int_U \int_U (f(z) - (f(w))^2 \mathbb{1}_{\{d(z, w) \geq 4A_2(\delta_U(z) \vee \delta_U(w))\}} \\ & \quad \int_{B(w, A_2\delta_U(w))} \int_{B(z, A_2\delta_U(z))} J(x, y) m(dx) m(dy) m(dz) m(dw) \\ & \lesssim \int_U \int_U (f(z) - (f(w))^2 \mathbb{1}_{\{d(z, w) \geq 4A_2(\delta_U(z) \vee \delta_U(w))\}} J(z, w) m(dz) m(dw), \end{aligned} \quad (2.56)$$

where the last line above follows from Lemma 2.2.

Note that if $x \in B(x_i, 3r_i)$ for some i , we have

$$\delta_U(x) \leq 3r_i + \delta_U(x_i) = \frac{1 + 4\epsilon}{1 + \epsilon} \delta_U(x_i),$$

and hence for $x \in B(x_i, 3r_i), y \in U$ such that $d(x, y) < A_3\delta_U(x)$, we have $d(x_i, y) \leq A_4\delta_U(x_i)$, where A_4 is as defined in (2.44). Since $\bigcup_{i \in I} B(x_i, 3r_i) = U$, by (2.44) and (2.38), for all $x, y \in U$, we have

$$\mathbb{1}_{\{d(x, y) < A_3(\delta_U(x) \wedge \delta_U(y))\}} \leq \sum_{i \in I} \mathbb{1}_{B(x_i, A_4\delta_U(x_i))}(x) \mathbb{1}_{B(x_i, A_4\delta_U(x_i))}(y). \quad (2.57)$$

Hence

$$\begin{aligned} & \int_U \int_U (h(x) - h(y))^2 \mathbf{1}_{\{d(x,y) < A_3(\delta_U(x) \wedge \delta_U(y))\}} J(x, y) m(dx) m(dy) \\ & \stackrel{(2.57)}{\lesssim} \sum_{i \in I} \int_{B(x_i, A_4 \delta_U(x_i))} \int_{B(x_i, A_4 \delta_U(x_i))} (h(x) - h(y))^2 J(x, y) m(dx) m(dy) \end{aligned} \quad (2.58)$$

For each $i \in I$, $x, y \in B(x_i, A_4 \delta_U(x_i))$ and any $j \in \tilde{I}(x)$, we have

$$\delta_U(x_j) \stackrel{(2.39)}{\leq} \frac{1 - 5\epsilon}{1 + \epsilon} \delta_U(x) \leq \frac{1 - 5\epsilon}{1 + \epsilon} (1 + A_4) \delta_U(x_i)$$

and using the definition of A_5 in (2.44) along with triangle inequality, we have

$$B_j = B(x_j, r_j) \subset B(x_i, A_4 \delta_U(x_i) + 6r_j) \subset B(x_i, A_5 \delta_U(x_i)).$$

Define $I_1 := \{i \in I : B(x_i, A_4 \delta_U(x_i)) \subset B(x_0, (1 + \eta/2)r)\}$, so that $i \in I_1, x \in B(x_i, A_4 \delta_U(x_i))$ and $j \in I$ with $x \in B(x_j, 6r_j)$ implies $j \in \tilde{I}$.

Since $\sum_{j \in \tilde{I}} \psi_{B_j} \geq \mathbf{1}_{B(x_0, (1 + \eta/2)r) \setminus \overline{B(x_0, r)}}$, for any $i \in I_1$, and for all $x, y \in B(x_i, A_4 \delta_U(x_i))$, we have

$$\begin{aligned} (h(x) - h(y))^2 & \stackrel{(2.45)}{=} \left(\sum_{\substack{j \in \tilde{I}, \\ B_j \subset B(x_i, A_5 \delta_U(x_i))}} (f_{B_j} - f_{B(x_i, A_5 \delta_U(x_i))})(\psi_{B_j}(x) - \psi_{B_j}(y)) \right)^2 \\ & \stackrel{(2.38)}{\lesssim} \sum_{\substack{j \in \tilde{I}, \\ B_j \subset B(x_i, A_5 \delta_U(x_i))}} (f_{B_j} - f_{B(x_i, A_5 \delta_U(x_i))})^2 (\psi_{B_j}(x) - \psi_{B_j}(y))^2. \end{aligned} \quad (2.59)$$

