Conformal Assouad dimension as the critical exponent for combinatorial modulus

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Abstract

The conformal Assouad dimension is the infimum of all possible values of Assouad dimension after a quasisymmetric change of metric. We show that the conformal Assouad dimension equals a critical exponent associated to the combinatorial modulus for any compact doubling metric space. This generalizes a similar result obtained by Carrasco Piaggio for the Ahlfors regular conformal dimension to a larger family of spaces. We also show that the value of conformal Assouad dimension is unaffected if we replace quasisymmetry with power quasisymmetry in its definition.

1 Introduction

The Assouad dimension of a metric space is defined as the infimum of all numbers $\beta > 0$ such that there exists $C > 0$ so that every set of diameter $r$ can be covered by at most $C \epsilon^{-\beta}$ sets of diameter $\epsilon r$. Equivalently, Assouad dimension is the infimum of all numbers $\beta > 0$ such that there exists $C > 0$ so that every ball of radius $r$ has at most $C \epsilon^{-\beta}$ distinct points whose mutual distance is at least $\epsilon r$ [Hei, Exercise 10.17]. We recall the closely related notion of doubling metric space. A metric space is said to be doubling, if there exists $N \in \mathbb{N}$ such that every ball of radius $r$ can be covered by at most $N$ balls of radii $r/2$. It is easy to see that the Assouad dimension is finite if and only if the metric space is doubling. We refer to the recent book by Fraser [Fra] for a comprehensive background.

Assouad dimension is related to doubling measures as we recall below in Theorem 1.1. A non-zero Borel measure $\mu$ on a metric space $(X, d)$ is said to be doubling if there exists $C_D > 1$ such that

$$\mu(B(x, 2r)) \leq C_D \mu(B(x, r))$$

for all $x \in X, r > 0$, where $B(x, r) = \{y \in X : d(x, y) < r\}$ denotes the open ball of radius $r$ centered at $x$. A non-zero Borel measure is said to be $q$-homogeneous if there exists $C > 1$ such that

$$\mu(B(x, R)) \leq C \left(\frac{R}{r}\right)^q \mu(B(x, r)),$$

for all $x \in X, 0 < r \leq R$.

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It is evident that a measure is doubling if and only if it is \( q \)-homogeneous for some \( q \in (0, \infty) \). The fundamental relationship between Assouad dimension and doubling measures is given by the following theorem of Vol’berg and Konyagin [VK].

**Theorem 1.1.** [VK] Let \((X, d)\) be a compact metric space. The Assouad dimension of \((X, d)\) is the infimum of all \( q > 0 \) such that there exists a \( q \)-homogeneous measure on \((X, d)\).

Ahlfors regular measures are an important class of doubling measures. A Borel measure \( \mu \) on \((X, d)\) is said to be \( p \)-Ahlfors regular if there exists \( C \geq 1 \) such that

\[
C^{-1}r^p \leq \mu(B(x, r)) \leq Cr^p
\]

for all \( x \in X, 0 < r \leq \text{diam}(X, d) \).

Note that if such a \( p \)-Ahlfors regular \( \mu \) exists, \( \mu \) can be chosen to be the \( p \)-Hausdorff measure.

We recall the definition of the conformal gauge. The terminology is motivated by the fact that quasisymmetric maps are an analogue of conformal maps in the context of metric spaces.

**Definition 1.2** (Conformal gauge). Let \((X, d)\) be a metric space and \( \theta \) be another metric on \( X \). We say that \( d \) is *quasisymmetric* to \( \theta \), if there exists a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that

\[
\frac{\theta(x, a)}{\theta(x, b)} \leq \eta \left( \frac{d(x, a)}{d(x, b)} \right)
\]

for all triples of points \( x, a, b \in X, x \neq b \).

We say that \( d \) is *power quasisymmetric* to \( \theta \) if the homeomorphism \( \eta \) above can be chosen so that \( \eta(t) = C(t^\alpha \vee t^{1/\alpha}) \) for all \( t > 0 \), where \( C, \alpha \in [1, \infty) \). The *conformal gauge* of a metric space \((X, d)\) is defined as

\[
\mathcal{J}(X, d) := \{ \theta : X \times X \to [0, \infty) \mid \theta \text{ is a metric on } X, d \text{ is quasisymmetric to } \theta \}.
\] (1.1)

We also define the power quasisymmetric conformal gauge as

\[
\mathcal{J}_p(X, d) := \{ \theta \in \mathcal{J}(X, d) \mid \theta \text{ is power quasisymmetric to } d \}.
\] (1.2)

We recall the definitions of conformal Assouad dimension and Ahlfors regular conformal dimension. The Ahlfors regular conformal dimension of \((X, d)\) is defined as

\[
d_{\text{ARC}}(X, d) = \inf\{p > 0 : \theta \in \mathcal{J}(X, d) \text{ and } \mu \text{ is a } p\text{-Ahlfors regular measure on } (X, \theta)\}
\] (1.3)

and the conformal Assouad dimension of \((X, d)\) is defined as

\[
d_{\text{CA}}(X, d) = \inf\{d_A(X, \theta) : \theta \in \mathcal{J}(X, d)\},
\] (1.4)

where \( d_A(X, \theta) \) denotes the Assouad dimension of \((X, \theta)\).
In (1.3) and (1.4) we adopt the convention that \( \inf \emptyset = \infty \). Ahlfors regular conformal dimension is a well-studied notion in complex dynamics and hyperbolic groups; see for example [BK05, BM, CM, HP09, PT, Par]. These notions of conformal dimensions are slight variant of the one introduced by Pansu [Pan89] and we refer the reader to [MT10] for more background and applications.

We now discuss the class of metric spaces for which conformal Assouad dimension and Ahlfors regular conformal dimensions are finite. Recall that a metric space \((X,d)\) is said to be uniformly perfect if there exists \(C > 1\) such that whenever \(B(x,r) \neq X\), we have \(B(x,r) \setminus B(x,r/C) \neq \emptyset\). The following lemma is well-known.

**Lemma 1.3.** Let \((X,d)\) be a compact metric space. Then

(a) \(d_{CA}(X,d)\) is finite if and only if \((X,d)\) is doubling.

(b) \(d_{ARC}(X,d)\) is finite if and only if \((X,d)\) is doubling and uniformly perfect.

(c) If \((X,d)\) is doubling and uniformly perfect, then \(d_{ARC}(X,d) = d_{CA}(X,d)\) [MT10, Proposition 2.2.6].

One motivation for this work is that conformal Assouad dimension is better behaved than Ahlfors regular conformal dimension. The above lemma shows that conformal Assouad dimension is a meaningful quasisymmetry invariant for a larger class of metric spaces. If \((X,d)\) is a compact metric space and \(Y \subset X\), then it is easy to see that

\[
d_{CA}(Y,d) \leq d_{CA}(X,d).
\]

An unpleasant fact is that the above inequality is not always true for Ahlfors regular conformal dimension because a subset of uniformly perfect metric space is not necessarily uniformly perfect. Nevertheless, if \((X,d)\) is a compact, doubling, uniformly perfect metric space and \(Y \subset X\) is also compact, doubling and uniformly perfect, then

\[
d_{ARC}(Y,d) \leq d_{ARC}(X,d).
\]

One way to show (1.5) is to use Lemma 1.3(c) and the analogous inequality for conformal Assouad dimension. Another more involved approach would be to use [Car, Theorem 1.3] and careful choices of hyperbolic fillings for \(X\) and \(Y\). The direct approach of restricting an Ahlfors regular metric in \(J(X,d)\) to \(Y\) does not work because the restriction of an Ahlfors regular metric on a subset need not be Ahlfors regular. Doubling measures are quasisymmetry invariant\(^1\) but the same is not true for Ahlfors regular measures. Since doubling measures determine Assouad dimension by Theorem 1.1, conformal Assouad dimension is a more natural quasisymmetry invariant than Ahlfors regular conformal dimension. We hope that the above discussion convinces the reader that conformal Assouad dimension is better behaved than Ahlfors regular conformal dimension.

\(^1\)More precisely, if \(\mu\) is a doubling measure on \((X,d)\) and \(\theta \in J(X,d)\), then \(\mu\) is a doubling measure on \((X,\theta)\)
A consequence of our results is that

\[ d_{CA}(X, d) = \inf \{ d_{A}(X, \theta) : \theta \in \mathcal{J}_p(X, d) \}. \]  

(1.6)

Since \( \mathcal{J}_p(X, d) \subset \mathcal{J}(X, d) \), the upper bound on \( d_{CA}(X, d) \) is obvious but the other inequality is non-trivial as it is possible that \( \mathcal{J}_p(X, d) \neq \mathcal{J}(X, d) \). Trotsenko and Väisälä characterize metric spaces for which \( \mathcal{J}_p(X, d) = \mathcal{J}(X, d) \). To recall their characterization, we recall the notion of weakly uniformly perfect spaces. We say that a metric space is weakly uniformly perfect if there exists \( C > 1 \) such that if \( B(x, r) \neq X \) for some \( x \in X, r > 0 \), then either \( B(x, r) = \{x\} \) or \( B(x, r) \setminus B(x, r/C) \neq \emptyset \). The Trotsenko-Väisälä theorem states that a compact metric space \((X, d)\) satisfies \( \mathcal{J}_p(X, d) = \mathcal{J}(X, d) \) if and only if \((X, d)\) is weakly uniformly perfect [TV, Theorems 4.11 and 6.20]. Since our results apply to doubling metric spaces that are not necessarily weakly uniformly perfect, (1.6) is non-trivial.

Next, we describe the critical exponent associated to the combinatorial modulus. The combinatorial \( p \)-modulus of a family of curves \( \Gamma \) in a graph \( G = (V, E) \) is defined as

\[ \Mod_p(\Gamma, G) = \inf \left\{ \sum_{v \in V} \rho(v)^p \mid \rho : V \to [0, \infty), \sum_{v \in \gamma} \rho(v) \geq 1 \text{ for all } \gamma \in \Gamma \right\}. \]

Fix parameters \( a, \lambda, L > 1 \). We choose a sequence \( X_k, k \geq 0 \) such that \( X_k \) is a maximal \( a^{-k} \)-separated subset of \((X, d)\) and \( X_k \subset X_{k+1} \) for all \( k \geq 0 \). For each \( k \), we define a graph \( G_k \) whose vertex set is \( X_k \) and there is an edge between two distinct vertices \( x, y \in X_k \) if and only if \( B(x, \lambda a^{-k}) \cap B(y, \lambda a^{-k}) \neq \emptyset \). We think of \( G_k \) as a sequence of combinatorial approximations of \((X, d)\) at scale \( a^{-k} \). We define

\[ M_{p,k}(L) = \sup \{ \Mod_p(\Gamma_{k,L}(x), G_{k+n}) \mid x \in X_n, n \geq 0 \} \text{ and } M_p(L) = \liminf_{k \to \infty} M_{p,k}. \]

The critical exponent corresponding to combinatorial modulus is defined as

\[ Q(L) = \inf\{ p > 0 : M_{p,L} = 0 \}. \]

It is not difficult to show that \( Q(L) \) is well-defined in the sense that \( Q(L) \) does not depend on the precise choices of \( a, L, \lambda \in (1, \infty) \) and also on the choices of \( X_k \) (see Proposition 4.2). It only depends on the metric space \((X, d)\).

### 1.1 Results and outline of the work

The main result of this work is that the conformal Assouad dimension coincides with the critical exponent corresponding to the combinatorial modulus for any compact doubling metric space (see Theorem 4.4) ; that is,

\[ d_{CA}(X, d) = Q(L). \]

(1.7)

A similar result was shown by Carrasco Piaggio for the Ahlfors regular conformal dimension and independently in an unpublished work of Keith and Kleiner. By Lemma 1.3,
the above result generalizes [Car, Theorem 1.3] to doubling metric spaces that are not necessarily uniformly perfect. We refer to [Kig20, Sha] for expositions to Carrasco's work.

The proof of (1.7) follows a similar approach as [Car] but the proofs of both $d_{CA}(X,d) \leq Q(L)$ and $d_{CA}(X,d) \geq Q(L)$ require new ideas. To explain this, we recall the approach of [Car]. We remark that many ideas in [Car] is based upon the ‘weight-loss program’ of Keith and Laakso [KL, §5.2]. To show the estimate $d_{CA}(X,d) \leq Q(L)$, we construct a graph which is Gromov hyperbolic called the hyperbolic filling (see §2.3). A theorem of Bonk and Schramm implies that a quasi-isometric change of metric on the hyperbolic filling induces a power quasisymmetric change of metric on its boundary. Roughly speaking, a quasi-isometric change of metric is done using the optimal functions for the combinatorial modulus. This is done in [Car, Theorems 1.1 and 1.2] where the author introduces hypotheses on weight functions on the graph that defines a bi-Lipschitz change of metric in the hyperbolic filling. However the hypotheses introduced in [Car, Theorem 1.1] implies that $(X,d)$ is uniformly perfect as pointed in [Sha, Lemma 6.2]. Since the metric spaces we consider is not necessarily weakly uniformly perfect, we modify one of the hypothesis so that it is more suitable for bounding the conformal Assouad dimension (see hypothesis (H4) in Theorem 3.2). The key new tool is a modification of a lemma of Vol’berg and Konyagin to construct a $p$-homogeneous measure on $(X, \theta)$, where $\theta$ is power quasisymmetric to $d$ and $p > Q(L)$ (see Lemma 3.5 and Proposition 3.8). This along with Theorem 1.1 implies the bound $d_{CA}(X,d) \leq Q(L)$. Another key distinction from [Car] is that the metric space is not necessarily uniformly perfect. Therefore by the Trotsenko-Väisälä theorem, this approach need not construct all possible metrics in $\mathcal{J}(X,d)$. Nevertheless, this approach provides the sharp upper bound and also leads to (1.6).

For the other bound $Q(L) \leq d_{CA}(X,d)$, we use a $p$-homogeneous measure $\mu$ in $(X, \theta)$ and $\theta \in \mathcal{J}(X,d)$ for $p > d_{CA}(X,d)$ and define an function $\rho$ for the combinatorial modulus that is similar to [Car, (3.7)]. However some modifications are needed because [Car] uses the uniform perfectness in an essential way to control $\rho$. Some of the parameters and constants in [Car] depend on the constant associated with the uniform perfectness property. Much of the work is about removing such dependence on uniform perfectness.

2 Hyperbolic filling of a compact metric space

2.1 Gromov hyperbolic spaces and its boundary

Let $(Z,d)$ be a metric space. We recall some basic notions regarding Gromov hyperbolic spaces and refer the reader to [BH, CDP, GH90, Gro87, Vä05] for a detailed exposition. Given three points $x, y, p \in Z$, we define the Gromov product of $x$ and $y$ with respect to the base point $w$ as

$$(x|y)_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)).$$
By the triangle inequality, Gromov product is always non-negative. We say that a metric space \((Z,d)\) is \(\delta\)-hyperbolic, if for any four points \(x, y, z, w \in Z\), we have
\[
(x|z)_w \geq (x|y)_w \wedge (y|z)_w - \delta.
\]
We say that \((Z,d)\) is hyperbolic (or \(d\) is a hyperbolic metric), if \((Z,d)\) is hyperbolic for some \(\delta \in [0, \infty)\). If the above condition is satisfied for a fixed base point \(w \in Z\), and arbitrary \(x, y, z \in Z\), then \((Z,d)\) is \(2\delta\)-hyperbolic \([CDP, \text{Proposition 1.2}]\).

We recall the definition of the boundary of a hyperbolic space. Let \((Z,d)\) be a hyperbolic space and let \(w \in Z\). A sequence of points \(\{x_i\} \subset Z\) is said to converge at infinity, if
\[
\lim_{i,j \to \infty} (x_i|x_j)_w = \infty.
\]
The above notion of convergence at infinity does not depend on the choice of the base point \(w \in Z\), because by the triangle inequality \(|(x|y)_w - (x|y)_w'| \leq d(w, w')\).

Two sequences \(\{x_i\}, \{y_i\}\) that converge at infinity are said to be equivalent, if
\[
\lim_{i \to \infty} (x_i|y_i)_w = \infty.
\]
This defines an equivalence relation among all sequences that converge at infinity \([CDP, \S 1, \text{Chapter 2}]\). As before, is easy to check that the notion of equivalent sequences does not depend on the choice of the base point \(w\). The boundary \(\partial Z\) of \((Z,d)\) is defined as the set of equivalence classes of sequences converging at infinity under the above equivalence relation. If there are multiple hyperbolic metrics on the same set \(Z\), to avoid confusion, we denote the boundary of \((Z,d)\) by \(\partial(Z,d)\). The notion of Gromov product can be defined on the boundary as follows: for all \(a, b \in \partial Z\)
\[
(a|b)_w = \sup \left\{ \lim \inf_{i \to \infty} (x_i|y_i)_w : \{x_i\} \in a, \{y_i\} \in b \right\}.
\]
By \([GH90, \text{Remarque 8, Chapitre 7}]\), if \(\{x_i\} \in a, \{y_i\} \in b\), we have
\[
(a|b)_w - 2\delta \leq \lim \inf_{i \to \infty} (x_i|y_i)_w \leq (a|b)_w.
\]
The boundary \(\partial Z\) of the hyperbolic space \((Z,d)\) carries a family of metrics. A metric \(\rho : \partial Z \times \partial Z \to [0, \infty)\) on \(\partial Z\) is said to be a visual metric with base point \(w \in Z\) and visual parameter \(\alpha \in (1, \infty)\) if there exists \(k_1, k_2 > 0\) such that
\[
k_1\alpha^{-(a|b)_w} \leq \rho(a, b) \leq k_2\alpha^{-(a|b)_w}.
\]
If a visual metric with base point \(w\) and visual parameter \(\alpha\) exists, then it can be chosen to be
\[
\rho_{\alpha,w}(a, b) := \inf \sum_{i=1}^{n-1} \alpha^{-(a_i|a_{i+1})_w},
\]
where the infimum is over all finite sequences \(\{a_i\}_{i=1}^n \subset \partial Z, n \geq 2\) such that \(a_1 = a, a_n = b\). Any other visual metric with the same basepoint and visual parameter is bi-Lipschitz equivalent to \(\rho_{\alpha,w}\).
Visual metrics exist on hyperbolic metric spaces as we recall now. A metric space \((Z, d)\) is said to be proper if all closed balls are compact. For any \(\delta\)-hyperbolic space \((Z, d)\), there exists \(\alpha_0 > 1\) (\(\alpha\) depends only on \(\delta\)) such that if \(\alpha \in (1, \alpha_0)\), then there exists a visual metric with parameter \(\alpha\) [GH90, Chapitre 7], [BoSc, Lemma 6.1]. It is well-known that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry on their boundaries (the notion of almost geodesic space is given in Definition 2.1). Since this plays a central role in our construction of metric, we recall the relevant definitions and results below.

