

Martingale and analytic dimensions

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Abstract

These notes introduce the martingale dimension of a strongly local regular Dirichlet form and the analytic dimension of a Lipschitz differentiability space. We explain why these dimensions coincide under two-sided Gaussian heat kernel estimates and discuss their relationships with other notions of dimension. We also introduce the energy image density property, its recent resolution, and its applications to martingale dimension. Finally, we show that spaces carrying diffusions with two-sided sub-Gaussian heat kernel estimates of walk dimension greater than two are not Lipschitz differentiability spaces.

Strongly local regular Dirichlet forms provide an analytic framework for describing symmetric diffusions through their associated Markov semigroups. Lipschitz differentiability spaces, on the other hand, arise from extending Rademacher’s theorem on the almost everywhere differentiability of Lipschitz functions to metric measure spaces. These different motivations have led the two theories to develop largely independently.

Despite their different origins, martingale dimension for strongly local regular Dirichlet spaces and analytic dimension for Lipschitz differentiability spaces can both be interpreted as notions of “tangent-space dimension” in nonsmooth settings. The energy image density property, which originated in Malliavin calculus, and related absolute-continuity results provide a connection between the two theories and play an important role in understanding these dimensions.

The aim of these notes is to introduce these two notions of dimension, explain their connections, and highlight their similarities and differences. Throughout, we discuss open problems that illustrate how incomplete our understanding of these dimensions remains. The first three sections largely follow the three lectures delivered by the second-named author at the 18th MSJ-SI in Fukuoka. The final section presents a new result showing that the underlying metric measure spaces of a broad class of Dirichlet spaces arising in the study of diffusions on fractals are not Lipschitz differentiability spaces.

1 Lecture 1: Definitions of martingale and analytic dimensions

In this lecture, we introduce the notions of martingale and analytic dimensions, using Euclidean space and the Heisenberg group as our main examples. These examples illustrate how both notions recover the dimension of the tangent space in Euclidean space and of the horizontal tangent space in the Heisenberg group.

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1.1 Strongly local Dirichlet forms

The notion of martingale dimension is formulated in terms of the diffusion associated with a regular strongly local Dirichlet form, whose definition we recall below.

Let X be a locally compact, separable, metrizable topological space, and let m be a Radon measure on X with full support. A pair $(\mathcal{E}, \mathcal{F})$ is called a *Dirichlet form* on $L^2(X, m)$ if $\mathcal{F}$ is a dense linear subspace of $L^2(X, m)$ and $\mathcal{E}: \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ is a symmetric, non-negative definite bilinear form that is closed and Markovian. Here, closedness means that $\mathcal{F}$ is a Hilbert space with respect to the inner product

$$\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \langle f, g \rangle_{L^2(X, m)}, \quad f, g \in \mathcal{F},$$

and the Markov property means that, for every $f \in \mathcal{F}$,

$$f^+ \wedge 1 \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(f^+ \wedge 1, f^+ \wedge 1) \leq \mathcal{E}(f, f).$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_c(X)$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(C_c(X), \|\cdot\|_{\sup})$. It is called *strongly local* if

$$\mathcal{E}(f, g) = 0$$

whenever $f, g \in \mathcal{F}$, $\text{supp}_m[g]$ is compact, and there exists $a \in \mathbb{R}$ such that

$$\text{supp}_m[f - a\mathbf{1}_X] \cap \text{supp}_m[g] = \emptyset.$$

Here, $\text{supp}_m[f]$ denotes the support of the measure $f \cdot m$.

The *energy measure* $\Gamma(f, f)$ of $f \in \mathcal{F}$ associated with the strongly local regular form $(\mathcal{E}, \mathcal{F})$ is defined, first for $f \in \mathcal{F} \cap L^\infty(X, m)$ as the unique positive Borel measure on X such that

$$\int_X g d\Gamma(f, f) = \mathcal{E}(f, fg) - \frac{1}{2}\mathcal{E}(f^2, g) \quad \text{for all } g \in \mathcal{F} \cap C_c(X), \quad (1.1)$$

and then by $\Gamma(f, f)(A) := \lim_{n \rightarrow \infty} \Gamma((-n) \vee (f \wedge n), (-n) \vee (f \wedge n))(A)$ for each Borel subset A of X for general $f \in \mathcal{F}$. The signed measure $\Gamma(f, g)$ for $f, g \in \mathcal{F}$ is defined by polarization.