Since $\mathcal{E}(\psi_{B_j}, \psi_{B_j}) \lesssim \frac{m(B_j)}{\phi(r_j)}$, by the same argument in (2.20) and (2.21), we obtain using (2.59) that

$$\begin{aligned} & \sum_{i \in I_1} \int_{B(x_i, A_4 \delta_U(x_i))} \int_{B(x_i, A_4 \delta_U(x_i))} (h(x) - h(y))^2 J(x, y) m(dx) m(dy) \\ & \lesssim \sum_{i \in I_1} \int_{B(x_i, A_5 \delta_U(x_i))} \int_{B(x_i, A_5 \delta_U(x_i))} (f(x) - f(y))^2 J(x, y) m(dx) m(dy) \\ & \lesssim \int_U \int_U (f(x) - f(y))^2 J(x, y) m(dx) m(dy), \end{aligned} \quad (2.60)$$

where we use (2.44) and (2.38) in the last line. It remains to consider the case when $i \in I \setminus I_1$. In this case, if $i \in I \setminus I_1, x \in B(x_i, A_4 \delta_U(x_i))$ and $j \in \tilde{I}(x)$, then by the triangle inequality (2.39), we have

$$\delta_U(x_j) > \frac{1 + \epsilon}{1 + 7\epsilon} \delta_U(x) > \frac{(1 + \epsilon)(1 - A_4)}{1 + 7\epsilon} \delta_U(x_i) \stackrel{(2.39), (2.45)}{>} \frac{(1 + \epsilon)(1 - A_4)\eta}{3(1 + 7\epsilon)(1 + A_4)} r,$$

and hence

$$r_j > ar, \quad \text{where } a := \frac{\epsilon(1 - A_4)\eta}{3(1 + 7\epsilon)(1 + A_4)}.$$

Instead of (2.59), we use the estimate

$$\begin{aligned}
(h(x) - h(y))^2 &= \left(\sum_{\substack{j \in \tilde{I}, \\ B_j \subset B(x_i, A_5 \delta_U(x_i))}} f_{B_j}(\psi_{B_j}(x) - \psi_{B_j}(y)) \right)^2 \\
&\stackrel{(2.38)}{\lesssim} \sum_{\substack{j \in \tilde{I}, \\ B_j \subset B(x_i, A_5 \delta_U(x_i))}} (f_{B_j})^2 (\psi_{B_j}(x) - \psi_{B_j}(y))^2.
\end{aligned} \tag{2.61}$$

Thus we estimate

$$\begin{aligned}
&\sum_{i \in I \setminus I_1} \int_{B(x_i, A_4 \delta_U(x_i))} \int_{B(x_i, A_4 \delta_U(x_i))} (h(x) - h(y))^2 J(x, y) m(dx) m(dy) \\
&\lesssim \sum_{j \in I, r_j > ar} f_{B_j}^2 \mathcal{E}(\psi_{B_j}, \psi_{B_j}) \lesssim \frac{1}{\phi(r)} \sum_{j \in I, r_j > ar} \int_{B_j} f^2 dm \quad (\text{by Jensen's inequality and } r_j > ar) \\
&\lesssim \frac{1}{\phi(r)} \int_U f^2 dm.
\end{aligned} \tag{2.62}$$

Combining (2.56), (2.58), (2.60) and (2.62), we obtain

$$\begin{aligned}
&\int_U \int_U (h(x) - h(y))^2 J(x, y) m(dx) m(dy) \\
&\lesssim \int_U \int_U (f(x) - f(y))^2 J(x, y) m(dx) m(dy) + \frac{1}{\phi(r)} \int_U f^2 dm.
\end{aligned} \tag{2.63}$$