We say that a map (not necessarily continuous) \(f : (X_1, d_1) \to (X_2, d_2)\) between two metric spaces is a quasi-isometry if there exists constants \(A, B > 0\) such that

\[
A^{-1}d_1(x, y) - A \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + A,
\]

for all \(x, y \in X_1\), and

\[
\sup_{x_2 \in X_2} d(x_2, f(X_1)) = \sup_{x_2 \in X_2} \inf_{x_1 \in X_1} d(x_2, f(x_1)) \leq B.
\]

**Definition 2.1.** A metric space \((Z, d)\) is \(k\)-almost geodesic, if for every \(x, y \in Z\) and every \(t \in [0, d(x, y)]\), there is some \(z \in Z\) with \(|d(x, z) - t| \leq k\) and \(|d(y, z) - (d(x, y) - t)| \leq k\). We say that a metric space is almost geodesic if it is \(k\)-almost geodesic for some \(k \geq 0\).

Quasi-isometry between hyperbolic spaces induce quasisymmetries on their corresponding boundaries. We recall a result due to Bonk and Schramm below.

**Proposition 2.2 ([BoSc, Theorem 6.5 and Proposition 6.3]).** Let \((Z_1, d_1)\) and \((Z_2, d_2)\) be two almost geodesic, \(\delta\)-hyperbolic metric spaces. Let \(f : (Z_1, d_1) \to (Z_2, d_2)\) be a quasi-isometry.

(a) If \(\{x_i\} \subset Z_1\) converges at infinity, then \(\{f(x_i)\} \subset Y\) converges at infinity. If \(\{x_i\}\) and \(\{y_i\}\) are equivalent sequences in \(X\) converging at infinity, then \(\{f(x_i)\}\) and \(\{f(y_i)\}\) are also equivalent.

(b) The map \(\partial f : \partial Z_1 \to \partial Z_2\) given by \(\partial f (\{x_i\}) = \{f(x_i)\}\) is well-defined, and is a bijection.

(c) Let \(p_1 \in Z_1\) be a base point in \(Z_1\), and let \(f(p_1)\) be a corresponding base point in \(Z_2\). Let \(\rho_1, \rho_2\) denote visual metrics (with not necessarily the same visual parameter) on \(\partial Z_1, \partial Z_2\) with base points \(p_1, f(p_1)\) respectively. Then the induced boundary map \(\partial f : (\partial Z_1, \rho_1) \to (\partial Z_2, \rho_2)\) is a power quasisymmetry.

### 2.2 Geodesic hyperbolic spaces

Let \((Z, d)\) by a geodesic \(\delta\)-hyperbolic metric space. Recall that \((Z, d)\) is geodesic if for any \(x, y \in X\), there exists a curve \(\gamma : [0, d(x, y)] \to Z\) such that \(\gamma(0) = x, \gamma(d(x, y)) = y\) and \(d(\gamma(s), \gamma(t)) = |s - t|\) for all \(s, t \in [0, d(x, y)]\). Such a curve is called a geodesic between \(x\)
and $y$. For $x, y \in Z$, we denote by $[x, y]$ a geodesic between $x$ and $y$. For $x, y, z \in Z$, we denote by $[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$ a geodesic triangle in $Z$. Recall that a tripod is a metric tree with three edges arising from a common ‘center’ vertex. By Given a geodesic triangle $\Delta = [x, y, z]$, there exists a map $f:\Delta \to T_\Delta$ from $\Delta$ to a tripod $T_\Delta$ such that the restriction of $f_\Delta$ to each side of the triangle is an isometry [GH90, Proposition 2 and Definition 16, Chapitre 2]. The preimages of the ‘central vertex’ is called the inscribed triple.

Unlike a tripod, a geodesic triangle $\Delta$ need not have a canonical center. However, it has a reasonable notion of approximate center. We introduce the notion of a $K$-approximate center of a geodesic triangle. A point $c \in Z$ is a $K$-approximate center of a geodesic triangle $[x, y, z]$ if $c$ is at a distance at most $K$ from each of the three sides; that is, $d(c, [x, y]) \vee d(c, [y, z]) \vee d(c, [z, x]) \leq K$. The following proposition concerns a few properties of approximate center.

**Proposition 2.3.** Let $(Z, d)$ be a geodesic, $\delta$-hyperbolic metric space.

(a) Each point of the inscribed triple of a geodesic triangle is a $4\delta$-approximate center.

(b) Any two $K$-approximate centers $c$ and $c'$ of a geodesic triangle $[x, y, z]$ satisfies $d(c, c') \leq 8K$.

(c) If $c$ is a $K$-approximate center of a geodesic triangle $[x, y, z]$ then

$$|d(x, c) - (y|z)| \leq 4K.$$

(d) If $f: (Z_1, d_1) \to (Z_2, d_2)$ is a quasi-isometry between two geodesic $\delta$-hyperbolic metric spaces and if $c$ is a $K_1$-approximate center of $[x, y, z]$ then $f(c)$ is a $K_2$-approximate center of any geodesic triangle $[f(x), f(y), f(z)]$, where $K_2$ depends only on $K_1, \delta$ and the constants associated with the quasi-isometry $f$. In particular,

$$|d_2(f(x), f(c)) - (f(y)|f(z))| \leq 4K_2.$$

**Proof.**

(a) By [GH90, Proposition 21, Chapitre 2] each point of the inscribed triple is a $4\delta$-approximate center.

(b,c) Let $c$ denote a $K$-approximate center of $[x, y, z]$. Let $p_1, p_2, p_3$ be the points of the inscribed triple on $[x, y], [y, z], [z, x]$ respectively. Similarly, let $q_1, q_2, q_3$ be three points on $[x, y], [y, z], [z, x]$ respectively such that $d(c, q_i) \leq A$ for all $i = 1, 2, 3$. This implies that $d(q_i, q_j) \leq 2K$ for all $i, j$. By the argument in [GH90, Proof of Lemme 20, Chapitre 2] we have

$$d(p_i, q_i) \leq 3K \quad \text{for all } i = 1, 2, 3. \quad (2.1)$$

Since $d(x, p_1) = (y|z)x$ and $d(p_1, q_1) \leq 3K$, we obtain

$$|d(x, c) - (y|z)| = |d(x, c) - d(x, p_1)| \leq d(p_1, q_1) + d(c, q_1) \leq 3K + K = 4K.$$

This concludes the proof of (b). Similarly, $d(c, p_i) \leq d(p_i, q_i) + d(c, q_i) \leq 4K$ for all $i = 1, 2, 3$. Therefore $d(c, c') \leq d(c, p_1) + d(c', p_1) \leq 8K$. 

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This is an immediate consequence of the geodesic stability under quasi-isometries [GH90, Théorème 11, Chapitre 5] and (c).

2.3 Construction of hyperbolic filling

In this section, we recall the construction of a hyperbolic filling of a compact metric space. Let \((X, d)\) be a compact metric space. The construction below is due to A. Bjorn, J. Bjorn and Shanmugalingam [BBS]. Earlier versions of this construction are due to Elek, Bourdon and Pajot [Ele, BP]. Let \(\lambda, a \in (1, \infty)\) be two parameters which we call the horizontal and vertical parameter of the hyperbolic filling respectively. We assume that the diameter is normalized so that \(\text{diam}(X, d) = \frac{1}{2}\). Let \(X_n, n \in \mathbb{N}_{\geq 0}\) be an increasing sequence of maximal \(a^{-n}\)-separated subsets of \(X\). In other words, \(X_n \subset X_m\) for all \(n < m\), any two distinct points in \(X_n\) have mutual distance at least \(a^{-n}\) and any set strictly larger than \(X_n\) has two distinct points whose distance is strictly less than \(a^{-n}\). The vertex set of the graph is \(S = \bigcup_{n \geq 0} S_n\), where \(S_n = \{(x, n) : x \in X_n\}\). Two distinct vertices \((x, n), (y, m) \in S\) are joined by an edge if and only if either \(n = m\) and \(B(x, \lambda a^{-n}) \cap B(y, \lambda a^{-m}) \neq \emptyset\) or if \(|n - m| = 1\) and \(B(x, a^{-n}) \cap B(y, a^{-m}) \neq \emptyset\). This defines a metric \(D_1\) on \(S\) which is the combinatorial (graph) distance defined by the above set of edges. That is \(D_1((x, n), (y, m))\) is the minimal number \(k\) such that \((x, n) = (x_0, n_0), (x_1, n_1), \ldots, (x_k, n_k) = (y, m)\), where \((x_i, n_i), (x_{i+1}, n_{i+1}) \in S\) is joined by an edge for all \(S, i = 0, 1, \ldots, k - 1\). It is evident that \((S, D_1)\) is 1-almost geodesic metric space.

We now construct a graph with fewer vertical edges. For each \((x, n) \in S\), we choose \((y, n - 1) \in S_n\) such that \(d(x, y) = \min_{z \in X_{n-1}} d(x, z)\). In this case, we say that \((y, n - 1)\) is the parent of \((x, n)\) or equivalently, \((x, y)\) is a child of \((y, n - 1)\). Such a choice of \(y \in X_{n-1}\) is not unique but we fix this choice for the remainder of this work. Since \(X_{n-1}\) is a maximal \(a^{-(n-1)}\) separated subset of \(X\), \(d(x, y) \leq a^{-(n-1)}\). Hence \(x \in B(x, a^{-n}) \cap B(y, a^{-(n-1)}) \neq \emptyset\). In other words every parent and their child is connected by an edge in the graph associated with \((S, D_1)\). We define a new graph whose edges consists of all of the edges between parent and child and those between \((x, n), (y, n) \in S\) where \(B(x, \lambda a^{-n}) \cap B(y, \lambda a^{-n}) \neq \emptyset, x \neq y\). The corresponding graph distance is denoted by \(D_2\).

If \((x, n+1)\) and \((y, n+1)\) share an edge in \(D_2\) and if \((x_0, n)\) and \((y_0, n)\) are their respective parents, then \(d(x_0, y_0) \leq d(x, y) + d(x, x_0) + d(y, y_0) < 2a^{-n} + 2\lambda a^{-n-1}\). Under the assumption \(\lambda \geq 2 + 2\lambda a^{-1}\), we have \(D_2((x, n+1), (y, n+1)) \leq 1, D_2((x_0, n), (y_0, n)) \leq 1\) whenever \((x_0, n), (y_0, n)\) are the parents of \((x, n + 1), (y, n + 1)\) respectively. We say that \((x, n)\) is a descendant of \((y, k)\) if \(n > k\), and there exists \((z_j, n_j) \in S\) for \(j = 0, \ldots, n - k\) such that \((z_0, n_0) = (y, k), (z_{n-k}, n_{n-k}) = (x, n)\), where \((z_{i+1}, n_{i+1})\) is a child of \((z_i, n_i)\) for all \(i = 0, \ldots, n - k - 1\). Combining the above observations yields the following lemma which is an analogue of [Car, Lemma 2.2].

Lemma 2.4. Let \(\lambda, a > 1\) be horizontal and vertical parameters of the hyperbolic filling respectively.
(a) If \((z,n+1)\) is a child of \((x,n)\), then \(d(x,z) < a^{-n}\). If \((y,k)\) is a descendant of \((x,n)\) (for some \(k > n\)), then
\[
d(x,y) < \frac{a}{a-1} a^{-n}.
\]

(b) If \(\lambda \geq 2 + 2\lambda a^{-1}\) and \(D_2((x,n+1),(y,n+1)) \leq 1\), then \(D_2((x_0,n),(y_0,n)) \leq 1\), where \((x_0,n),(y_0,n)\) are the parents of \((x,n+1),(y,n+1)\) respectively. Similarly, if \(\lambda \geq 2 + 4\lambda a^{-1}\) and \(D_2((x,n+1),(y,n+1)) \leq 1\), then \(D_2((x_0,n),(z_0,n)) \leq 1\), \(D_2((y_0,n),(z_0,n)) \leq 1\), where \((x_0,n),(y_0,n),(z_0,n)\) are the parents of \((x,n+1),(y,n+1),(z,n+1)\).

(c) Let \(\lambda \geq 6\). If \((x,n+1),(y,n+1) \in S_{n+1}\) such that \(d(x,y) \leq 4a^{-n}\). If \((x_0,n),(y_0,n) \in S_n\) are the parents of \((x,n+1),(y,n+1)\) respectively, then \(D_2((x_0,n),(y_0,n)) \leq 1\).

(d) If \(\lambda > 1 + a^{-1}\), we have \(D_1 \leq D_2 \leq 2D_1\).

(e) Let \(\lambda > 1 + a^{-1}\). Let \(w \in S_{n+1}\) and \(u,v \in S_n\) be such that \(D_1(u,w) = 1\) and \(D_2(v,w) = 1\). Then \(D_2(u,v) \leq 1\).

Proof.

(a) Since \(X_n\) is maximal \(a^{-n}\)-subset of \(X\), every point \(z \in X\) satisfies \(d(z,X_n) < a^{-n}\). This shows the first claim. If \((y,k)\) is a descendant of \((x,n)\) by the first claim and triangle inequality \(d(x,y) \leq \sum_{i=n}^{k} a^{-i} < \frac{a}{a-1} a^{-n}\).

(b) If \((x,n+1)\) and \((y,n+1)\) share an edge in \(D_2\) and if \((x_0,n)\) and \((y_0,n)\) are their respective parents, then \(d(x_0,y_0) \leq d(x,y) + d(x,x_0) + d(y,y_0) < 2a^{-n} + 2\lambda a^{-n-1}\). Hence \(D_2((x_0,n),(y_0,n)) \leq 1\) (since \(\{x_0,y_0\} \subset B(x_0,\lambda a^{-n}) \cap B(y_0,\lambda a^{-n}) \neq \emptyset\) The other claim follows from a similar argument.

(c) Since \(d(x_0,y_0) \leq d(x,y) + d(x,x_0) + d(y,y_0) < 4a^{-n} + a^{-n} + a^{-n} = 6a^{-n}\), we have \(\{x_0,y_0\} \subset B(x_0,\lambda a^{-n}) \cap B(y_0,\lambda a^{-n}) \neq \emptyset\) whenever \(\lambda \geq 6\). Therefore \(D_2((x_0,n),(y_0,n)) \leq 1\).

(d),(e) Since every edge in the graph corresponding to \((\mathcal{S},D_2)\) is contained in the graph corresponding to \((\mathcal{S},D_2)\), we have \(D_2 \geq D_1\).

On the other hand, if there is an edge in \((\mathcal{S},D_1)^2\) which is not present in \((\mathcal{S},D_2)\), then it must be between some \((x,n),(y,n+1)\) such that \(n \in \mathbb{N}_{\geq 0}\) and that the parent of \((y,n+1)\) is \((z,n)\) where \(z \neq x\). In this case \(d(x,y) \leq a^{-n-1} + a^{-n}\) (since \(B(x,a^{-n}) \cap B(y,a^{-n-1}) \neq \emptyset\)). Therefore if \(\lambda > 1 + a^{-1}\), there would be an edge between \((x,n)\) and \((z,n)\) in both graphs (since \(y \in B(x,\lambda a^{-n}) \cap B(z,\lambda a^{-n})\)). This implies that
\[
D_1 \leq D_2 \leq 2D_1, \quad \text{whenever } \lambda \geq 1 + a^{-1}.
\]  

\footnote{Here we abuse notation and use \((\mathcal{S},D_i)\) to denote the graph, for \(i = 1,2\).}
We recall the relevant properties the metric spaces \((S, D_1)\) and \((S, D_2)\). By the choice of the diam\((X, d)\), there is an unique point \(x_0 \in X_0\). We choose \(v_0 := (x_0, 0)\) as the base point of the metric spaces \((S, D_1)\) and \((S, D_2)\). We denote the Gromov product with respect to the basepoint \(v_0\) in \((S, D_1)\) and \((S, D_2)\) by \(\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2\) respectively. The key point in the following result is that the hyperbolicity constant \(\delta\) depends only on \(a\) and \(\lambda\) unlike the analogous result in [BP, Car] where \(\delta\) also depends on the constant associated with the uniform perfectness property (see [Car, Remark after Proposition 2.1]).

**Proposition 2.5.** (Cf. [BBS, Lemma 3.3 and Theorem 3.4]) Let \((X, d)\) be a compact metric space and let \(a, \lambda\) denote the vertical and horizontal parameters respectively of the hyperbolic filling. Then the hyperbolic filling \((S, D_1)\) satisfies the following properties

(a) For any \(v = (z, n), w = (y, m) \in S\), we have

\[
\frac{a - 1}{4\lambda a} (d(z, y) + a^{-n} + a^{-m}) \leq a^{-\langle v | w \rangle_1} \leq \frac{a^{5/2}}{\lambda - 1} (d(z, y) + a^{-n} + a^{-m})
\]

In particular, if \(a, \lambda \in [2, \infty)\) and \(a \geq \lambda\), then

\[
\left| \langle v | w \rangle_1 + \frac{\log(d(x, y) + a^{-m} + a^{-n})}{\log a} \right| \leq 4.
\]

(b) \((S, D_1)\) is \(\delta\)-hyperbolic, where \(\delta = \frac{\log\left(\frac{3\lambda a^{7/2}}{(a - 1)(\lambda - 1)}\right)}{\log a}\). In particular, if \(a, \lambda \in [2, \infty)\) and \(a \geq \lambda\) implies that \(\delta\) can be chosen to be 15.