Case 4: $x \in K, y \in U$. We divide this into two sub-cases depending on whether or not $y \in B(x_0, (1 + (\eta/2)r)\overline{B(x_0, r)})$. In this case, by (2.45)

$$\begin{aligned}
(h(x) - h(y))^2 &= (f(x) - \sum_{i \in \tilde{I}} f_{B_i} \psi_{B_i}(y))^2 = \left(\sum_{i \in \tilde{I}} (f(x) - f_{B_i}) \psi_{B_i}(y) \right)^2 \\
&\lesssim \sum_{i \in \tilde{I}} (f(x) - f_{B_i})^2 \psi_{B_i}(y)^2 \lesssim \sum_{i \in \tilde{I}(y)} (f(x) - f_{B_i})^2.
\end{aligned}$$

If $i \in \tilde{I}(y)$, then

$$B_i \subset B(y, 7r_i) \stackrel{(2.39)}{\subset} B\left(y, \frac{7\epsilon}{1-5\epsilon} \delta_U(y)\right) = B(y, A_1 \delta_U(y)).$$

Hence by Jensen's inequality, the above two displays, and (2.38), we have

$$\begin{aligned}
(h(x) - h(y))^2 &\lesssim \sum_{i \in \tilde{I}(y)} (f(x) - f_{B_i})^2 \\
&\lesssim \int_{B(y, A_1 \delta_U(y))} \frac{(f(x) - f(z))^2}{V(y, A_1 \delta_U(y))} m(dz) \lesssim \int_{B(y, A_1 \delta_U(y))} \frac{(f(x) - f(z))^2}{V(z, 2A_2 \delta_U(z))} m(dz).
\end{aligned} \tag{2.64}$$

If $z \in B(y, A_1\delta_U(y))$, then $B(y, A_1\delta_U(y)) \subset B(z, 2A_1\delta_U(y)) \subset B(z, 2A_2\delta_U(z))$, and hence

$$\mathbb{1}_{B(y, A_1\delta_U(y))}(z) \leq \mathbb{1}_{B(z, 2A_2\delta_U(z))}(y). \quad (2.65)$$

Hence

$$\begin{aligned} & \int_K \int_{B(x_0, (1+(\eta/2))r) \setminus \overline{B(x_0, r)}} (h(x) - h(y))^2 J(x, y) m(dx) m(dy) \\ & \stackrel{(2.64)}{\lesssim} \int_K \int_U \int_{B(y, A_1\delta_U(y))} \frac{(f(x) - f(z))^2}{V(z, 2A_2\delta_U(z))} J(x, y) m(dz) m(dy) m(dx) \\ & \stackrel{(2.65)}{\lesssim} \int_K \int_U (f(x) - f(z))^2 \int_{B(z, 2A_2\delta_U(z))} \frac{J(x, y)}{V(z, 2A_2\delta_U(z))} m(dy) m(dz) m(dx) \\ & \lesssim \int_K \int_U (f(x) - f(z))^2 J(x, z) m(dz) m(dx) \quad (\text{by } A_2 < 1/8, \delta_U(z) \leq d(x, z), \text{ Lemma 2.2}). \end{aligned} \quad (2.66)$$