(c) A sequence of vertices \(\{(x_i, n_i)\} \in S\) converges at infinity if and only if \(\lim n_i = \infty\) and \(\{(x_i)\}\) is a convergent sequence in \((X, d)\). Two sequences \(\{(x_i, n_i)\}\) and \(\{(y_i, m_i)\}\) that converge at infinity are equivalent if and only if \(\lim_{i \to \infty} x_i = \lim_{i \to \infty} y_i\) and \(\lim_{i \to \infty} n_i = \lim_{i \to \infty} m_i = \infty\). In particular, the map \(p : \partial(S, D_1) \to X\) defined by \(p(\{(x_i, n_i)\}) = \lim_{i \to \infty} x_i\) is well defined and is a bijection. Further more, \(d\) is a visual metric on \(\partial(S, D_1)\) with visual parameter \(a\) and base point \(v_0\) in the following sense:

\[
\frac{a - 1}{4\lambda a} d(x, y) \leq a^{-(p^{-1}(x)|p^{-1}(y))_1} \leq \frac{a^{5/2}}{\lambda - 1} d(x, y), \quad \text{for all } x, y \in X.
\]

**Proof.**

(a) The first estimate follows from [BBS, Proof of Lemma 3.3]. The second conclusion is a consequence of the estimate

\[
\max\left(\frac{\log(a^{5/2}/(\lambda - 1))}{\log a}, \frac{4\lambda a}{a - 1}\right) \leq 4 \quad \text{whenever } a \geq \lambda \geq 2.
\]
(b) The $\delta$-hyperbolicity follows from the proof of [BBS, Theorem 3.4] along with [CDP, Proposition 1.2]. For the second conclusion, observe that

$$2 \frac{\log \left( \frac{8\lambda a^{7/2}}{(a-1)(\lambda-1)} \right)}{\log a} \leq 2 \frac{\log(32a^{5/2})}{\log a} \leq 15, \quad \text{whenever } a \geq \lambda \geq 2.$$  

(c) This is an easy consequence of (a).

\[ \square \]

We would like to use Proposition 2.3 to estimate Gromov product in the hyperbolic filling. Since $(S, D_1)$, we embed into a geodesic space by replacing each edge with an isometric copy of the unit interval to obtain a metric space $(\tilde{S}, D_1)$ where we view $S \subset \tilde{S}$ and $D_1$ on $\tilde{S}$ is an extension of $D_1$ on $S$. The following lemma identifies approximate centers of certain geodesic triangles in $\tilde{S}$.

**Lemma 2.6.** Let the parameters of the hyperbolic filling satisfy

$$a \geq \lambda \geq 2, \quad \text{and} \quad \lambda \geq 1 + \frac{a}{a-1}.$$  

For $x, y \in X$, let $c(x, y)$ denote the set of all $(\tilde{z}, \tilde{n}) \in S$ such that $\tilde{n}$ is the maximal integer that satisfies $\{x, y\} \subset B(z, 2a^{-\tilde{n}})$. If $(z, m), (w, n) \in S$ such that $x \in B(z, a^{-m}), y \in B(w, a^{-n})$ and such that $a^{-m} + a^{-n} < a^{-2}d(x, y)$. Then any $(w, k) \in c(x, y)$ is a $K$-approximate center for any geodesic triangle $[v_0, (z, m), (w, n)]$ in $(\tilde{S}, D_1)$, where $K = 80$.

**Proof.** Since every point in $\tilde{S}$ is at most distance $\frac{1}{2}$ away from a point in $S$, by replacing points in $\tilde{S}$ with the corresponding closest points in $S$, we obtain that $(\tilde{S}, D_1)$ is $(\delta + 3)$-hyperbolic whenever $(S, D_1)$ is $\delta$-hyperbolic.

Let $(\tilde{z}, k) \in c(x, y)$. Since $\{x, y\} \subset B(\tilde{z}, 2a^{-k})$, we obtain $d(x, y) \leq 4a^{-k}$. Choose $w \in X$ such that $(w, k+1) \in S_{k+1}$ and $d(x, w) < a^{-(k+1)}$. By the maximality of $k$, we have that $y \notin B(w, 2a^{-(k+1)})$ and hence $d(x, y) \geq d(w, y) - d(w, x) > a^{-(k+1)}$. In particular,

$$a^{-(k+1)} < d(x, y) \leq 4a^{-k}. \quad (2.3)$$

If $a \geq 2$, we have $a^{-(k+1)} < d(x, y) \leq 4a^{-n} \leq a^{-k+2}$, which implies

$$-1 \leq k + \frac{\log d(x, y)}{\log a} \leq 2, \quad \text{whenever } a \geq 2. \quad (2.4)$$

Since $a^{-m} < a^{-2}d(x, y)$, we have $m + \frac{\log d(x, y)}{\log a} > 2$ which along with (2.4) implies that $m \geq k$. Choose $(w, k)$ such that $(z, m)$ is a descendant of $(w, k)$. By Lemma 2.4(a), we have $d(x, w) \leq d(x, z) + d(w, z) < a^{-m} + \frac{a}{a-1}a^{-k} \leq (1 + \frac{a}{a-1})a^{-k}$. Therefore if $\lambda \geq 1 + \frac{a}{a-1}$, we have $x \in B(w, \lambda a^{-k}) \cap B(\tilde{z}, \lambda a^{-k}) \neq \emptyset$. Hence $(w, k)$ and $(\tilde{z}, k)$ are either equal or horizontal neighbors.
Note that $|d(x, y) - d(z, w)| \leq a^{-m} + a^{-n} < a^{-2}d(x, y)$, which implies
\[(1 - a^{-2})d(x, y) \leq d(z, w) + a^{-m} + a^{-n} \leq (1 + 2a^{-2})d(x, y)\]
Therefore if $a \geq \lambda \geq 2$, we have
\[\left|\frac{\log(d(x, y)) - \log(d(z, w) + a^{-m} + a^{-n})}{\log a}\right| \leq 1.\]
Combining with 2.4 and Proposition 2.5(a), we obtain that
\[\|((z, m)|(w, n))_1 - k| \leq 7.\]
This along with Proposition 2.3(a) and the fact that $(\tilde{z}, k)$ is a neighbor of $(w, k)$, we obtain that $(\tilde{z}, k)$ is $K$-approximate center of the geodesic triangle $[v_0, (z, m), (w, n)]$, where $K = 1 + 7 + 4(15 + 3) = 80.$

\[\square\]

**Remark 2.7.** The assumption
\[a \geq \lambda \geq 6 \quad (2.5)\]
implies the estimates assumed on $a, \lambda$ in Lemma 2.4, Proposition 2.5, and Lemma 2.6 hold. For this reason we assume (2.5) for much of this work. The analogous estimate [Car, (2.8)] is more complicated because it involves the constant in the definition of uniform perfectness.

### 3 Construction of metric and homogeneous measure

In this section, we construct metric and homogeneous measure using a weight function on the hyperbolic filling $\mathcal{S}$ as constructed in §2.3.

Given a vertex $v$ let $\pi_1 : \mathcal{S} \to X, \pi_2 : \mathcal{S} \to \mathbb{N}_{\geq 0}$ denote the projection maps such that $v = (\pi_1(v), \pi_2(v))$. We say that an edge between two vertices $v$ and $w$ is horizontal if $\pi_2(v) = \pi_2(w)$. For a vertex $v \in \mathcal{S}$, by $B_v$ we denote the metric ball $B(\pi_1(v), a^{-\pi_2(v)})$.

Given a vertex $v \in \mathcal{S}$, we define the genealogy $g(v)$ as a sequence of vertices $(v_0, v_1, \ldots, v_k)$ where $v_k = v$ and $v_i$ is the parent of $v_{i+1}$ for all $i = 0, \ldots, k - 1$ and $v_0$ is the unique vertex in $\mathcal{S}_0$. Given a function $\rho : \mathcal{S} \to (0, \infty)$, we define $\pi : \mathcal{S} \to (0, \infty)$ as
\[\pi(v) = \prod_{w \in g(v)} \rho(w). \quad (3.1)\]

A path $\gamma$ in $(\mathcal{S}, D_2)$ is a sequence of vertices $\gamma = (w_1, \ldots, w_n)$ where there is an edge between $w_i$ and $w_{i+1}$ (that is, $D_2(w_i, w_{i+1}) = 1$) for each $i = 1, \ldots, n - 1$. The $\rho$-length of a path $\gamma$ is defined by
\[L_\rho(\gamma) = \sum_{v \in \gamma} \pi(v). \quad (3.2)\]
The following two families of paths will play an important role in this work. A path is said to be horizontal if it only consists of horizontal edges. Given $x, y \in X$ and $n \in \mathbb{N}_{\geq 0}$, we define

$$\Gamma_n(x, y) = \left\{ \gamma = (v_1, \ldots, v_k) \mid \gamma \text{ is a path in } (\mathcal{S}, D_2), \pi_2(v_1) = \pi_2(v_k) = n, \text{ and } x \in B_{v_1}, y \in B_{v_k}, k \in \mathbb{N} \right\}. \quad (3.3)$$

For a vertex $v \in \mathcal{S}_k$, we define

$$\Gamma_k(v) = \inf \left\{ \gamma = (v_1, v_2, \ldots, v_n) \mid \gamma \text{ is a horizontal path with } \pi_2(v_i) = k + 1 \text{ for all } i, n \in \mathbb{N}, \pi_1(v_1) \in B_v, \pi_1(v_n) \notin 2 \cdot B_v \right\}. \quad (3.4)$$

For $x, y \in X$, we define

$$\pi(c(x, y)) = \max \{ \pi(w) : w \in c(x, y) \}, \quad (3.5)$$

where $c(x, y)$ is as defined in Lemma 2.6. We recall the conditions imposed on the weight function $\rho : \mathcal{S} \to (0, 1)$.

**Assumption 3.1.** A weight function $\rho : \mathcal{S} \to (0, 1)$ may satisfy some of the following hypotheses:

(H1) There exist $0 < \eta_\ast \leq \eta_+ < 1$ so that $\eta_\ast \leq \rho(B) \leq \eta_+$ for all $B \in \mathcal{S}$.

(H2) There exists a constant $K_0 \geq 1$ such that for all $v, w \in \mathcal{S}$ that share a horizontal edge, we have

$$\pi(B) \leq K_0 \pi(B'),$$

where $\pi$ is as defined in (3.1).

(H3) There exists a constant $K_1 \geq 1$ such that for any pair of points $x, y \in X$, there exists $n_0 \geq 1$ such that if $n \geq n_0$ and $\gamma$ is a path in $\Gamma_n(x, y)$, then

$$L_\rho(\gamma) \geq K_1^{-1} \pi(c(x, y)),$$

where $\Gamma_n(x, y), L_\rho, \pi(c(x, y))$ are as defined in (3.3), (3.2), and (3.5) respectively.

(H4) There exists $p > 0$ such that for all $v \in \mathcal{S}_m$, and $n > m$, we have

$$\sum_{w \in D_n(v)} \pi(w)^p \leq \pi(v)^p,$$

where $D_n(v)$ denotes the descendants of $v$ in $\mathcal{S}_n$. Clearly, it suffices to impose the above condition only for $n = m + 1$.

The following is an analogue of [Car, Theorem 1.1] but the hypothesis (H4) is different from that of [Car, Theorem 1.1]. As explained in the introduction, [Sha, Lemm 6.2] implies that (H1) along with the version of (H4) in [Car] can hold only on a a uniformly perfect
metric space. Since we consider metric spaces that are not necessarily uniformly perfect, we need to change (H4) as above. This hypothesis plays a key role in the upper bound on Assouad dimension. Another distinction from [Car] is that the weights can be used to construct essentially all metrics in \( J(X, d) \). Since this method which relies on Proposition 2.2(c) can only construct metrics in \( J_p(X, d) \), we cannot obtain such a result.

**Theorem 3.2.** (Cf. [Car, Theorem 1.1]) Let \((X, d)\) be a compact doubling metric space and let \( S \) be a hyperbolic filling with vertical and horizontal parameters \( a, \lambda \) such that \( a \geq \lambda \geq 6 \). Let \( \rho : S \to (0, 1) \) be a weight function satisfying the hypothesis (H1), (H2), (H3), and (H4). Then there exists \( \Theta_\rho \in J(X, d) \) such that the identity map \( \text{Id} : (X, d) \to (X, \Theta_\rho) \) is a power quasisymmetry and \( d_A(X, \Theta_\rho) \leq p \).

The weight function \( \rho \) which satisfies the hypotheses (H1)-(H4) induces a metric \( D_\rho \) on \( S \). Each edge \( e = (v, w) \) has a length \( \ell_\rho(e) \) where

\[
\ell_\rho(e) = \begin{cases} 
2 \max\{-\log \eta_-, (-\log \eta_+)^{-1}, \log K_0\} & \text{if } e \text{ is horizontal}, \\
\log \frac{1}{\rho(v)} & \text{if } w \text{ is the parent of } v,
\end{cases}
\]

where \( \eta_-, \eta_+, K_0 \) are as defined in (H1) and (H2). This defines a metric

\[
D_\rho(v, w) = \inf \sum_{e \in \gamma} \ell_\rho(e),
\]

where \( \gamma \) varies over all paths in the graph \((S, D_2)\) from \( v \) to \( w \) and \( e \) varies over all edges in \( \gamma \). Similar to \((\tilde{S}, D_1)\), by replacing each edge \( e \) is an isometric copy of the interval \([0, \ell_\rho(e)]\), we can construct a geodesic metric space \((\tilde{S}, D_\rho)\) such that \( S \subset \tilde{S} \) and the restriction of \( D_\rho \) on \( \tilde{S} \) coincides with that of \( S \).

**Lemma 3.3.** Let \((X, d)\) and \( S \) be as in the statement of Theorem 3.2. Let \( \rho : S \to (0, \infty) \) which satisfies hypotheses (H1) and (H2). Let \( D_\rho \) denote the metric defined in (3.6).

(a) \((S, D_\rho)\) is approximately geodesic. The identity map \( \text{Id} : (S, D_1) \to (S, D_\rho) \) is a quasi-isometry.

(b) An path of the form \(((x_i, n_i))_{1 \leq i \leq k} \) such that \((x_i, n_i)\) is the parent of \((x_{i+1}, n_{i+1})\) for all \( i = 1, \ldots, k - 1 \) defines a shorted path in the \( D_\rho \) metric and hence

\[
D_\rho((x_1, n_1), (x_k, n_k)) = \left| \log \frac{1}{\pi((x_1, n_1))} - \log \frac{1}{\pi((x_k, n_k))} \right|.
\]

(c) A sequence of vertices converges at infinity in \((S, D_1)\) if and only if it converges at infinity \((S, D_\rho)\). Two sequences that converge at infinity are equivalent in \((S, D_1)\) if and only if they are equivalent in \((S, D_\rho)\). In particular, there is a canonical bijection between \( \partial(S, D_1) \) and \( \partial(S, D_\rho) \). The Gromov product satisfies

\[
e^{-\langle p^{-1}(x)\rho^{-1}(y)\rangle} \leq \pi(c(x, y)), \quad \text{for all } x, y \in X,
\]

where \( \langle \cdot, \cdot \rangle_\rho \) is the Gromov product on \((S, D_\rho)\) with base point \( v_0 \in S_0 \).
(d) There exists $C > 0$ such that the following holds: for any pair of distinct points $x, y \in X$, there exists $n_0$ such that whenever $n \geq n_0$ and $u, v \in S_n$ such that $x \in B_u, y \in B_v$, there exists a path $\gamma = (w_i)_{i=0,\ldots,k}$ in the graph $(S, D_2)$ such that $L_\rho(\gamma) \leq C\pi(c(x, y))$.

Proof.

(a) It is easy to check that $2 \max \{-\log \eta_-, (\log \eta_+)^{-1}, \log K_0\}$-approximately geodesic, since the horizontal edges are the longest edges. The fact that the identity map is a quasi-isometry is because there exist constants $C_1, C_2$ such that $C_1 \leq \ell_\rho(e) \leq C_2$ for all edges $e$. This along with Lemma 2.4(d) implies that $C_1D_1 \leq C_1D_2 \leq D_\rho \leq C_2D_2 \leq 2C_2D_1$.

(b) This follows from the same argument as [Car, Proof of Lemma 2.3] which uses Lemma 2.4(b).

(c) The first three claims follow from Proposition 2.2. Let $(v_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}$ be two sequences of vertices such that $x \in B_{v_n}, y \in B_{v_n}, v_n \in S_n, w_n \in S_n$ for all $n \in \mathbb{N}$. By Lemma 2.6, every vertex in $c(x, y)$ is a 80-approximate center for the geodesic triangle $[v_0, v_n, w_n]$ (in $(\tilde{S}, D_1)$) for all large enough $n$. By (a) and Proposition 2.3(d), every vertex in $c(x, y)$ is a $K'$-approximate center pf the geodesic triangle $[v_0, v_n, w_n]$ (in $(\tilde{S}, D_\rho)$) for some $K' > 0$. By Proposition 2.3(d), we obtain the desired estimate.