If $x \in K, y \in U \setminus B(x_0, (1 + (\eta/2))r)$, then for any $i \in \tilde{I}(y)$, we have

$$\delta_U(x_i) \geq \frac{1 + \epsilon}{1 + 7\epsilon} \delta_U(y) \stackrel{(2.45)}{\geq} \frac{(1 + \epsilon)\eta r}{3(1 + 7\epsilon)}, \quad B(x_i, r_i) \subset B(x_0, (1 + (2\eta)/3)r) \setminus B(x_0, (1 + (\eta)/3)r)$$

and hence

$$\begin{aligned} (h(x) - h(y))^2 &= (f(x) - \sum_{i \in \tilde{I}} f_{B_i} \psi_{B_i}(y))^2 \lesssim f(x)^2 + \sum_{i \in \tilde{I}(y)} f_{B_i}^2 \\ &\lesssim f(x)^2 + \frac{1}{m(K)} \int_{B(x_0, (1+(2\eta)/3)r) \setminus B(x_0, (1+(\eta)/3)r)} f^2 dm. \end{aligned} \quad (2.67)$$

Since for any $x \in K, y \in U \setminus B(x_0, (1 + (\eta/2))r)$, we have $J(x, y) \asymp \frac{1}{m(K)\phi(r)}$, we have

$$\begin{aligned} & \int_K \int_{B(x_0, (1+(\eta/2))r) \setminus \overline{B(x_0, r)}} (h(x) - h(y))^2 J(x, y) m(dx) m(dy) \\ & \stackrel{(2.67), (1.1)}{\lesssim} \int_K \int_U \frac{f(x)^2}{m(K)\phi(r)} m(dy) m(dx) + \frac{1}{\phi(r)} \int_{B(x_0, (1+(2\eta)/3)r) \setminus B(x_0, (1+(\eta)/3)r)} f^2 dm \\ & \stackrel{(1.1)}{\lesssim} \frac{1}{\phi(r)} \int_{K \cup U} f^2 dm. \end{aligned} \quad (2.68)$$

Combining (2.66) and (2.68), we obtain

$$\begin{aligned} & \int_K \int_U (h(x) - h(y))^2 J(x, y) m(dx) m(dy) \\ & \lesssim \int_K \int_U (f(x) - f(z))^2 J(x, z) m(dz) m(dx) + \frac{1}{\phi(r)} \int_{K \cup U} f^2 dm. \end{aligned} \quad (2.69)$$

Case 5: $x \in K \cup U, y \in F$. In this case, we have $h(y) = 0$ for all $y \in F$ and $h(x) = 0$ for all $x \in U \setminus B(x_0, (1 + (2\eta)/3)r)$. Thus it suffices to consider $x \in B(x_0, (1 + (2\eta)/3)r)$ and $y \in F$, which implies $d(x, y) \geq \frac{\eta r}{3}$. Hence we have

$$\int_{K \cup U} \int_F (h(x) - h(y))^2 J(x, y) m(dy) m(dx)$$

$$\begin{aligned}
&= \int_{K \cup U} \int_F h(x)^2 J(x, y) \mathbf{1}_{\{d(x, y) \geq \eta r/3\}} m(dy) m(dx) \\
&\leq \int_{K \cup U} h(x)^2 \int_{B(x, \eta r/3)^c} J(x, y) m(dy) m(dx) \\
&\stackrel{(2.26), (1.2)}{\lesssim} \frac{1}{\phi(r)} \int_{K \cup U} h^2 dm \stackrel{(2.47)}{\lesssim} \frac{1}{\phi(r)} \int_{B(x_0, (1+\eta)r)} f^2 dm. \tag{2.70}
\end{aligned}$$

Combining (2.49), (2.50), (2.63), (2.69), and (2.70), we obtain

$$\begin{aligned}
&\int_X \int_X (h(x) - h(y))^2 J(x, y) m(dy) m(dx) \\
&\lesssim \int_{B(x_0, (1+\eta)r)} \int_{B(x_0, (1+\eta)r)} (f(x) - f(y))^2 J(x, y) m(dy) m(dx) + \frac{1}{\phi(r)} \int_{B(x_0, (1+\eta)r)} f^2 dm.
\end{aligned}$$

By Proposition 2.1 this implies that $h \in \mathcal{F}$ and that (2.41) holds for the case $g = 0$.