(d) Let $x, y \in X$ and $u, v \in S_n$ be as in the statement of the lemma. We choose $n_0$ be the integer such that $c(x, y) \subset S_{n_0}$. Let $\tilde{u}, \tilde{v} \in S_{n_0}$ be the vertices such that $u, v$ are descendants of $\tilde{u}, \tilde{v}$ respectively. Let $\tilde{c} \in c(x, y)$. As shown in the proof of Lemma 2.6, either $\tilde{u}$ (resp. $\tilde{v}$) is equal to $\tilde{c}$ or is a horizontal neighbor of $\tilde{u}$ (resp. $\tilde{v}$). Therefore by (H2), $\pi(\tilde{u}) \vee \pi(\tilde{v}) \leq K_0\pi(c(x, y))$. We now construct the desired path $\gamma$ from $u$ to $v$ as follows. We join $u$ to $\tilde{u}$ and $\tilde{v}$ to $v$ using the geneology. We can connect $\tilde{u}$ to $\tilde{v}$ using $\tilde{c}$ if necessary. By (H1) and (H2), the length of $\gamma$ is bounded by

$$\pi(c(x, y)) \left(1 + \frac{2K_0}{1 - \eta_+}\right).$$

\[\square\]

The following proposition provides a bound on visual parameter on $\partial(S, D_\rho)$ and relies crucially on (H3). This construction of metric is slightly different from that of [Car] and [Sha, Theorem 5.1].

**Proposition 3.4** (Visual parameter control). Let $(X, d)$ and $S$ be as in the statement of Theorem 3.2. Let $\rho : S \to (0, \infty)$ which satisfies hypotheses (H1), (H2), and (H3). Then there exists a visual metric $\theta_\rho$ on $\partial(S, D_\rho)$ with base point $v_0 \in S_0$ with visual parameter $e$, where $D_\rho$ is as defined in (3.6). This metric satisfies

$$\theta_\rho(p^{-1}(x), p^{-1}(y)) \asymp e^{-p^{-1}(x)p^{-1}(y)} \asymp \pi(c(x, y)).$$

Furthermore, the map $p : (\partial(S, D_\rho), \theta_\rho) \to (X, d)$ is a power quasisymmetry.
Proof. The second estimate follows from Lemma 3.3(c). It remains to show that such a metric exists. To this end, we define

\[ \theta_\rho(p^{-1}(x), p^{-1}(y)) = \inf \left\{ \sum_{i=0}^{k-1} \pi(c(x_i, x_{i+1})) : k \in \mathbb{N}, x_0 = x, x_k = y, x_i \in X \text{ for all } i \right\}. \]

Clearly, \( \theta_\rho \) is non-negative, satisfies the triangle inequality and \( \theta_\rho(p^{-1}(x), p^{-1}(y)) \leq \pi(c(x, y)) \) for all \( x, y \in X \). It suffices to show

\[ \theta_\rho(p^{-1}(x), p^{-1}(y)) \gtrsim \pi(c(x, y)) \]

To this end consider a sequence \( x_0, \ldots, x_k \) such that \( x_0 = x, x_k = y \). Without loss of generality, we may assume that \( x_i \neq x_{i+1} \) for all \( i = 0, \ldots, k - 1 \). We choose a \( n \) large enough so that we can apply Lemma 3.3(d) to each pair \( x_i, x_{i+1}, i = 0, \ldots, k - 1 \). Choose \( v_i \in \mathcal{S}_n \) such that \( x_i \in B_{v_i} \) for all \( i = 0, \ldots, k \). By concatenating all points obtained by applying Lemma 3.3(d) to each pair \( x_i, x_{i+1} \), we obtain a path \( \gamma \in \Gamma_n(x, y) \) such that

\[ L_\rho(\gamma) \leq C \sum_{i=0}^{k-1} \pi(c(x_i, x_{i+1})), \]

where \( C \) is the constant from Lemma 3.3(d). Combining the above estimate with (H3), we obtain

\[ (K_1 C)^{-1} \pi(c(x, y)) \leq \theta_\rho(p^{-1}(x), p^{-1}(y)) \leq \pi(c(x, y)), \]

(3.7)

where \( K_1 \) is the constant in (H3). The conclusion that \( \rho : (\partial(S, D_\rho), \theta_\rho) \to (X, d) \) is a power quasisymmetry follows from Proposition 2.5(c), Lemma 3.3(a) and Proposition 2.2. \( \square \)

Under the hypotheses of Lemma 3.4, the metric \( \Theta_\rho : X \times X \to [0, \infty) \) defined by

\[ \Theta_\rho(x, y) := \theta_\rho(p^{-1}(x), p^{-1}(y)) \] (3.8)

is the conformal gauge and the identity map \( \text{Id} : (X, d) \to (X, \Theta_\rho) \) is a power quasisymmetry. Next, we need to control the Assouad dimension of \((X, \Theta_\rho)\). We will use Theorem 1.1 to obtain an upper bound on the Assouad dimension of \((X, \Theta_\rho)\). To this end, we construct a doubling measure on \( X \) using the weight function \( \rho \). We need the following modification of a lemma of Vol’berg and Konyagin [VK, Lemma, p. 631] which plays a key role in the construction of doubling measures.

**Lemma 3.5.** Let \((X, d)\) and \( \mathcal{S} \) be as in the statement of Theorem 3.2. Let \( \rho : \mathcal{S} \to (0, \infty) \) be function that satisfies hypotheses (H1) and (H4). Let \( C \geq \eta_-^p \), where the constants \( \eta_- \), \( p \) are as given in the hypotheses (H1) and (H4). Let \( k \in \mathbb{N}_{\geq 0} \), and let \( \mu_k \) be a probability mass function on \( \mathcal{S}_k \) such that

\[ \frac{\mu_k(u)}{\pi(u)^p} \leq C^2 \frac{\mu_k(u')}{\pi(u')^p} \quad \text{for all } u, u' \in \mathcal{S}_k \text{ with } D_2(u, u') \leq 1. \] (3.9)

Then there exists a probability mass function \( \mu_{k+1} \) on \( \mathcal{S}_{k+1} \) such that the following hold.
(1) For all \( v, v' \in S_{k+1} \) with \( D_2(v, v') \leq 1 \) we have
\[
\frac{\mu_{k+1}(v)}{\pi(v)^p} \leq C_2 \frac{\mu_{k+1}(v')}{\pi(v')^p}.
\] (3.10)

(2) For all points \( u \in S_k \) and \( v \in S_{k+1} \) such that \( u \) is the parent of \( v \),
\[
C^{-1} \mu_k(u) \leq \frac{\mu_{k+1}(v)}{\pi(v)^p} \leq \frac{\mu_k(u)}{\pi(u)^p}.
\] (3.11)

(3) The construction of the measure \( \mu_{k+1} \) from the measure \( \mu_k \) can be regarded as the transfer of masses from the points of \( X_k \) to those of \( X_{k+1} \), with no mass transferred over a distance greater than \( (1 + 2\lambda a^{-1})a^{-k} \). More precisely, there is a probability measure \( \mu_{k,k+1} \) on \( X \times X \) which coupling of the probability measures
\[
\tilde{\mu}_k := \sum_{w \in S_k} \mu_k(u) \delta_{\pi_1(x)}, \quad \tilde{\mu}_{k+1} := \sum_{v \in S_{k+1}} \mu_{k+1}(v) \delta_{\pi_1(v)}
\] such that
\[
\mu_{k,k+1} \left( \{(x_1, x_2) \in X \times X : d(x_1, x_2) \geq (1 + 2\lambda a^{-1})a^{-k} \} \right) = 0,
\]
where \( \delta_x \) denotes the Dirac measure at \( x \in X \). Here by a coupling we mean the projection maps from \( X \to X \) to \( X \) to the first and second component pushes forward the measure \( \mu_{k,k+1} \) to \( \tilde{\mu}_k \) and \( \tilde{\mu}_{k+1} \) respectively.

Proof. Let \( k \in \mathbb{N}_{\geq 0} \), and let \( \mu_k \) be any probability measure on \( S_k \) such that (3.9) holds. The transfer of mass is accomplished in two steps. In the first step we distribute the mass \( \mu_k(v) \) to all its children such that the mass distributed to each child \( w \) is proportional to \( \pi(w)^p \) (or equivalently \( \rho(w)^p \)); that is
\[
f_0(w) = \frac{\pi(w)^p}{\sum_{w' \in C(v)} \pi(w')^p} \mu_k(v),
\]
for all \( v \in S_k \) and \( g \in C(v) \), where \( C(v) \) denotes the set of children of \( v \) \( (C(v) \subset S_{k+1}) \).

By (H4), (H1) and the fact that every vertex has at least one child, we obtain
\[
\eta_p \pi(v)^p \leq \sum_{w' \in C(v)} \pi(w')^p \leq \pi(v)^p.
\]

Therefore, we have
\[
C^{-1} \frac{\mu_k(v)}{\pi(v)^p} \leq \eta_p \frac{\mu_k(v)}{\pi(v)^p} \leq \frac{f_0(w)}{\pi(w)^p} \leq \frac{\mu_k(v)}{\pi(v)^p},
\] (3.12)
for all points \( v \in S_k \) and \( w \in C(v) \). Since every point \( v \in S_k \) has at least one child\footnote{Since \( (\pi_1(v), \pi_2(v) + 1) \) is always a child of \( v \) for any \( v \in S \).} \( f_0 \) is probability mass function on \( S_{k+1} \). If the measure \( f_0 \) on \( N_{k+1} \) satisfies condition (1) of the Lemma, we set \( \mu_{k+1} = f_0 \). This is the desired probability mass function. Condition (2) is satisfied by (3.12), and (3) is satisfied since every \( w \in C(v), v \in S_k \) satisfies \( d(\pi_1(v), \pi_1(w)) < a^{-k} \). The second step is not necessary in this case.
But if \( f_0 \) does not satisfy condition (1) of the Lemma, then we proceed as follows at the second step. Let \( p_1, \ldots, p_T \) be the indexed pairs of points \( \{w, w'\} \) with \( w, w' \in S_{k+1} \) and \( D_2(w, w') = 1 \). Take the pair \( p_1 = \{w_1, w'_1\} \). If \( f_0(w_1) \pi(w_1)^p \leq C^2 f_0(w'_1) \pi(w'_1)^p \) and \( f_0(w_1) \pi(w_1)^p \leq C^2 f_0(w'_1) \pi(w'_1)^p \), then we set \( f_1 = f_0 \). Assume one of the inequalities is violated, say \( f_0(w_1) \pi(w_1)^p > C^2 f_0(w'_1) \pi(w'_1)^p \). Then we construct a measure \( f_1 \) from \( f_0 \) such that

\[
\begin{align*}
    f_1(w_1) &= f_0(w_1) - \alpha_1, \\
    f_1(w'_1) &= f_0(w'_1) + \alpha_1, \\
    f_1(w) &= f_0(w), \quad w \neq w_1, w'_1,
\end{align*}
\]

where \( \alpha_1 > 0 \) is chosen such that

\[
\alpha_1 \left( \frac{C^2}{\pi(w_1)^p} + \frac{1}{\pi(w_1)} \right) = \frac{f_0(w_1)}{\pi(w_1)} - \frac{C^2 f_0(w'_1)}{\pi(w'_1)},
\]

so that \( f_1(w_1) \pi(w_1)^p = C^2 f_1(w'_1) \pi(w'_1)^p \).

The next step is the construction of a measure \( f_2 \) from \( f_1 \) in exactly the same way that \( f_1 \) was constructed from \( f_0 \). Here we consider the pair \( p_2 \). A measure \( f_2 \) is next constructed from \( f_2 \) and so on. We claim that \( \mu_{k+1} = f_T \) is the desired measure in the lemma.

We first verify that for all \( v \in S_k \), for all \( w \in C(v) \) and for all \( s = 0, 1, \ldots, T \), we have

\[
C^{-1} \frac{\mu_k(v)}{\pi(v)^p} \leq \frac{f_s(w)}{\pi(w)} \leq \frac{\mu_k(v)}{\pi(v)^p}.
\]

By (3.12), it is clear that (3.13) holds for \( s = 0 \). We now show (3.13) by induction. Suppose (3.13) holds for \( s = j \), we will verify it for \( s = j + 1 \). Let \( p_{j+1} = \{w_{j+1}, w'_{j+1}\} \), and let \( v_{j+1} \) and \( v'_{j+1} \) be parents of \( w_{j+1}, w'_{j+1} \) respectively. If \( f_j = f_{j+1} \), there is nothing to prove. But if \( f_{j+1} \neq f_j \), then assume without loss of generality, that

\[
\frac{f_j(w_{j+1})}{\pi(w_{j+1})^p} > C^2 \frac{f_j(w'_{j+1})}{\pi(w'_{j+1})^p}.
\]

By (3.14) and the construction, we have

\[
f_{j+1}(w_{j+1}) < f_j(w_{j+1}), \quad f_{j+1}(w'_{j+1}) > f_j(w'_{j+1}).
\]

Therefore by the induction hypothesis (3.13) for \( s = j \) and (3.15), we have

\[
\frac{f_{j+1}(w_{j+1})}{\pi(w_{j+1})^p} \leq \frac{\mu_k(v_{j+1})}{\pi(v_{j+1})^p}, \quad \frac{f_{j+1}(w'_{j+1})}{\pi(w'_{j+1})^p} \geq C^{-1} \frac{\mu_k(v'_{j+1})}{\pi(v'_{j+1})^p}.
\]

Therefore it suffices to verify that

\[
\frac{f_{j+1}(w_{j+1})}{\pi(w_{j+1})^p} \geq C^{-1} \frac{\mu_k(v_{j+1})}{\pi(v_{j+1})^p}, \quad \frac{f_{j+1}(w'_{j+1})}{\pi(w'_{j+1})^p} \leq \frac{\mu_k(v'_{j+1})}{\pi(v'_{j+1})^p}.
\]

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Suppose the first inequality in (3.16) fails to be true, then by construction, (3.15) and the induction hypothesis (3.13) for \( s = j \), we have

\[
C^{-1} \frac{\mu_k(v_{j+1})}{\pi(v_{j+1})^p} > \frac{f_{j+1}(w_{j+1})}{\pi(w_{j+1})^p} = C^2 \frac{f_{j+1}(w'_{j+1})}{\pi(w'_{j+1})^p} > C^2 \frac{f_j(w'_{j+1})}{\pi(w'_{j+1})^p} > C^2 \mu_k(v'_{j+1}) \frac{v'_{j+1}}{\pi(v'_{j+1})^p},
\]

which implies \( \frac{\mu_k(v_{j+1})}{\pi(v_{j+1})^p} > C^2 \frac{\mu_k(v'_{j+1})}{\pi(v'_{j+1})^p} \). However, by Lemma 2.4(b) it implies that \( D_2(v_{j+1}, v'_{j+1}) \leq 1 \) and therefore the above estimate contradicts the induction hypothesis. This proves the first inequality in (3.16). The proof of the second inequality in (3.16) is similar. Indeed, assume to the contrary that \( \frac{f_{j+1}(w'_{j+1})}{\pi(w'_{j+1})^p} > \frac{\mu_k(v'_{j+1})}{\pi(v'_{j+1})^p} \); then we have

\[
\frac{\mu_k(v_{j+1})}{\pi(v_{j+1})^p} > \frac{f_{j+1}(w_{j+1})}{\pi(w_{j+1})^p} > \frac{f_j(w_{j+1})}{\pi(w_{j+1})^p} = C^2 \frac{f_{j+1}(w'_{j+1})}{\pi(w'_{j+1})^p} > C^2 \frac{f_j(w'_{j+1})}{\pi(w'_{j+1})^p},
\]

which again implies \( \frac{\mu_k(v_{j+1})}{\pi(v_{j+1})^p} > C^2 \frac{\mu_k(v'_{j+1})}{\pi(v'_{j+1})^p} \). Therefore (3.13) follows by induction. In particular, \( \mu_{k+1} = f_T \) satisfies condition (2) of the lemma.

We now verify condition (1) for \( \mu_{k+1} = f_T \). For this, it suffices to prove the following assertion: if

\[
C^{-2} \frac{f_j(w')}{\pi(w')^p} \leq \frac{f_j(w)}{\pi(w)^p} \leq C^2 \frac{f_j(w')}{\pi(w')^p}
\]

holds for a pair of points \( w, w' \in S_{k+1} \) such that \( D_2(w, w') = 1 \), then the same inequalities hold when \( f_j \) is replaced by \( f_{j+1} \).

We now prove this. If \( p_{j+1} = \{w, w'\} \), then \( f_{j+1} = f_j \) and there is nothing to prove. If \( \{w, w'\} \cap p_{j+1} = \emptyset \), then again there is nothing to prove. Let \( p_{j+1} = \{w_{j+1}, w'_{j+1}\} \). Without loss of generality, we assume \( p_{j+1} \cap \{w, w'\} = \{w_{j+1}\} \) where \( w_{j+1} = w \). First we consider the case, \( f_j(w)/\pi(w)^p > C^2 f_j(w'_{j+1})/\pi(w'_{j+1})^p \). Then

\[
\frac{f_{j+1}(w)}{\pi(w)^p} = \frac{f_{j+1}(w_{j+1})}{\pi(w_{j+1})^p} = C^2 \frac{f_{j+1}(w'_{j+1})}{\pi(w'_{j+1})^p}, \quad f_{j+1}(w) < f_j(w), \quad f_{j+1}(w') = f_j(w').
\]

Therefore, only the first inequality in (3.19) can fail for \( f_{j+1} \). Suppose that this happens, that is

\[
\frac{f_{j+1}(w')}{\pi(w')^p} > C^2 \frac{f_{j+1}(w)}{\pi(w)^p}.
\]

Let \( v', v'_{j+1} \in S_k \) be parents of \( w', w_{j+1} \) respectively. Then by (3.21), (3.20) and (3.13)

\[
\frac{\mu_k(v')}{\pi(v')^p} \geq \frac{f_{j+1}(w')}{\pi(w')^p} > C^2 \frac{f_{j+1}(w)}{\pi(w)^p} = C^2 \frac{f_{j+1}(w'_{j+1})}{\pi(w'_{j+1})^p} \geq C^2 \frac{\mu_k(v'_{j+1})}{\pi(v'_{j+1})^p} > C^2 \frac{\mu_k(v'_{j+1})}{\pi(v'_{j+1})^p},
\]

which contradicts (3.9) (since \( D_2(v', v'_{j+1}) \leq 1 \) by Lemma 2.4(b) and \( \lambda \geq 2 + 4\lambda a^{-1} \)). The case \( f_j(w)/\pi(w)^p < C^{-2} f_j(w'_{j+1})/\pi(w'_{j+1})^p \) is analyzed similarly and therefore the assertion given by (3.19) is proved. It remains to observe that this assertion proves condition (1) of the lemma for the measure \( \mu_{k+1} = f_T \). Along the path from \( f_0 \) to \( f_T \),
we “correct” the measure at all pairs of points where condition (1) is violated, and the
assertion given by (3.19) shows that once a pair is corrected, it remains corrected when
further changes are made.

It remains to verify condition (3). Since every \( w \in C(v), v \in S_k \) satisfies \( d(\pi_1(v), \pi_1(w)) < a^{-k} \), there was a mass transfer over a distance of at most \( a^{-k} \) while passing from \( \mu_k \) to \( f_0 \). Therefore it suffices to verify that while passing from \( f_0 \) to \( f_T = \mu_{k+1} \) there is a transfer over a distance of at most \( 2\lambda a^{-k-1} \).

We will now verify this. Recall that whenever \( w, w' \) share a horizontal edge, it satisfies
\( d(\pi_1(w), \pi_1(w')) < 2\lambda a^{-k} \). Therefore, it suffices to verify that there are no pairs \( p_i = \{ w_1, w_2 \}, p_n = \{ w_2, w_3 \}, l, n \in \mathbb{Z} \cap [1, T], l < n \), such that mass is transferred from \( w_1 \) to \( w_2 \) (in the transition from \( f_{l-1} \) to \( f_l \)) and then mass is transferred from \( g_2 \) to \( g_3 \) (in the transition from \( f_{n-1} \) to \( f_n \)). Assume the opposite.

Using the assumption that there are a mass transfer from \( w_1 \) to \( w_2 \) followed by a mass
transfer from \( w_2 \) to \( w_3 \) and the assertion given in (3.19), we have
\[
\frac{f_0(w_1)}{\pi(w_1)^p} > C^2 \frac{f_0(w_2)}{\pi(w_2)^p}, \quad \frac{f_0(w_2)}{\pi(w_2)^p} > C^2 \frac{f_0(w_3)}{\pi(w_3)^p}.
\]
(3.23)
If \( v_1, v_3 \) denote the parents of \( w_1, w_3 \) respectively, then by Lemma 2.4(b)
\[
D_2(v_1, v_3) \leq 1.
\]
Consequently by assumption, \( \mu_k(v_1)/\pi(v_1)^p \leq C^2 \mu_k(v_3)/\pi(v_3)^p \). However the inequalities
(3.23) and (3.13) imply the opposite inequality \( \mu_k(v_1)/\pi(v_1)^p \geq C^2 \mu_k(v_3)/\pi(v_3)^p \). We have
arrived at the desired contradiction and the proof of the lemma is complete. \( \square \)

The following lemma provides a sequence in \( p^{-1}(x) \in \partial(S, D_2) \) with desirable
properties and is useful for approximating balls centered at \( x \in X \) in different metrics.

**Lemma 3.6.** Let \( (X, d) \) be a compact doubling metric space and let \((S, D_2)\) denote
the corresponding hyperbolic filling with vertical and horizontal parameters \( a, \lambda \) respectively
such that \( a \geq \lambda \geq 6 \). For any \( x \in X \), there exists a sequence of vertices \( v_n \in S_n, n \geq 0 \)
such that \( v_n \) is the parent of \( v_{n+1} \) and \( d(x, \pi_1(v_n)) \leq \frac{1}{1-a^{-1}} a^{-n} \) for all \( n \in \mathbb{N}_{\geq 0} \).

**Proof.** For each \( n \), we choose \( w_n \in S_n \) be such that \( d(w_n, x) < a^{-n} \) (this is possible since
\( X_n \) is a maximal \( a^{-n} \)-separated subset). Consider the sequence of genealogies \( g(w_n) \) for
each \( n \in \mathbb{N}_{\geq 0} \). By a diagonal argument the sequence of genealogies \( g(w_n) \) converge along
a subsequence to yield the a sequence \( v_n \in S_n \) such that \( v_n \) is the parent of \( v_{n+1} \) for all \( n \in \mathbb{N}_{\geq 0} \). If \( v_n \in S_n \) is in the genealogy of \( w_k \), \( k > n \), we have \( d(x, \pi_1(v_n)) < a^{-k} + \sum_{i=n}^{k} a^{-k} \).
Letting \( k \to \infty \) along a subsequence yields the desired bound \( d(x, \pi_1(v_n)) \leq \frac{1}{1-a^{-1}} a^{-n} \). \( \square \)

We construct a doubling measure on \((X, d)\) using Lemma 3.5 below.

**Lemma 3.7** (Construction of doubling measure). Let \((X, d)\) be a compact doubling metric
space and let \( S \) denote a hyperbolic filling with vertical and horizontal parameters \( a, \lambda \}
respectively with $a \geq \lambda \geq 6$. Let $\rho : S \to (0, \infty)$ denote a weight function that satisfies the hypotheses (H1), (H2), and (H4). Let $\mu_0$ denote the (unique) probability measure on $S_0 = \{v_0\}$. Let $\mu_k$ denote the probability measure on $S_k$ for all $k \in \mathbb{N}$ constructed inductively using Lemma 3.5. Let

$$\tilde{\mu}_k = \sum_{v \in S_k} \mu_k(v)\delta_{\pi_1(v)}, \quad \text{for all } k \in \mathbb{N},$$

denote a sequence of probability measures on $X$ associated with the above construction. Then any sub-sequential probability limit $\mu$ of $(\tilde{\mu}_k)_{k \in \mathbb{N}}$ is a doubling measure on $(X, d)$.

**Proof.** Observe that such a sub-sequential limit $\mu$ exists by Prokhorov’s theorem along with the compactness of $(X, d)$.

We obtain two sided bounds on $\mu(B(x, r))$ as $\mu_n(v_n)$ for a suitably chosen value of $n$. Since $\text{diam}(X, d) = \frac{1}{2}$, it suffices to consider $r < 1$. For $x \in X$ choose a sequence $\{v_n\}$ as given in Lemma 3.6. To describe this let $n \in \mathbb{N}_{\geq 0}$ denote the largest integer such that $a^{-n} \geq r$. We claim that

$$\mu(B(x, r)) \leq \mu_n(v_n) \quad (3.24)$$

where the constants of comparison are independent of $x \in X$, $r \in (0, 1)$. Let us first show the upper bound. If mass from $\mu_n(v), v \in S_n$ contributes to $\mu(B(x, r))$, then by Lemma 3.5(3) we have

$$d(\pi_1(v), x) \leq r + \sum_{k=n}^{\infty} (1 + 2\lambda a^{-1})a^{-k} = (1 + (1 + 2\lambda a^{-1})(1 - a^{-1})^{-1}) a^{-n}$$

Since $\lambda > ((1 - a^{-1})^{-1}) (1 + 2\lambda a^{-1})$, we have that $x \in B(v, \lambda a^{-n}) \cap B(v_n, \lambda a^{-n})$ and hence $D_2(v, v_n) \leq 1$. Therefore

$$\mu(B(x, r)) \leq \sum_{v \in S_{n} : D_2(v, v_n) \leq 1} \mu_n(v).$$

If $D_2(v, v_n) \leq 1$ and $v \in S_n$, then by Lemma 3.5(1) and (H2), we obtain $\mu_n(v_n) \propto \mu(v)$ for any pair of such vertices. Furthermore since $(X, d)$ satisfies the metric doubling property, the number of neighbors of each vertex is uniformly bounded above [BBS, Proposition 4.5]. Combining the above estimates yields the upper bound in (3.24).

For the lower bound, we consider $\mu_{n+2}(v_{n+2})$. By Lemma 3.5(3) and $d(\pi_1(v_{n+2}), x) < (1 + 2\lambda a^{-1})a^{-(n+2)}$, we note that the mass from $v_{n+2}$ stays within $B(x, s)$ where

$$s = (1 + 2\lambda a^{-1}) (1 + (1 - a^{-1})^{-1}) a^{-(n+2)} < r$$

(since $a^{n-1} < r$ and $a^{-1}(1 + 2\lambda a^{-1})(1 - a^{-1})^{-1} < 1$). This implies that $\mu(B(x, r)) \geq \mu_{n+2}(v_{n+2})$. This along with Lemma 3.5(2) and (H1), we obtain $\mu_{n+2}(v_{n+2}) \propto \mu_n(v_n)$. Combining these estimates yields the lower bound for $\mu(B(x, r))$ in (3.24).

Next, we show that (3.24) implies the desired doubling property. For the remainder of the proof we assume $r \in (0, 1/2)$. The case $r \geq 1/2$ is similar and easier. Let $N \in \mathbb{N}_{\geq 0}$
Let denote the largest integer such that \(a^{-N} \geq 2r\). This implies \(a^{-(N+2)} < 2^{a^{-1}}r < r\). This implies that \(n = N\) or \(n = N + 1\). Therefore by the same argument as above (using Lemma 3.5(2) and (H1)), we have \(\mu_n(v_n) \approx \mu_N(v_N)\). This along with (3.24) shows that \(\mu\) is a doubling measure on \((X,d)\).

Let \(\Theta_\rho\) denote the metric defined in (3.8). In the following proposition, we obtain an upper bound on the Assouad dimension of \((X,\Theta_\rho)\). We establish this by showing that the measure \(\mu\) in Lemma 3.7 is \(p\)-homogeneous in \((X,\Theta_\rho)\). This along with Theorem 1.1 show that \(d_A(X,\Theta_\rho) \leq p\).

**Proposition 3.8.** Let \((X,d)\) be a compact doubling metric space and let \(S\) denote a hyperbolic filling with vertical and horizontal parameters \(a,\lambda\) respectively such that \(a \geq \lambda \geq 6\). Let \(\rho : S \to (0,\infty)\) denote a weight function that satisfies the hypotheses (H1), (H2), (H3), and (H4). Let \(\Theta_\rho\) denote the metric defined in (3.8) using Proposition 3.4. Then the measure \(\mu\) defined in Lemma 3.7 is \(p\)-homogeneous in \((X,\Theta_\rho)\), where \(p\) is the constant in (H4). In particular, \(d_A(X,\Theta_\rho) \leq p\).

**Proof.** For ease of notation, we abbreviate \(\Theta_\rho\) by \(\Theta\). By (3.7), there exists \(L > 1\) such that

\[
\frac{1}{L} \pi(c(x,y)) \leq \Theta(x,y) \leq \pi(c(x,y)) \quad \text{for all } x,y \in X. \tag{3.25}
\]

Let \(\eta_-, \eta_+, K_0\) denote the constants in (H1) and (H2).

Let \(x \in X\) be a arbitrary. Choose a sequence \(\{v_n\}\) such that \(v_n \in S_n\) as given in Lemma 3.6. We will compare balls in \((X,\Theta)\) using balls in \((X,d)\). We denote the respectively balls by \(B_\Theta(\cdot, \cdot), B_d(\cdot, \cdot)\) respectively. To this end, we show the following:

\[
B_d(\pi_1(v_k), 2a^{-k}) \subset B_\Theta(x,r), \quad \text{whenever } v_k \text{ satisfies } \pi(v_k) < K_0^{-1}r, \tag{3.26}
\]

and

\[
B_\Theta(x,r) \subset B_d(\pi_1(v_k), 2(\lambda + 2)a^{-k}), \tag{3.27}
\]

whenever \(k \in \mathbb{N}\) is such that \(\pi(v_k) \leq K_0Lr\) and \(\pi(v_{k-1}) > K_0Lr\).

First, we show (3.26). Let \(k \in \mathbb{N}_{\geq 0}\) be such that \(\pi(v_k) < K_0^{-1}r\) and let \(y \in B_d(\pi_1(v_k), 2a^{-k})\). Let \((z,l) \in c(x,y)\). Since \(\{x,y\} \subset B_d(\pi_1(v_k), 2a^{-k})\), we have \(l \leq k\). Note that \(D_2((z,l), v_l) \leq 1\), since \(d(\pi_1(v_l), x) \leq d(\pi_1(v_l), \pi_1(v_k)) + d(\pi_1(v_k), x) \leq (1-a^{-1})^{-1}a^{-l} + (1-a^{-1})^{-1}a^{-k} \leq 2(1-a^{-1})^{-1}a^{-l} < \lambda a^{-l}\) and \(x \in B(z, 2a^{-l}) \subset B(z, \lambda a^{-l})\). This along with (H2) implies that

\[
\pi(c(x,y)) \leq K_0\pi(v_l) \leq K_0\pi(v_k) < r.
\]

This estimate along with (3.25) implies (3.26).

Next, we show (3.27). Let \(k \in \mathbb{N}\) be such that \(\pi(v_k) \leq K_0Lr\) and \(\pi(v_{k-1}) > K_0Lr\) and let \(y \in B_\Theta(x,r)\). By (3.25), \(\pi(c(x,y)) \leq Lr\). Let \((z,l) \in c(x,y)\). Note that \(D_2(v_l, (z,l)) \leq 1\) which along with (H2) implies that

\[
\pi(v_l) \leq K_0\pi(c(x,y)) \leq K_0Lr.
\]
The choice of $k$ implies that $l \geq k$. Hence by Lemma 3.6
\[
d(\pi_1(v_k), y) \leq d(\pi_1(v_k), \pi_1(v_l)) + d(\pi_1(v_l), z) + d(z, y) < (1 - a^{-1})^{-1}a^{-k} + 2\lambda a^{-l} + 2a^{-l} \quad \text{(since $D_2(v_l, (z, l)) \leq 1$ and $(z, l) \in c(x, y)$)}
\leq (2 + (1 - a^{-1})^{-1} + 2\lambda) a^{-k} < 2(2 + \lambda)a^{-k} \quad \text{(since $k \geq l$).} \tag{3.28}
\]
This completes the proof of (3.27).

Next, we show that $\mu$ is $p$-homogeneous in $(X, \Theta)$; that is, there exists $C > 1$ such that
\[
\frac{\mu(B_\Theta(x, r))}{\mu(B_\Theta(x, s))} \leq C \left(\frac{r}{s}\right)^p \quad \text{for all $x \in X$, $0 < s < r$.} \tag{3.29}
\]
Let $0 < s < r$ and $x \in X$. Choose a sequence $\{v_n\}$ such that $v_n \in \mathcal{S}_n$ for all $n \in \mathbb{N}_{\geq 0}$ as given in Lemma 3.6. Let $k \in \mathbb{N}_{\geq 0}$ be the smallest non-negative integer such that $\pi(v_k) < K_0^{-1}s$. By (3.26), Lemma 3.7 and (3.24), we have
\[
\mu(B_\Theta(x, s)) \gtrsim \mu_k(v_k). \tag{3.30}
\]
If $k = 0$, then it we have $1 \geq \mu(B_\Theta(x, r)) \leq \mu(B_\Theta(x, s)) \gtrsim 1$ which implies (3.29). So it suffices to consider the case $k \geq 1$. The choice of $k$ along with (H1) implies that
\[
\eta_\mu K_0^{-1}r \leq \eta_\mu \pi(v_{k-1}) \leq \pi(v_k) < K_0^{-1}r. \tag{3.31}
\]
Next, we bound $\mu(B_\Theta(x, r))$ from above. We consider two cases depending on whether or not $r < (K_0L)^{-1}\pi(v_0)$. If $r < (K_0L)^{-1}\pi(v_0)$, there exists $l \in \mathbb{N}$ such that $\pi(v_l) \leq K_0Lr$ and $\pi(v_{l-1}) > K_0Lr$. Hence by (H1), we have
\[
\eta_\mu K_0Lr < \eta_\mu \pi(v_{l-1}) \leq \pi(v_l) \leq K_0Lr \tag{3.32}
\]
By (3.27), Lemma 3.7, and (3.24), we have
\[
\mu(B_\Theta(x, r)) \leq \mu(B_d(\pi_1(v_l)), 2(\lambda + 2)a^{-l}) \lesssim \mu(B_d(\pi_1(v_l)), a^{-l}) \lesssim \mu_l(v_l). \tag{3.33}
\]
Since $r > s$, we have $k < l$. Therefore by Lemma 3.5(2), we have
\[
\frac{\mu_l(v_l)}{\pi(v_l)^p} \leq \frac{\mu_k(v_k)}{\pi(v_k)^p}, \quad \text{for any $k \leq l$.} \tag{3.34}
\]
By (3.30), (3.33), (3.31), (3.32) and (3.34), we have
\[
\frac{\mu(B_\Theta(x, r))}{\mu(B_\Theta(x, s))} \lesssim \frac{\mu_l(v_l)}{\mu_k(v_k)} \lesssim \frac{\pi(v_l)^p}{\pi(v_k)^p} \lesssim \frac{r^p}{s^p}.
\]
This implies (3.29) in the case $r < (K_0L)^{-1}\pi(v_0)$.

On the other hand, if $r \geq (K_0L)^{-1}\pi(v_0)$ we use the trivial bound $\mu(B_\Theta(x, r)) \leq 1 = \mu_0(v_0)$. By (3.30), (3.31), (3.34) and the bound $1 \leq \pi(v_0) \lesssim r$, we have
\[
\frac{\mu(B_\Theta(x, r))}{\mu(B_\Theta(x, s))} \lesssim \frac{\mu_0(v_0)}{\mu_k(v_k)} \lesssim \frac{\pi(v_0)^p}{\pi(v_k)^p} \lesssim \frac{r^p}{s^p}.
\]
This completes the proof of (3.29). By Theorem 1.1, we obtain the desired bound on Assouad dimension.\qed

Proof of Theorem 3.2. This follows immediately from Propositions 3.4 and 3.8. \qed

The following is an analogue of [Car, Theorem 1.2]. The hypotheses (S1) and (S2) below are identical to [Car, Theorem 1.2] but in the conclusion we bound $d_{CA}$ instead of $d_{AR}$.  