It remains to show that $h \in C(X)$. By the bounded overlap property (2.38), (2.40), and the definition of h , it follows that the restriction of h on $\overline{B(x_0, r)}$ and on $\overline{B(x_0, r)}$ is continuous. Therefore it suffices to show that h is continuous at each point of $\overline{B(x_0, r)} \setminus B(x_0, r)$. This follows from the fact that $\{\psi_{B_j}\}$ is a partition of unity satisfying (2.40) and [Mur24, Proposition 3.2-(c)] using the same argument given in [Mur24, Proof of Lemma 5.9]. Indeed, by that argument, for any $\xi \in \overline{B(x_0, r)} \setminus B(x_0, r)$, there exists $s_0 > 0, K_0 > 1$ such that for all $s \in (0, s_0)$, we have

$$\sup_{y \in B(\xi, s)} |h(y) - h(\xi)| \leq \sup_{y \in B(\xi, K_0 s)} |f(y) - f(\xi)|.$$

By letting $s \downarrow 0$ and using the continuity of f , we obtain that h is continuous. \square

2.6 Self-improvement of a simplified cutoff Sobolev inequality

We prove the following version of cutoff Sobolev inequality using the Whitney blending result. This result can be viewed as a proof of the non-local analogue of simplified cutoff Sobolev inequality introduced in [Mur24, Definition 6.1] using the Whitney blending method of [Eri].

Proposition 2.10. *Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump Dirichlet form on $L^2(X, m)$ that satisfies $\mathbf{J}(\phi)$ and $\mathbf{Cap}(\phi)_{\leq}$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a scale function. Then there exists $C > 0$ such that for any $r, \rho > 0, f \in C(X) \cap \mathcal{F}$, the equilibrium potential ψ for $\overline{B(x, r)} \subset B(x, 2r)$ for the truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ satisfies*

$$\begin{aligned}
\int_X f^2 d\Gamma^{(\rho)}(\psi, \psi) &= \int_{B(x, 2r+\rho)} f^2 d\Gamma^{(\rho)}(\psi, \psi) \\
&\leq C \int_{B(x, 2r+2\rho)} \int_{B(x, 2r+2\rho)} (f(y) - f(z))^2 J(y, z) m(dy) m(dz) \\
&\quad + \frac{C}{\phi(r)} \int_{B(x, 2r)} f^2 dm \tag{2.71}
\end{aligned}$$

Proof. If $y \notin B(x, 2r + \rho)$, we have $\phi(y) - \phi(z) = 0$ for m -a.e. $z \in \overline{B}(y, \rho)$, then

$$\Gamma^{(\rho)}(\psi, \psi)(B(x, 2r + \rho)^c) = 0.$$

This implies the first equality in (2.40).

Now by using Lemma 2.9, we find two functions $f_1, f_2 \in \mathcal{F} \cap C(X)$ and $C_1 > 0$ such that

$$f_1|_{\overline{B}(x, 3r/2)} \equiv f|_{\overline{B}(x, 3r/2)}, \quad f_1|_{B(x, 7r/4)^c} \equiv 0, \quad f_2|_{B(x, 3r/2)^c} \equiv f|_{B(x, 3r/2)^c}, \quad f_2|_{\overline{B}(x, 5r/4)} = 0, \quad (2.72)$$

and for $i = 1, 2$,

$$\begin{aligned} & \int_{B(x, 2r+2\rho)} \int_{B(x, 2r+2\rho)} (f_i(y) - f_i(z))^2 J(y, z) m(dy) m(dz) \\ & \leq C_1 \left(\int_{B(x, 2r+2\rho)} \int_{B(x, 2r+2\rho)} (f(y) - f(z))^2 J(y, z) m(dy) m(dz) + \frac{\int_{B(x, 2r)} f^2 dm}{\phi(r)} \right). \end{aligned} \quad (2.73)$$

Note that (2.72) implies

$$|f(y)| \leq \max(|f_1(y)|, |f_2(y)|), \quad \text{for all } y \in X. \quad (2.74)$$

By Lemma 2.6 and (2.74), we have

$$\begin{aligned} \int_X f^2 d\Gamma^{(\rho)}(\psi, \psi) &= \int_{B(x, 2r+\rho)} f^2 d\Gamma^{(\rho)}(\psi, \psi) \stackrel{(2.74)}{\leq} \int_{B(x, 2r+\rho)} (|f_1|^2 + |f_2|^2) d\Gamma^{(\rho)}(\psi, \psi) \\ &\stackrel{(2.34)}{\leq} 48 \sum_{i=1}^2 \Gamma^{(\rho)}(f_i, f_i)(B(x, 2r + \rho)) \\ &\leq 48 \sum_{i=1}^2 \int_{B(x, 2r+\rho)} \int_{B(x, 2r+2\rho)} (f_i(y) - f_i(z))^2 J(y, z) m(dy) m(dz). \end{aligned} \quad (2.75)$$

Combining (2.73) and (2.75), we obtain (2.71) with $C = 48C_1$. \square

In view of Proposition 2.10, the argument below, completing the proof of Theorem 1.6, may be regarded as a non-local analogue of [Mur24, Lemma 6.2]. As shown below, the truncated Dirichlet form plays a crucial role in adapting the argument of [Mur24] to the non-local setting.

Proof of Theorem 1.6. Let $x \in X, 0 < r < R$ be arbitrary. If $B(x, R+r) = X$ we simply choose $\psi \equiv 1$ as the cutoff function. In this case, we have $\Gamma(\psi, \psi) \equiv 0$.

It remains to consider the case $B(x, R+r) \neq X$. In this case $2r < R+r \leq \text{diam}(X, d)$ implies $r < \text{diam}(X, d)/2$. Let $A_1, A_2, C_1 > 1$ denote the constants such that $\text{Cap}(\phi) \leq$ holds. Define

$$\rho := r/44. \quad (2.76)$$

Let $N = \{z_i : i \in I\}$ denote a maximal ρ -separated subset of X . For each $i \in I$, let $B_i = \overline{B}(z_i, \rho), B_i^* = B(z_i, 2\rho), B_i^\# = B(z_i, 4\rho)$. By Proposition 2.10, there exists $C_1 > 0$ such that for any $i \in I$ and for all $f \in \mathcal{F} \cap C(X)$, the equilibrium potential ψ_i for $B_i \subset B_i^*$ corresponding to the truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ satisfies

$$\int_X f^2 d\Gamma(\psi_i, \psi_i) \leq C_1 \int_{B_i^\# \times B_i^\#} (f(y) - f(z))^2 J(dy, dz) + \frac{C_1}{\phi(\rho)} \int_{B_i^*} f^2 dm. \quad (2.77)$$

Define $\tilde{I} \subset I$ as

$$\tilde{I} := \{i \in I : B(z_i, \rho) \cap B(x, R + r/2) \neq \emptyset\},$$

so that

$$\bigcup_{i \in \tilde{I}} B_i \supset B(x, R + r/2), \quad \bigcup_{i \in \tilde{I}} B_i^* \subset B(x, R + (r/2) + 3\rho), \quad \bigcup_{i \in \tilde{I}} B_i^\# \subset B(x, R + (r/2) + 5\rho). \quad (2.78)$$

Therefore the function

$$\psi := \max_{i \in \tilde{I}} \psi_i$$

satisfies (by [FOT, Theorem 1.4.2](i),(ii) and triangle inequality)

$$\psi \in \mathcal{F}, \quad 0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ on } B(x, R + r/2), \quad \psi \equiv 0 \text{ on } B(x, R + r/2 + 3\rho)^c,$$

and hence

$$\Gamma^{(\rho)}(\psi, \psi)(X) = \Gamma^{(\rho)}(B(x, R + r/2 + 4\rho) \setminus B(x, R + r/2 - \rho)). \quad (2.79)$$