**Theorem 3.9.** (cf. [Car, Theorem 1.2]) Let $(X, d)$ be a compact doubling metric space and and let $S$ denote a hyperbolic filling with vertical and horizontal parameters $a, \lambda$ respectively such that $a \geq \lambda \geq 6$. Let $p > 0$. There exists $\eta_0 \in (0, 1)$ which depends only on $p, \lambda$ and the doubling constant of $(X, d)$ (but not on the vertical parameter $a$) such that if there exists a function $\sigma : S \to [0, \infty)$ that satisfies:

(S1) for all $v \in S_k$ and $k \geq 0$, if $\gamma \in \Gamma_k(v)$, then

$$\sum_{w \in \gamma} \sigma(v) \geq 1,$$

where $\Gamma_k$ is as defined in (3.4), and

(S2) for all $k \geq 0$ and all $v \in S_k$, we have

$$\sum_{w \in C(v)} \sigma(v)^p \leq \eta_0,$$

where $C(v) \subset S_{k+1}$ denotes those vertices $w$ whose parent is $v$ (that is, $C(v) = D_{k+1}(v)$), then there exists $\Theta \in \mathcal{J}(X, d)$ such that $d_{A}(X, \Theta) \leq p$ and the identity map $\text{Id} : (X, d) \to (X, \Theta)$ is a power quasisymmetry. Therefore the conformal Assouad dimension is less than or equal to $p$.

The proof of the above theorem is very similar to that of [Car, Theorem 1.2] except for the use of Theorem 3.2 instead of [Car, Theorem 1.1]. For the convenience of the reader, we provide further details since the hypothesis (H4) is different from that of [Car]. To the reader who is familiar with Carrasco’s work, we point out that the estimate in [Car, (2.52)] implies our version of (H4) for small enough $\eta_0$. The proof of other three hypothesis is exactly similar. Readers who are familiar with the proof of [Car, Theorem 1.2] may want to skip the proof of Theorem 3.9.

Let $\rho : S \to [0, \infty)$ be a function. We define $\rho^* : S \to [0, \infty)$ as

$$\rho^*(v) = \min\{\rho(w) : w \in S : \pi^*(w) = \pi_2(v), D_2(v, w) \leq 1\}, \quad \text{for all } v \in S.$$ (3.35)

Similarly, we define $\pi^* : S \to [0, \infty)$ as

$$\pi^*(v) = \min\{\pi(w) : w \in S : \pi_2(w) = \pi_2(v), D_2(v, w) \leq 1\}, \quad \text{for all } v \in S.$$ (3.36)
If \( \gamma = (v_1, \ldots, v_N) \) is a horizontal path, we define
\[
L_h(\gamma, \rho) = \sum_{j=1}^{N-1} \rho^*(v_j) \wedge \rho^*(v_{j+1}).
\] (3.37)

The following is a version of [Car, Proposition 2.9] and provides a useful sufficient condition for \((H3)\).

**Proposition 3.10.** Let \((X, d)\) be a compact doubling metric space. Let \((S, D_2)\) denote the hyperbolic filling with horizontal and vertical parameters \(\lambda, a\) respectively that satisfy \(a \geq \lambda \geq 6\). Assume that there exists \(p > 0\) and a function \(\rho : S \to (0, \infty)\) which satisfy the hypotheses \((H1), (H2)\), and also

\[(H3') \text{ for all } k \geq 1, \text{ for all } v \in S_k \text{ and for all } \gamma \in \Gamma_{k+1}(v), \text{ it holds } L_h(\gamma, \rho) \geq 1,\]

where \(L_h(\gamma, \rho)\) is as defined in (3.37). Then the function \(\rho\) also satisfies \((H3)\).

The proof of Proposition 3.10 requires several lemmas. We say that a path \(\gamma = (v_1, \ldots, v_N)\) is of level \(k\) (resp. level at most \(k\)) if \(\pi_2(v_i) = k\) (resp. \(\pi_2(v_i) \leq k\)) for all \(i = 1, \ldots, N\).

**Lemma 3.11.** (Cf. [Car, Lemma 2.10]) Let \((X, d)\) and \((S, D_2)\) be as given in Proposition 3.10. Let \(k \geq 0\) and \(v \in S_k\). Assume that \(\rho\) satisfies \((H3')\). Consider a horizontal path \(\gamma = (v_1, \ldots, v_n)\) of level \(k + 1\) such that \(\pi_1(v_i) \in B(\pi_1(v), 3a^{-k})\) for all \(i = 1, \ldots, N\), \(\pi_1(v_1) \in B(\pi_1(v), a^{-k})\) and \(\pi_1(v_N) \notin B(\pi_1(v), 2a^{-k})\). Let \(w\) denote the parent of \(z_1\). Then
\[
\sum_{i=1}^{N-1} \pi^*(v_i) \wedge \pi^*(v_{i+1}) \geq \max\{\pi^*(v), \pi^*(w)\}.
\]

**Proof.** First, we show that for all \(j = 1, \ldots, N\),
\[
\pi^*(v_j) \geq \max\{\pi^*(v), \pi^*(w)\} \min\{\rho(w_j) : w_j \in S_{k+1}, D_2(w_j, v_j) \leq 1\}
\] (3.38)
Let \(\tilde{w}_j \in S_{k+1}\) be such that \(\pi^*(v_j) = \pi(\tilde{w}_j)\) and \(D_2(\tilde{w}_j, v_j) \leq 1\). Let \(u_j \in S_k\) be the parent of \(w_j\). Then by Lemma 2.4(a),
\[
d(\pi_1(v), \pi_1(u_j)) \leq d(\pi_1(v), \pi_1(v_j)) + d(\pi_1(v_j), \pi_1(\tilde{w}_j)) + d(\pi_1(\tilde{w}_j), \pi_1(u_j))
\leq 3a^{-k} + 2\lambda a^{-k-1} + a^{-k} = (4 + 2\lambda a^{-1})a^{-k} < \lambda a^{-k},
\]
d(\pi_1(w), \pi_1(v)) \leq d(\pi_1(w), \pi_1(v_1)) + d(\pi_1(v_1), \pi_1(v)) < a^{-k} + a^{-k} < \lambda a^{-k}

The above estimates imply that \(D_2(v, u_j) \leq 1\) and \(D_2(w, u_j) \leq 1\). Therefore \(\pi(u_j) \geq \max\{\pi^*(v), \pi^*(w)\}\) and hence
\[
\pi^*(v_j) = \pi(\tilde{w}_j) = \pi(u_j) \rho(\tilde{w}_j) \\
\geq \max\{\pi^*(v), \pi^*(w)\} \min\{\rho(w_j) : w_j \in S_{k+1}, D_2(w_j, v_j) \leq 1\}.
\]
This completes the proof of (3.38). By (3.38) and (H3’), we have
\[
\sum_{i=1}^{N-1} \pi^*(v_i) \wedge \pi^*(v_{i+1}) \geq \max\{\pi^*(v), \pi^*(w)\} \sum_{i=1}^{N-1} \rho^*(v_i) \wedge \rho^*(v_{i+1})
\]
\[
= \max\{\pi^*(v), \pi^*(w)\} L_{\delta}(\gamma, \rho) \geq \max\{\pi^*(v), \pi^*(w)\}.
\]

\[\square\]

We introduce a different notion of length on paths. For any edge \(e = \{u, v\}\) we define
\[
\hat{\ell}_1(e) = \begin{cases} 
\pi^*(u) \wedge \pi^*(v) & \text{if } e = \{u, v\} \text{ is a horizontal edge,} \\
K_0 \eta^{-1} \pi^*(v) & \text{if } e = \{u, v\} \text{ and } u \text{ is a parent of } v,
\end{cases}
\]
and for a path \(\gamma = (v_1, \ldots, v_N)\), we define
\[
\hat{\ell}_1(\gamma) = \sum_{i=1}^{N-1} \hat{\ell}_1(e_i), \text{ where } e_i = \{v_i, v_{i+1}\}.
\] (3.39)

If \(w \in S_{k+1}, u, v \in S_k\) such that \(D_2(u, v) = D_2(u, w) = 1\), then by (H1) and (H2), we have
\[
\hat{\ell}_1(\{u, v\}) \leq \pi^*(u) \leq \pi(u) \leq \eta^{-1} \pi(w) \leq K_0 \eta^{-1} \pi^*(w) \leq \hat{\ell}_1(\{u, w\}).
\] (3.41)

**Lemma 3.12.** [Cf. [Car, Lemma 2.11]] Let \((X, d)\) and \((\mathcal{S}, D_2)\) be as given in Proposition 3.10. Assume that \(\rho : \mathcal{S} \to [0, \infty)\) satisfies the hypotheses (H1), (H2), (H3’). Let \(u, v \in S_{k+1}\) be such that \(d(\pi_1(u), \pi_1(v)) > 4a^{-k}\). Let \(\gamma = (v_1, \ldots, v_N)\) be a path of level at most \(k + 1\) from \(v_1 = u\) to \(v_N = v\). Then there exists a path \(\gamma' = (u_1, \ldots, u_M)\) of level at most \(k\) such that:

1. \(u_1, u_M\) are parents of \(v_1\) and \(v_N\) respectively, and
2. \(\hat{\ell}_1(\gamma') \leq \hat{\ell}_1(\gamma)\).

**Proof.** Let \(\gamma = (v_1, \ldots, v_N)\) be a path of level at most \(k + 1\) as given in the statement of the lemma. We decompose \(\gamma\) into sub-paths of level at most \(k\) or level equal to \(k + 1\). Let \(s_1 = 1\). Define inductively positive integers \(s_i, t_i\) as
\[
t_i = \min\{j > s_i : \pi_2(v_j) \leq k \text{ or } j = N\},
\]
\[
s_{i+1} = \min\{j \geq t_i : \pi_2(v_{j+1}) = k + 1\}.
\]
We stop when \(t_i = N\) for some \(i = L\). Note that \(\pi_2(v_{s_1}) = \pi_2(v_{t_k}) = k + 1\), and \(\pi_2(v_{s_i}) = \pi_2(v_j) = k\) for \(i \neq 1\) and \(j \neq L\). Since we are trying to bound \(\hat{\ell}_1(\gamma)\) from below, we may assume that path \(\gamma\) has no self-intersections; that is \(v_i \neq v_j\) for all \(i \neq j\). In particular, \(v_{s_i} \neq v_{t_i}\) for all \(i\).

For each \(i = 1, \ldots, L\), let \(\gamma_i\) denote the sub-path \((v_{s_i}, \ldots, v_{t_i-1})\). We will replace each path \(\gamma_i\) with \(\gamma'_i\) such that \(\hat{\ell}_1(\gamma'_i) \leq \hat{\ell}_1(\gamma_i)\).
First, we consider $2 \leq i \leq L - 1$ and postpone the cases $i = 1, L$ to the end. Let $2 \leq i \leq L - 1$. We consider two cases.

**Case 1:** $\pi_1(v_j) \in B(\pi_1(v_{s_i}), 2a^{-k})$ for all $j = s_i + 1, \ldots, t_i - 1$.

In this case, $v_{s_i}, v_{t_i} \in S_k$ and are the parents of $v_{s_{i+1}}, v_{t_{i-1}}$ respectively. By Lemma 2.4(c), $D_2(v_{s_i}, v_{t_i}) = 1$ and hence we replace $\gamma_i$ with $\gamma'_i = (v_{s_i}, v_{t_i})$. From (3.41), we obtain

$$\hat{\ell}_1(\gamma'_i) \leq \hat{\ell}_1(\{v_{s_i}, v_{s_{i+1}}\}) \leq \hat{\ell}_1(\gamma_i).$$

**Case 2:** There exists $j_1 \in \{s_i + 1, t_i - 1\}$ such that $\pi_1(v_{j_1}) \notin B(\pi_1(v_{s_i}), 2a^{-k})$. We assume $j_1$ is the first index with this property. We denote $j_0 = s_i + 1, w_0 = v_{s_i} \in S_k$. Suppose $j_l, w_l$ are defined, and if $j_l < t_i - 1$, we define

$$j_{l+1} = \min\{j_l < j < t_i - 1 : \pi_1(v_j) \notin B(\pi_1(w_l), 2a^{-k}) \text{ or } j = t_i - 1\},$$

and let $w_{l+1} \in S_k$ be the parent of $v_{j_l+1} \in S_{k+1}$. Let $L_i$ be such that $j_{L_i} = t_i - 1$.

If $l \in \{0, \ldots, L_i - 2\}$, we have $\pi_1(v_{j_l+1}) \notin B(\pi_1(w_l), 2a^{-k})$. Since $a > 2\lambda$, we have

$$d(\pi_1(w_l), \pi_1(v_{j_l+1}) \leq d(\pi_1(w_l), \pi_1(v_{j_l})) + d(\pi_1(v_{j_l}), (v_{j_l+1})) < 2a^{-k} + 2a^{-k-1} < 3a^{-k}.$$  

Therefore by Lemma 3.11, Lemma 2.4(c), and (3.41), we have

$$\hat{\ell}_1((w_l, w_{l+1})) \leq \pi^*(w_l) \leq \hat{\ell}_1((v_{j_l}, \ldots, v_{j_{l+1}})), \quad \text{for all } l = 0, \ldots, L_i - 2. \quad (3.42)$$

The above estimate (3.42) is also true for $j = L_i - 1$ by combining the above argument and with that of case 1 by considering depending on whether or not $\pi_1(v_{j_l}) \notin B(\pi_1(v_{s_i}), 2a^{-k})$.

Hence $\gamma'_i = (w_0, \ldots, w_{L_i-1}, w_{L_i})$, where $w_0 = v_{s_i}, w_{L_i} = v_{t_i}$. By (3.42) along with the above remark, we obtain

$$\hat{\ell}_1(\gamma'_i) \leq \hat{\ell}_1(\gamma_i), \quad \text{for } i \in \{2, \ldots, L - 1\}.$$ 

The case $i = 1$ is also similar to above. Let $u_1$ be the parent of $v_1$. Similar to argument above, we consider two cases depending on whether or not $\pi_1(v_j) \in B(\pi_1(u_1), 2a^{-k})$ for all $j = 1, \ldots, t_i - 1$ as explained in [Car, proof of Lemma 2.11]. This yields a path $\gamma'_1$ from $u_1$ to $v_1$. The case $i = L$ is exactly same as $i = 1$ after reversing the order in which the vertices of $\gamma_L$ appear. By concatenating the paths $\gamma'_1, \ldots, \gamma'_L$, we obtain the path $(u_1, \ldots, u_M)$ with desired properties. $\square$

**Lemma 3.13.** (Cf. [Car, Lemma 2.12]) Let $(X, d)$ and $(S, D_2)$ be as given in Proposition 3.10. Assume that $\rho : S \to [0, \infty)$ satisfies the hypotheses (H1), (H2), and (H3'). There exists a constant $K_2 \geq 1$ such that the following property: for all $x, y \in X$, there exists $k_0$ depending on $x, y$ such that for all $k \geq k_0$, if $u, v \in S_k$ such that $x \in B_u, y \in B_v$, then any path $\gamma$ joining $u$ and $v$ satisfies

$$\hat{\ell}_1(\gamma) \geq K_2^{-1}\pi(c(x, y)),$$

where $\pi(c(x, y))$ is as defined in (3.5).
Proof. Let \( u, v \in \mathcal{S}_k \) be such that \( x \in B_u \), \( y \in B_v \). Let \( m \) be such that \( \pi_2(w) = m \) for some (or equivalently, for all) \( w \in c(x, y) \). By (2.3), we have \( d(x, y) > a^{-m-1} \). For \( k \geq m + 2 \), we have (using \( a \geq 12 \))

\[
d(\pi_1(u), \pi_1(v)) \geq d(x, y) - d(x, \pi_1(u)) - d(y, \pi_1(v)) > a^{-m-1} - 2a^{-m-2} \geq 10a^{-m-2}. \tag{3.43}
\]

The idea is to use Lemma to find a path of level at most \( m + 2 \) whose \( \ell_1 \) length is larger than \( \ell_1(\gamma) \). We consider two cases.

Case 1: The path \( \gamma \) is of level at most \( k \), where \( k \geq m + 2 \). By (3.43), we can apply Lemma 3.11. Set \( \gamma_k = \gamma \). Let \( u_i, v_i \in \mathcal{S}_k \) be such that \( u, v \) are descendants of \( u_i, v_i \) respectively. By Lemma 2.4(a), for all \( l \geq m + 2 \), we have

\[
d(\pi_1(u_l), \pi_1(v_l)) \geq d(\pi_1(u_l), \pi_1(v_l)) - d(\pi_1(u_l), \pi_1(u_l)) - d(\pi_1(v_l), \pi_1(v_l)) \\
\geq 10a^{-m-2} - 2a^{-\frac{a}{a-1}}a^{-l} > 6a^{-m-2}.
\]

Using the above estimate, and applying Lemma 3.11 repeatedly we obtain path \( \gamma_{m+2} \) of level at most \( m+2 \) from \( u_{m+2} \) to \( v_{m+2} \) such that \( \ell_1(\gamma) \geq \ell_1(\gamma_{m+2}) \). This along with (3.41), (H1), (H2), implies

\[
\ell_1(\gamma) \geq K_0^{-2}\pi(u_{m+2}) \geq K_0^{-2}\eta_\gamma^2\pi(u_m) \geq K_0^{-3}\eta_\gamma^2\pi(c(x, y)).
\]

In the last estimate, we used \( D_2(u_m, w) \leq 1 \) for any \( w \in c(x, y) \) (since \( x \in B(\pi_1(u_m), \lambda a^{-m}) \cap B(\pi_1(w), \lambda a^{-m}) \neq \emptyset \)).