Let $\hat{I} := \{i \in \tilde{I} : B_i^* \cap (B(x, R + r/2 + 5\rho) \setminus B(x, R + r/2 - 2\rho)) \neq \emptyset\}$, so that

$$\psi = \hat{\psi}, \quad \text{on } B(x, R + r/2 + 5\rho) \setminus B(x, R + r/2 - 2\rho),$$

so that the energy measure $\Gamma^{(\rho)}(\psi, \psi)$ and $\Gamma^{(\rho)}(\hat{\psi}, \hat{\psi})$ coincide on $B(x, R + r/2 + 4\rho) \setminus B(x, R + r/2 - \rho)$; that is,

$$\Gamma^{(\rho)}(\psi, \psi)(A) = \Gamma^{(\rho)}(\hat{\psi}, \hat{\psi})(A), \quad \text{for all } A \subset B(x, R + r/2 + 4\rho) \setminus B(x, R + r/2 - \rho). \quad (2.80)$$

Furthermore, by the triangle inequality and the choice of ρ in (2.76), we have

$$\bigcup_{i \in \hat{I}} B_i^\# \subset (B(x, R + r/2 + 11\rho) \setminus B(x, R + r/2 - 8\rho)) \subset B(x, R + (3r)/4) \setminus B(x, R + (r/4)). \quad (2.81)$$

For any $n \in \mathbb{N}$ and for any $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$, we have

$$\left(\max_{1 \leq i \leq n} a_i - \max_{1 \leq i \leq n} b_i \right)^2 \leq \max_{1 \leq i \leq n} (a_i - b_i)^2 \leq \sum_{i=1}^n (a_i - b_i)^2.$$

Hence by integrating the above pointwise bound, we have

$$\Gamma^{(\rho)}(\hat{\psi}, \hat{\psi}) \leq \sum_{i \in \hat{I}} \Gamma^{(\rho)}(\psi_i, \psi_i). \quad (2.82)$$

By (2.25), (2.82), and (1.2), there exists $C_2 > 0$ such that for any $A \subset X$, we have

$$\begin{aligned} \Gamma(\psi, \psi)(A) &\leq \Gamma^{(\rho)}(\psi, \psi)(A) + \frac{C_2}{\phi(r)} m(A) \\ &\stackrel{(2.79)}{=} \Gamma^{(\rho)}(\psi, \psi)(A \cap (B(x, R + r/2 + 4\rho) \setminus B(x, R + r/2 - \rho))) + \frac{C_2}{\phi(r)} m(A) \\ &\stackrel{(2.80)}{=} \Gamma^{(\rho)}(\hat{\psi}, \hat{\psi})(A \cap (B(x, R + r/2 + 4\rho) \setminus B(x, R + r/2 - \rho))) + \frac{C_2}{\phi(r)} m(A) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(2.82)}{\leq} \sum_{i \in \hat{I}} \Gamma^{(\rho)}(\psi_i, \psi_i)(A \cap (B(x, R + r/2 + 4\rho) \setminus B(x, R + r/2 - \rho))) + \frac{C_2}{\phi(r)} m(A) \\
&\leq \sum_{i \in \hat{I}} \Gamma^{(\rho)}(\psi_i, \psi_i)(A) + \frac{C_2}{\phi(r)} m(A).
\end{aligned} \tag{2.83}$$

Hence, for any function $f \in \mathcal{F} \cap C(X)$,

$$\begin{aligned}
&\int_{B(x, R+r)} f^2 d\Gamma(\psi, \psi) \\
&\leq \frac{C_2}{\phi(r)} \int_{B(x, R+r)} f^2 dm + \sum_{i \in \hat{I}} \int_X f^2 d\Gamma^{(\rho)}(\psi_i, \psi_i) \quad (\text{by (2.83)}) \\
&\leq \frac{C_2}{\phi(r)} \int_{B(x, R+r)} f^2 dm + \sum_{i \in \hat{I}} \left(C_1 \int_{B_i^\# \times B_i^\#} (f(y) - f(z))^2 J(dy, dz) + \frac{C_1}{\phi(\rho)} \int_{B_i^*} f^2 dm \right).
\end{aligned} \tag{2.84}$$