Case 2: \( \gamma \) is not a path of level at most \( k \). Let \( n > k \) be the smallest integer such that \( \gamma \) is a path of level at most \( n \). Let \( k_0 \geq m + 2 \) be large enough so that

\[
K_0^{-3}\eta_\gamma^2\eta_n^m \geq 4K_0\eta_{k_0}^{-1} \sum_{i=k_0}^{\infty} \eta_i^i.
\]

Let \( \tilde{u}_n, \tilde{v}_n \in \mathcal{S}_n \) be such that \( x \in B_{\tilde{u}_n} \), \( y \in B_{\tilde{v}_n} \) and let \( \tilde{u}_k, \tilde{v}_k \in \mathcal{S}_k \) be the ancestors of \( \tilde{u}_n, \tilde{v}_n \) respectively. By Lemma 2.4(a), \( D_2(\tilde{u}_n, u_n) \leq 1 \) and \( D_2(\tilde{v}_n, v_n) \leq 1 \). Let \( \gamma_u \) denote the path from \( \tilde{u}_n \) to \( u \) formed by concatenating the genealogy from \( \tilde{u}_n \) to \( \tilde{u}_k \) and adding an edge from \( \tilde{u}_k \) to \( u_k \) if necessary. Similarly, let \( \gamma_v \) denote the path from \( v \) to \( \tilde{v}_n \) formed in a similar fashion. By concatenating \( \gamma_u, \gamma, \gamma_v \) we obtain a path \( \tilde{\gamma} \) from \( \tilde{v}_n \) to \( \tilde{u}_n \) whose level is at most \( n \). Using the first case, we obtain

\[
\ell_1(\tilde{\gamma}) \geq K_0^{-3}\eta_\gamma^2\pi(c(x, y)) \geq K_0^{-3}\eta_\gamma^2\eta_n^m \geq 4K_0\eta_{k_0}^{-1} \sum_{i=k_0}^{\infty} \eta_i^i \geq 2\ell_1(\gamma_u) + 2\ell_1(\gamma_v).
\]

This implies

\[
\ell_1(\gamma) \geq \frac{1}{2}K_0^{-3}\eta_\gamma^2\pi(c(x, y))
\]

for any \( k \geq k_0 \).
Proof of Proposition 3.10. By (H1), (H2), there exists \( c > 0 \) such that
\[
L_\rho(\gamma) \geq c \hat{c}_1(\gamma), \quad \text{for all paths } \gamma \text{ in } (S, D_2).
\]
This estimate along with Lemma 3.13 implies (H3). \( \square \)

The statement of the lemma below is slightly different from that of [Car, Lemma 2.13] and the proof is omitted as it is similar to [Car].

Lemma 3.14. ([Car, Lemma 2.13]) Let \((X, d)\) and \((S, D_2)\) be as given in Theorem 3.9. Suppose we have a function \( \pi_0 : S_k \to (0, \infty) \) such that
\[
\frac{1}{K} \leq \frac{\pi_0(v)}{\pi_0(w)} \leq K \quad \text{for all } v, w \in S_k \text{ such that } D_2(v, w) \leq 1,
\]
where \( K \geq 1 \) is a constant. Suppose also that there is function \( \pi_1 : S_{k+1} \to (0, \infty) \) such that for any \( u \in S_k \) and for any \( v \in S_{k+1} \) such that \( u \) is the parent of \( v \), we have
\[
1 \leq \frac{\pi_0(u)}{\pi_0(v)} \leq K.
\]

Let \( \hat{\pi}_1 : S_{k+1} \to [0, \infty) \) be defined as
\[
\hat{\pi}_1(w) = \pi_1(w) \lor \left( \frac{1}{K} \max\{\pi_1(v) : v \in S_{k+1}, D_2(v, w) \leq 1\} \right).
\]
Then for all \( w_1, w_2 \in S_{k+1} \) such that \( D_2(w_1, w_2) \leq 1 \), we have
\[
\frac{1}{K} \leq \frac{\hat{\pi}_1(w_1)}{\hat{\pi}_1(w_2)} \leq K.
\]

Lemma 3.15. (see [Car, Lemma 2.14]) Let \( G = (V, E) \) be a graph whose vertices has a degree bounded by \( K \) and let \( p > 0 \). Let \( \Gamma \) be a family of paths of \( G \). Let \( \tau : V \to [0, \infty) \) that satisfies
\[
\sum_{i=1}^{N-1} \tau(z_i) \geq 1, \quad \text{for all paths } \gamma = (v_1, \ldots, v_N) \in \Gamma.
\]
Let \( d_G : V \times V \to [0, \infty) \) denote the combinatorial graph distance metric on \( V \). Let \( \hat{\tau} : V \to [0, \infty) \) be defined as
\[
\hat{\tau}(v) = 2 \max\{\tau(w) : w \in V, d_G(w, v) \leq 2\}.
\]
Then
\[
\sum_{i=1}^{N-1} \hat{\tau}^*(v_i) \land \hat{\tau}^*(v_{i+1}) \geq 1 \quad \text{for all paths } \gamma = (v_1, \ldots, v_N) \in \Gamma,
\]
where \( \hat{\tau}^*(v) = \min\{\hat{\tau}(w) : d_G(w, v) \leq 1\} \), and such that
\[
\sum_{v \in V} \hat{\tau}(v)^p \leq 2^p(K^2 + 1) \sum_{v \in V} \tau(v)^p.
\]
The statement of Lemma 3.15 is slightly different from that of [Car, Lemma 2.14] where the term $K^2 + 1$ was replaced by $K^2$. This is because the estimate $\#\{w \in V : d_G(w, v) \leq 2\} \leq K^2$ for all $v \in V$ in [Car] must be replaced by $\#\{w \in V : d_G(w, v) \leq 2\} \leq K^2 + 1$. The proof is otherwise identical and is omitted.

Proof of Theorem 3.9. Let $\eta_0 \in (0, 1)$ whose value will be determined later. Since $(X, d)$ is doubling there exists $M_1 \in \mathbb{N}$, depending only on $a, \lambda$ and the doubling constant such that the number of neighbors of each vertex in $(\mathcal{S}, D_1)$ is bounded by $M_1$, and in particular the number of children of each vertex uniformly bounded above [BBS, Proposition 4.5].

Set
$$\eta_\sim = (\eta_0 M_1^{-1})^{1/p} \in (0, 1).$$

Let $\sigma : \mathcal{S} \to [0, \infty)$ satisfy (S1) and (S2). We define $\tau : \mathcal{S} \to [0, \infty)$ as $\tau : (\sigma^p + \eta_\sim^p)^{1/p} \geq \eta_\sim$, which also satisfies (S1). The function $\tau$ satisfies
$$\sum_{v \in C(u)} \tau(v)^p \leq \sum_{v \in C(u)} (\sigma(v)^p + \eta_\sim^p) \leq 2\eta_0, \quad \text{for all } u \in \mathcal{S}. \quad (3.48)$$

By [Hei, Exercise 10.17], there exists $M_2$, depending only on $\lambda$ and doubling constant of $(X, d)$, such that
$$\#\{w \in \mathcal{S} : D_2(v, w) = 1, \pi_2(w) = \pi_2(v)\} \leq M_2 \quad \text{for all } v \in \mathcal{S}. \quad (3.49)$$

That is, the number of horizontal edges at any vertex is uniformly bounded in $M_2$. By Lemma 3.15, there exists $\hat{\tau}$ satisfying condition (H3') such that
$$\sum_{v \in C(u)} \hat{\tau}(v)^p \leq 2^{p+1}(M_2^2 + 1)\eta_0. \quad (3.50)$$

We construct a function $\hat{\rho} : \mathcal{S} \to [0, \infty)$ that satisfies

1. $\rho(v) \geq \hat{\tau}(v)$ for all $v \in \mathcal{S}$.
2. $\rho$ satisfies (H2) with $K_0 = \eta_\sim^{-1}$.
3. $\rho(v) \leq \max\{\hat{\tau}(w) : D_2(w, v) \leq 1, \pi_2(w) = \pi_2(v)\}$ for all $v \in \mathcal{S}$.

The idea behind the proof is to inductively construct $\hat{\rho}$ on $\mathcal{S}_k$ for $k = 0, 1, \ldots$. Since the conditions (2) and (3) depend only on horizontal edges this inductive construction works well. We pick $\rho(v_0) = \hat{\tau}(v_0)$ where $v_0 \in \mathcal{S}_0$. Clearly, this satisfies (1), (2), (3) on $\mathcal{S}_0$ because $\mathcal{S}_0$ is a singleton set. Suppose we have constructed $\rho$ on $\cup_{j=0}^k \mathcal{S}_j$; we construct $\rho$ on $\mathcal{S}_{i+1}$ as follows. Define $\pi_0 : \mathcal{S}_i \to (0, \infty), \pi_1 : \mathcal{S}_{i+1} \to (0, \infty)$ as
$$\pi_0(u) = \prod_{w \in g(u)} \rho(w), \quad \pi_1(v) = \hat{\tau}(v) \prod_{w \in g(v), w \neq v} \rho(w) = \hat{\tau}(v) \pi_0(\tilde{v}),$$

for all $u \in \mathcal{S}_i, v \in \mathcal{S}_{i+1}$, where $\tilde{v} \in \mathcal{S}_i$ is the parent of $v \in \mathcal{S}_{i+1}$. Using the estimate $\hat{\tau} \geq \eta_\sim$ along with induction hypothesis, $\pi_0, \pi_1$ satisfy the hypotheses of Lemma 3.14 with $K = \eta_\sim^{-1}$. Consider the function $\hat{\pi}_1 : \mathcal{S}_{i+1} \to (0, \infty)$ defined by (3.46) as
$$\hat{\pi}_1(v) = \pi_1(v) \lor \left(\frac{1}{K} \max\{\pi_1(v) : v \in \mathcal{S}_{i+1}, D_2(v, w) \leq 1\}\right),$$

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and set \( \rho : S_{i+1} \to (0, \infty) \) as

\[
\rho(w) = \frac{\hat{\pi}_1(w)}{\pi_0(\tilde{w})}, \quad \text{for all } w \in S_{i+1}, \text{ where } \tilde{w} \text{ is the parent of } w.
\]

Since \( \hat{\pi}_1 \geq \pi_1 \) the condition (1) is satisfied. By Lemma 3.14, the condition (2) above is also satisfied on \( \bigcup_{j=0}^{i+1} S_j \). It only remains to check (3) on \( S_{i+1} \). For \( v \in S_{i+1} \), we have two possibilities for \( \hat{\pi}_1(v) \): either \( \hat{\pi}_1(v) = \pi_1(v) \) or \( \hat{\pi}_1(v) = K^{-1} \pi_1(w) \) for some \( w \in S_{i+1} \) such that \( D_2(v, w) = 1 \). The first possibility implies that \( \rho(v) = \hat{\tau}(v) \) and hence (3) is satisfied for \( v \). The other possibility is that \( \hat{\pi}_1(v) = K^{-1} \pi_1(w) \). In this case, let \( \tilde{v}, \tilde{w} \in S_i \) denote the parents of \( v, w \) respectively. By Lemma 2.4(c), \( D_2(\tilde{v}, \tilde{w}) \leq 1 \). Therefore by Lemma 3.14, we have

\[
\rho(v) = \frac{\hat{\tau}(w) \pi_0(\tilde{w})}{K \pi_0(\tilde{v})} \leq \hat{\tau}(w),
\]

which concludes the proof of condition (3) above. By induction, there exists a function \( \rho : S \to (0, \infty) \) which satisfies (1), (2), (3) above.

Next, we want to show that \( \rho \) satisfies the upper bound \( \rho \leq \eta_+ \) in (H1) for some \( \eta_+ \in (0,1) \) and the hypothesis (H4) whenever \( \eta_0 \) is small enough. To this end, consider

\[
\sum_{v \in C(u)} \rho(v)^p \leq \sum_{v \in C(u)} \sum_{w \in S : D_2(v, w) \leq 1, \pi_2(v) = \pi_2(w)} \hat{\tau}(w)^p \quad \text{(by condition (3))}
\]

\[
\leq (M_2 + 1) \sum_{\tilde{u} \in S : D_2(u, \tilde{w}) \leq 1, \pi_2(\tilde{w}) = \pi_2(u)} \sum_{v \in C(\tilde{u})} \hat{\tau}(v)^p \quad \text{(by (3.49) and Lemma 2.4(c))}
\]

\[
\leq 2^{p+1}(M_2 + 1)(M_2 + 1)^2 \eta_0 \quad \text{(by (3.49) and (3.50)).}
\]

By the above estimate, the choice \( \eta_0 \in (0,1) \) such that \( 2^{p+1}(M_2 + 1)(M_2 + 1)^2 \eta_0 = 2^{-p} \) implies the upper bound \( \rho \leq \eta_+ \) in (H1) for \( \eta_+ = \frac{1}{2} \in (0,1) \) and also (H4). Since \( \hat{\tau} \) satisfies (H3') and \( \rho \geq \hat{\tau} \), \( \rho \) also satisfies (H3'). This along with conditions (1), (2) above and Proposition 3.10 implies that \( \rho \) satisfies (H1), (H2), (H3), and (H4). The desired conclusion follows from Theorem 3.2. \( \Box \)

4 Critical exponent associated to the combinatorial modulus

Let \( G = (V, E) \) be a graph and let \( \Gamma \) be a family of paths in \( G \). Consider a function \( \rho : V \to [0, \infty) \) and for \( \gamma \in \Gamma \), we define its \( \rho \)-length as

\[
\ell_\rho(\gamma) := \sum_{v \in \gamma} \rho(v).
\]

For \( p > 0 \), we denote the \( p \)-mass of \( \rho \) by

\[
M_{\rho}(p) = \sum_{v \in V} \rho(v)^p.
\]
The \( p \)-combinatorial modulus of \( \Gamma \) is defined as
\[
\text{Mod}_p(\Gamma, G) = \inf_{\rho \in \text{Adm}(\Gamma)} M_p(\rho),
\]
where \( \text{Adm}(\Gamma) := \{ \rho : V \to [0, \infty) \mid \ell_\rho(\gamma) \geq 1 \text{ for all } \gamma \in \Gamma \} \) denote the set of \( \Gamma \)-admissible functions. If \( \Gamma = \emptyset \), we set \( \text{Mod}_p(\Gamma, G) = 0 \) by convention.

Let \( a, \lambda, L \in (1, \infty) \). Let \( X_k \) denote a maximal \( a^{-k} \) separated subset of \( X \) for all \( k \geq 0 \) and let \( S_k = \{(x, k) : x \in X_k\} \). In this section, we need not assume that \( X_k \) is increasing in \( k \). Let \( \pi_1 : S_k \to X, \pi_2 : S_k \to \mathbb{N}_{\geq 0} \) be the projection maps to the first and second components.

For each \( k \geq 1 \), we define a graph \( G_k \) whose vertex set is \( S_k \) and whose edges are the horizontal edges in \((S, D_2)\) (or equivalently \((S, D_1)\)); that is, there is an edge between distinct vertices \( v \) and \( w \) if and only if \( B(\pi_1(v), \lambda a^{-\pi_2(v)}) \cap B(\pi_1(v), \lambda a^{-\pi_2(w)}) \neq \emptyset \). Let \( p > 0, L > 1 \). For \( v \in S \), we define
\[
\Gamma_{k,L}(v) = \inf \left\{ \gamma = (v_1, v_2, \ldots, v_n) \mid \gamma \text{ is a path in } G_{\pi_2(v)+k} \text{ with } \pi_1(v_1) \in B_v, \pi_1(v_n) \notin B(\pi_1(w), La^{-\pi_2(w)}) \right\}. \tag{4.1}
\]
Define
\[
M_{p,k}(L) = \sup_{v \in S} \text{Mod}_p(\Gamma_{k,L}(v), G_{\pi_2(v)+k}),
\]
\[
M_p(L) = \liminf_{k \to \infty} M_{p,k}(L). \tag{4.2}
\]
The critical exponent of the combinatorial modulus is defined as
\[
Q(L) = \inf \{ p \in (0, \infty) : M_p(L) = 0 \}. \tag{4.3}
\]

If \( \rho \in \text{Adm}(\Gamma) \), then \( 1 \land \rho \in \text{Adm}(\Gamma) \). This shows that \( \text{Mod}_p(\Gamma, G) \) is non-increasing in \( G \) for any family of paths \( \Gamma \) and any graph \( G \). This shows that the set of \( p \) such that \( M_p(L) = 0 \) is an interval.

Strictly speaking \( Q(L) \) should be denoted as \( Q(a, \lambda, L, \{X_k : k \geq 0\}) \) since it might depend on all these choices. It is known that \( Q(L) \) does not depend on the choice of \( L > 1 \) [Car, Lemma 3.3]. We will show that it also does not depend on the choices of the \( a, \lambda \in (1, \infty) \) and \( \{X_k, k \geq 0\} \). To this end, we recall the following lemma.

Lemma 4.1. [Kig22, Lemma C.4] Let \( G = (V, E), G' = (V', E') \) be two graphs and let \( H : V \to 2^{V'} \) be a function so that \( \#H(v) < \infty \) for all \( v \in V \). Let \( \Gamma, \Gamma' \) be two families of paths in \( G, G' \) respectively such that for each \( \gamma \in \Gamma \), there exists \( \gamma' \in \Gamma' \) so that \( \gamma' \) is contained in \( \bigcup_{v \in \gamma} H(v) \). Then
\[
\text{Mod}_p(\Gamma, G) \leq \left( \sup_{v \in V} \#H(v) \right)^p \sup_{v' \in V'} \#\{ v \in V \mid v' \in H(v) \} \text{Mod}_p(\Gamma', G'). \tag{4.4}
\]

The following proposition shows that the critical exponent for the combinatorial modulus is well defined.
Proposition 4.2 (critical exponent is well defined). Let \( a, a', \lambda, \lambda', L, L' \in (1, \infty) \). Let \( X_k \) (resp. \( X'_k \)) denote a sequence of maximal \( a^{-k} \)-separated (resp. \( a'^{-k} \)-separated) subsets of \( X \). Let \( Q(L), Q'(L') \) denote the corresponding critical exponents be as defined in (4.3) for these two sets of parameters. Then

\[ Q(L) = Q'(L'). \]

Proof. Let \( M_{p,k}(L) \) and \( M'_{p,k}(L') \) be as defined in (4.2). Let \( G_k, G'_k, k \geq 0 \) be the corresponding graphs. By symmetry, it suffices to show that \( Q(L) \leq p \). To show this, we need an upper bound on \( \text{Mod}_p(\Gamma_{k,L}(v), G_{\pi_2(v)+k}) \) for \( v \in G_n, n \in \mathbb{N} \). Let \( n' \in \mathbb{N} \) be unique integer such that

\[ 2L'(a')^{-n'} < (L - 1)a^{-n'} \leq 2L'(a')^{-n'+1} \quad (4.5) \]

so that for any \( \gamma = (v_1, \ldots, v_{n'}) \in \Gamma_{n,k}(v) \), and for any \( x \in X \) such that \( \pi_1(v_1) \in B(x, a^{-n'}) \), we have \( \pi_1(v_N) \notin B(x, L'a^{-n'}) \). For any \( l, l' \in \mathbb{N} \), we set \( H_{l,l'} : S_n \rightarrow 2^{S'_n} \) as

\[ H_{l,l'}(v) = \left\{ w \in S'_{l'} : d(\pi_1(v), \pi_1(w)) = \min_{u \in S'_l} d(\pi_1(v), \pi_1(w)) \right\}. \]

Let \( \beta > d_A(X, d) \). From [Hei, Exercise 10.17], it is easy to see that

\[ \sup_{v \in S_l} \#H_{l,l'}(v) \leq 1 \lor \left( \frac{a^{-l}}{a'^{-l'}} \right)^{\beta}, \quad \sup_{v' \in S'_{l'}} \# \{ v \in S_l : v' \in H_{l,l'}(v) \} \leq 1. \quad (4.6) \]

Given \( k' \in \mathbb{N} \), let \( k \in \mathbb{N} \) be the smallest integer such that

\[ L'(\lambda' - 1)(a')^{-k'} > \lambda(L - 1)a^{-k}. \]

This along with (4.5) implies

\[ (\lambda' - 1)(a')^{-n'-k'} > 2\lambda a^{-n-k}, \quad (4.7) \]

so that the following property holds: for all neighbors \( u, v \) in \( G_{n+k} \) and for all \( u' \in H_{n+k,n'+k'}(u), v' \in H_{n+k,n'+k'}(v) \), either \( u' = v' \) or \( u' \) and \( v' \) are neighbors in \( G_{n'+k'} \). Let \( \varepsilon \in (0, L' - 1) \) be arbitrary. Combining the above property, (4.5), (4.6), (4.7) and Lemma 4.1, for all large enough \( k' \), we obtain

\[ \text{Mod}_p(\Gamma_{k,L}(v), G_{n+k}) \lesssim \text{Mod}_p(\bigcup_{v' \in H_{n,n'+k'}(v)} \Gamma_{k',L' - \varepsilon}(v'), G_{n'+k'}) \]

\[ \leq \sum_{v' \in H_{n,n'+k'}(v)} \text{Mod}_p(\Gamma_{k',L' - \varepsilon}(v'), G_{n'+k'}) \lesssim M'_{p,k'}(L' - \varepsilon). \quad (4.8) \]

This implies \( M_{p,k}(L) \lesssim M'_{p,k'}(L' - \varepsilon) \) for all large enough \( k' \in \mathbb{N} \). This along with [Car, Lemma 3.3] implies \( Q(L) \leq p \) for any \( p > Q'(L') \). \( \square \)

The following ‘reverse volume doubling estimate’ is known if the metric space is uniformly perfect [Hei, Exercise 13.1]. Since our metric space is not necessarily uniformly perfect, the following lemma provides a substitute for uniform perfectness at sufficiently many scales.
Lemma 4.3. Let $\mu$ be a be a doubling measure on a metric space $(X,d)$ such that $\mu(B(x,2r)) \leq C_D\mu(B(x,r))$ for all $x \in X, r > 0$. Let $x_0, \ldots, x_N$ be a set of points such that $d(x_i, x_{i+1}) < r/4$ for all $i = 0, \ldots, N - 1$ such that $d(x_0, x_N) > R > r$. There exists $c, \alpha > 0$ depending only on $C_D$ such that $\mu(B(x_0, R)) \geq c(R/r)^\alpha$.

Proof. If $s \in [r, R/2]$, then by triangle inequality there exists $x_j$ such that
\[
\frac{5}{4}s \leq d(x_0, x_j) \leq \frac{7}{4}s.
\]
By the doubling property, for such $x_j$, we have
\[
\mu(B(x_j, s/4)) \geq C_D^{-3}\mu(B(x_j, 4s)) \geq C_D^{-3}\mu(B(x_0, s))
\]
Therefore
\[
\mu(B(x_0, 2s)) \geq \mu(B(x_0, s)) + \mu(B(x_j, s/4)) \geq (1 + C_D^{-3})\mu(B(x_0, s))
\]
for all $s \in [r, R/2]$. Let $k$ be the largest integer such that $2^k r \leq R$. By iterating the above estimate
\[
\mu(B(x_0, R)) \geq \mu(B(x_0, 2^{k-1}r)) \geq (1 + C_D^{-3})^{k} \geq c \left( \frac{R}{r} \right)
\]
where $\alpha = \log(1 + C_D^{-3})/\log 2$ and $c = (1 + C_D^{-3})^{-1}$.

The following is the main result of this work.

Theorem 4.4. Let $(X,d)$ be a compact doubling metric space. Then the conformal Assouad dimension $d_{CA}(X,d)$ equals the critical exponent of the combinatorial modulus $Q$. Furthermore $d_{CA}(X,d) = \inf\{d_A(X,\theta) : \theta \in J_p(X,d)\}$.

Proof of Theorem 4.4. By Proposition 4.2, it suffices to consider the critical exponent $a \geq \lambda \geq 6$, where the maximal $a^{-n}$ separated subsets are increasing (similar to the definition of hyperbolic filling).

The inequality $d_{CA}(X,d) \leq Q(L)$ follows from the same argument as the proof of $d_{ARC}(X,d) \leq Q(L)$ in [Car, Theorem 1.3] where the use of [Car, Theorem 1.2] is replaced with Theorem 3.9. This yields the inequality
\[
d_{CA}(X,d) \leq \inf\{d_A(X,\theta) : \theta \in J_p(X,d)\} \leq Q(L).
\]

So it suffices to show $Q(L) \leq d_{CA}(X,d)$. Let $d_{CA}(X,d) < p$. We consider a hyperbolic filling $S$ of $(X,d)$ with parameters $a \geq \lambda \geq 6$. Then by Theorem 1.1, for any $q \in (d_{CA}(X,d), p)$, there exists $\theta \in J(X,d)$ and a $q$-homogeneous measure $\mu$ on $(X,\theta)$. Let $v \in S, k \in \mathbb{N}$. Define $\rho : S_{\pi_2(v) + k} \rightarrow [0, \infty)$ as
\[
\rho(w) = \begin{cases} 
\left( \frac{\mu(B_w)}{\mu(B_v)} \right)^{1/q} & \text{if } B_w \cap B_d(\pi_1(v), (L + 1)a^{-\pi_2(v)}) \neq \emptyset, w \in \gamma \text{ for some } \gamma \in \Gamma_{k,L}(v), \\
0 & \text{otherwise.}
\end{cases}
\]
We claim that there exists $c > 0$ so that
\[
\sum_{w \in \gamma} \rho(w) \geq c, \quad \text{for all } \gamma \in \Gamma_{k,L}(v), \, v \in S, \quad (4.11)
\]
for all large enough $k$. Let $\gamma \in \Gamma_{k,L}(v)$. To show (4.11), by choosing a sub-path if necessary, we may assume that $\gamma = (v_1, \ldots, v_N)$ and $\pi_1(v_1) \in B_v$, $\pi_1(v_j) \notin B(\pi_1(v), a^{-\pi_2(v)}) = B_v$ for all $j = 2, \ldots, N-1$, $\pi_1(v_i) \in B(\pi_1(v), L a^{-\pi_2(v)})$ for all $i = 1, \ldots, N-1$ and $v_N \notin \pi_1(v_i) \in B_d(\pi_1(v), L a^{-\pi_2(v)})$. Since
\[
d(\pi_1(v_i), \pi_1(v_{i+1})) < 2\lambda a^{-k-\pi_2(v)} \leq a^{-k-\pi_2(v)+1} \leq a^{-\pi_2(v)}, \quad \text{for all } i = 1, \ldots, N-1,
\]
we have $v_N \in B(v, (L + 1)a^{-\pi_2(v)})$. In particular,
\[
z_i \in B_d(\pi_1(v_i), 2\lambda a^{-\pi_2(v_i)}) \setminus B_d(\pi_1(v_i), a^{-\pi_2(v_i)}) \neq \emptyset \quad \text{for all } i = 1, \ldots, N,
\]
where $z_i = \pi_1(v_{i+1})$ for $i = 1, \ldots, N-1$ and $\pi_1(v_{i-1})$ for $i = N$. Let $\eta : [0, \infty) \to [0, \infty)$ be a distortion function such that the identity map $\text{Id} : (X, d) \to (X, \theta)$ is an $\eta$-quasisymmetry. This along with the choice of $z_i$ above this implies that
\[
B(\pi_1(v_i), c_1 \theta(\pi_1(v_i), z_i)) \subset B_d(\pi_1(v_i), a^{-\pi_2(v_i)}) = B_v \subset B(\pi_1(v_i), \eta(1) \theta(\pi_1(v_i), z_i)),
\]
for all $i = 1, \ldots, N$, where $c_1 = \lceil \eta(2\lambda) \rceil^{-1}$. Since $d(\pi_1(v_i), \pi_1(v_i)) < (L + 1)a^{-\pi_2(v)}$ for all $i = 1, \ldots, N$, we have
\[
B_v = B_d(\pi_1(v), a^{-\pi_2(v)}) \subset B_d(\pi_1(v_i), (L + 2)a^{-\pi_2(v)}). \quad \text{Choosing } w_i \in \{\pi_1(v), \pi_1(v_N)\} \text{ such that } 2(L + 2)a^{-\pi_2(v)} > d(\pi_1(v_i), w_i) \geq d(\pi_1(v), \pi_1(v_N))/2 \geq La^{-\pi_2(v)}/2, \quad \text{we have}
\]
\[
B_v \subset B_d(\pi_1(v_i), (L + 2)a^{-\pi_2(v)}) \subset B(\pi_1(v_i), C_2 \theta(\pi_1(v_i), w_i)),
\]
for all $i = 1, \ldots, N$, where $C_2 = \eta(2(L + 2)L^{-1})$. Furthermore
\[
B(\pi_1(v_i), \theta(\pi_1(v_i), w_i)) \subset B_d(\pi_1(v_i), \eta(1)d(\pi_1(v_i), w_i)) \subset B(\pi_1(v_i), C_3a^{-\pi_2(v)}),
\]
where $C_3 = (L + 1)(2\eta(1) + 1)$.

Since $\mu$ is $q$-homogeneous on $(X, \theta)$ and $\theta \in \mathcal{J}(X, d)$, $\mu$ is a doubling measure on $(X, d)$. Therefore
\[
\mu(B_v) \gtrsim \mu(B_d(\pi_1(v), C_3a^{-\pi_2(v)})) \gtrsim \mu(\theta(\pi_1(v_i), w_i)). \quad (4.17)
\]

\[
\theta(\pi_1(v_i), \pi_1(v_{i+1})) \leq \theta(\pi_1(v_i), w_i) \left[\eta^{-1} \left(4^{-1}L\lambda^{-1}a^k\right)\right]^{-1} \quad \text{for all } i = 1, \ldots, N-1. \quad (4.18)
\]
Pick $k_0 \in \mathbb{N}$ large enough so that
\[
c_1 \left[\eta^{-1} \left(4^{-1}L\lambda^{-1}a^{k_0}\right)\right]^{-1} \leq 1. \quad (4.19)
\]
Let \( k \geq k_0 \). Since \( \mu \) is a \( q \)-homogeneous measure on \((X, \theta)\) and \( \text{Id} : (X, d) \to (X, \theta) \) is an \( \eta \)-quasisymmetry, we have

\[
\sum_{i=1}^{N} \frac{\rho(v_i)}{\mu(B_{v_i})} = \frac{1}{\mu(B_v)} \left( \frac{\mu(B_{v_i})}{\mu(B_v)} \right)^{1/q} = \frac{\sum_{i=1}^{N} \frac{\mu(B_{v_i})}{\mu(B_v)}^{1/q}}{\mu(B_v)} \geq \frac{1}{\mu(B_v)} \left( \frac{\sum_{i=1}^{N} \mu(B_{v_i})}{\mu(B_v)} \right)^{1/q} \geq \frac{\mu(B_w)}{\mu(B_v)} \left( \frac{\sum_{i=1}^{N} \mu(B_{v_i})}{\mu(B_v)} \right)^{1/q} \geq \frac{\mu(B_w)}{\mu(B_v)} \left( \frac{\sum_{i=1}^{N} \mu(B_{v_i})}{\mu(B_v)} \right)^{1/q} 
\]

(by (4.14) and (4.15))

\[
\sum_{i=1}^{N} \frac{\mu(B_{v_i})}{\mu(B_v)}^{1/q} \geq \frac{\mu(B_w)}{\mu(B_v)} \left( \frac{\sum_{i=1}^{N} \mu(B_{v_i})}{\mu(B_v)} \right)^{1/q} \geq \frac{\mu(B_w)}{\mu(B_v)} \left( \frac{\sum_{i=1}^{N} \mu(B_{v_i})}{\mu(B_v)} \right)^{1/q} \geq \frac{\mu(B_w)}{\mu(B_v)} \left( \frac{\sum_{i=1}^{N} \mu(B_{v_i})}{\mu(B_v)} \right)^{1/q} \geq 1
\]

(by triangle inequality and \( \mu(B_{v_i}) \geq \mu(B_{v_N}) \).

This completes the proof of (4.11), where \( c > 0 \) depends only on \( \eta \), \( q \)-homogeneity constants of \( \mu \) and \( \lambda, a, L \).

For any \( v \in S \), \( w \in S_{\pi_2(v)+k} \) such that \( w \in \gamma \) for some \( (v_1, \ldots, v_N) = \gamma \in \Gamma_{k,L}(v) \). Note that \( d(\pi_1(w), \pi_1(v_1)) \vee d(\pi_1(w), \pi_1(v_N)) \geq \frac{1}{2}d(\pi_1(v_1), \pi_1(v_N)) \geq (L-1)a^{-\pi_2(v)/2} \).

Therefore there exists a sequence \( w = x_0, \ldots, x_M \) so that \( d(x_0, x_M) \geq (L-1)a^{-\pi_2(w)} \).

Choose \( k_1 \in \mathbb{N} \) such that \( (L-1)a^{-k_1} \leq \frac{1}{4} \). By the volume doubling property of \( \mu \) on \((X, d)\), we have \( \mu(B_v) \geq \mu(B(\pi_1(v), 2La^{-\pi_2(v)}) \geq \mu(B(\pi_1(v), (L-1)a^{-\pi_2(v)/2)}) \).

For all \( k \geq k_1 \), by Lemma 4.3, we have

\[
\frac{\mu(B_w)}{\mu(B_v)} \leq \frac{\mu(B_w)}{\mu(B(\pi_1(w), (L-1)a^{-\pi_2(v)/2}))} \leq Ca^{-k}, \quad \text{for all } v \in S, w \in \gamma, \gamma \in \Gamma_{k,L}(v),
\]

(4.20)

where \( C, \alpha \) only depends on \( \lambda, a \) the doubling constant of \( \mu \) in \((X, d)\). Since \( \mu \) is a doubling measure, we have

\[
\sum_{w \in S_{\pi_2(v)+k}} \rho(w)^q \leq \sum_{w \in S_{\pi_2(v)+k}, \pi_1(w) \in B(\pi_1(v), (L+1)a^{-\pi_2(v)})} \frac{\mu(B_w)}{\mu(B_d(\pi_1(v), (L+2)a^{-\pi_2(v)}))} \leq \sum_{w \in S_{\pi_2(v)+k}, \pi_1(w) \in B(\pi_1(v), (L+1)a^{-\pi_2(v)})} \frac{\mu(B_d(\pi_1(v), a^{-\pi_2(w)/2}))}{\mu(B_d(\pi_1(v), (L+2)a^{-\pi_2(v)}))} \leq 1 \quad \text{(since } B_d(\pi_1(w), a^{-\pi_2(w)/2}) \text{ pairwise disjoint)}
\]

(4.21)

By (4.11), \( c^{-1} \rho \in \text{Adm}(\Gamma_{k,L}(v)) \), and hence by (4.20) and (4.21), we have

\[
\text{Mod}_p(\Gamma_{k,L}(v)) \lesssim \sum_{w} \rho(w)^p(w) \leq \left( \sup_{w} \rho(w) \right)^{p-q} \sum_{w} \rho(w)^q \lesssim a^{-k \alpha(p-q)}.
\]

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This implies $M_{p,k}(L) \lesssim a^{-k\alpha(p-q)}$ for all large enough $k$ and hence $M_p(L) = 0$ and $Q(L) \leq d_{CA}(X, d)$. This along with (4.9) concludes the proof. \qed

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References


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