By (2.2), and (2.81), we have

$$\sum_{i \in \hat{I}} \mathbb{1}_{B_i^*}(y) \lesssim \mathbb{1}_{B(x, R+r)}(y), \tag{2.85}$$

and

$$\sum_{i \in \hat{I}} \mathbb{1}_{B_i^\#}(y) \mathbb{1}_{B_i^\#}(z) \lesssim \mathbb{1}_V(y) \mathbb{1}_V(z), \quad \text{where } V := B(x, R + (3r)/4) \setminus B(x, R + (r)/4). \tag{2.86}$$

Combining (2.84), (2.85), (2.86), and (1.2), for all $f \in \mathcal{F} \cap C(X)$, we have

$$\int_{B(x, R+r)} f^2 d\Gamma(\psi, \psi) \lesssim \frac{1}{\phi(r)} \int_{B(x, R+r)} f^2 dm + \int_V \int_V (f(y) - f(z))^2 J(y, z) m(dy) m(dz).$$

By regularity of the Dirichlet form, the above estimate also holds for all $f \in \mathcal{F}$ and hence $\text{CSJ}(\phi)$ holds. \square

2.7 Characterization of stable-like heat kernel estimate

Next, we complete the proof of Theorem 1.7 using Theorem 1.6. Indeed, if $\text{diam}(X, d) = \infty$ and (X, d, m) satisfies the stronger RVD the equivalence is an immediate consequence of Theorem 1.6, (1.4) and [CKW21, Theorem 1.13]. In the general case, we need some additional ingredients from [Mal, GHH23, GHH24, KM25+].

Proof of Theorem 1.7. The implication $(b) \Rightarrow (a)$ is well known and follows from [Mal, Theorem 1.20] together with (1.4) in the case $\text{diam}(X, d) = \infty$. When $\text{diam}(X, d) < \infty$, the same reasoning applies: the argument used to show $(1) \Rightarrow (4)$ in [CKW21, Theorem 1.13], as well as the construction and properties of the auxiliary space in [Mal, § 3], extend to the finite diameter case with only minor modifications.

Next, we show (a) implies (b). By Theorem 1.6, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies $\text{CSJ}(\phi)$ and $\text{J}(\phi)$. When $\text{diam}(X, d) = \infty$, the stable-like heat kernel estimate $\text{HK}(\phi)$ follows from obtaining the a similar heat kernel estimate for the auxiliary space in [Mal, §3] using [Mal, eq. (4.13),

Proposition 4.1] and [CKW21, Theorem 1.13]. The stable-like heat kernel estimate $\text{HK}(\phi)$ for the auxiliary space then implies the desired heat kernel estimate for $(\mathcal{E}, \mathcal{F})$ using [Mal, eq. (4.6) and (4.7), Proposition 4.1].

It remains to show (a) implies (b) when $\text{diam}(X, d) < \infty$. By adapting the arguments in [Mal] for the finite diameter case, and using the argument in the previous paragraph, it suffices to further assume in addition that (X, d, m) satisfies the reverse volume doubling property RVD . The proof of the implication (a) implies (b) in [CKW21, Theorem 1.13] relies crucially on the infinite diameter assumption to obtain on-diagonal upper bounds as explained in [KM25+, Remark 2.41]. In this case, the argument in [CKW21, §4.3] implies two-sided bounds on exit time which in turn is sufficient to obtain $\text{HK}(\phi)$ by using the equivalence in [KM25+, Theorem 2.40]. The proof in [KM25+, Theorem 2.40] relies on results obtained in [GHH23, GHH24]. An alternate approach to [KM25+, Theorem 2.40]. We refer to [CC, Remark 8.3] for a different approach to [KM25+, Theorem 2.40]. \square

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