We say that a positive σ -finite Borel measure ν is a *minimal energy dominant measure* if it satisfies the following:

- (a) (Energy dominance) $\Gamma(f, f) \ll \nu$ for every $f \in \mathcal{F}$, where $\Gamma(f, f)$ is the energy measure of f .
- (b) (Minimality) If $\tilde{\nu}$ satisfies (a), then $\nu \ll \tilde{\nu}$.

Note that (b) implies that any two minimal energy dominant measures are mutually absolutely continuous. Every strongly local, regular Dirichlet form has a minimal energy dominant measure [51, Lemma 2.2], [30, Lemma 2.2]. Next, we recall the definition of martingale dimension from [30].

1.2 Martingale dimension

Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(X, m)$. Denote its energy measures by $\Gamma(\cdot, \cdot)$, and let ν be a minimal energy dominant measure. The *pointwise index* of $(\mathcal{E}, \mathcal{F})$ is a measurable function $p_H: X \rightarrow \mathbb{N} \cup \{0, \infty\}$ satisfying the following conditions:

(a) For every $N \in \mathbb{N}$ and $f_1, \dots, f_N \in \mathcal{F}$,

$$\text{rank} \left(\frac{d\Gamma(f_i, f_j)}{d\nu}(x) \right)_{1 \leq i, j \leq N} \leq p_H(x)$$

for ν -almost every $x \in X$.

(b) If a measurable function $p'_H : X \rightarrow \mathbb{N} \cup \{0, \infty\}$ satisfies (a) in place of p_H , then $p_H \leq p'_H$ ν -almost everywhere.

These conditions determine p_H uniquely up to ν -null sets. The *martingale dimension* of $(\mathcal{E}, \mathcal{F})$ is

$$\nu\text{-esssup}_{x \in X} p_H(x).$$

The notion of martingale dimension and its terminology originate in the work of Motoo and Watanabe [47] on stochastic integral representations of martingale additive functionals of Markov processes (see also [20]). Roughly speaking, the martingale dimension of the associated diffusion is the smallest number k for which there is a fixed family of k martingale additive functionals of finite energy such that every martingale additive functional of finite energy admits a stochastic integral representation with respect to this family. If no finite family suffices, the martingale dimension is infinite. The equivalence between this probabilistic definition and the analytic definition above is due to Hino [30, Theorem 3.4], building on earlier work of Kusuoka [41, 42].

1.3 Examples: Euclidean space and the Heisenberg group

Let $\mathcal{L}_n$ denote Lebesgue measure on $\mathbb{R}^n$. The heat semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy, \quad t > 0,$$

is associated with the strongly local regular Dirichlet form

$$\mathcal{F} = W^{1,2}(\mathbb{R}^n), \quad \mathcal{E}(f, g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g d\mathcal{L}_n.$$

Here the gradients are understood in the weak sense. With this normalization,

$$P_t f(x) = \mathbb{E}_x[f(B_{2t})],$$

where $(B_t)_{t \geq 0}$ is standard Brownian motion.

The energy measures are given by

$$\Gamma(f, g) = (\nabla f \cdot \nabla g) \mathcal{L}_n, \quad f, g \in \mathcal{F}.$$

In particular, $\mathcal{L}_n$ is energy dominant. To see that it is minimal, fix a ball B and choose $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to one on a neighbourhood of $\bar{B}$. The functions $f_i(x) = \chi(x)x_i$, $1 \leq i \leq n$, belong to $\mathcal{F}$ and satisfy $\nabla f_i = e_i$ on B . Consequently,

$$\Gamma(f_i, f_i)|_B = \mathcal{L}_n|_B.$$

Every energy dominant measure therefore dominates $\mathcal{L}_n|_B$. Taking a countable collection of balls covering $\mathbb{R}^n$ shows that $\nu = \mathcal{L}_n$ is minimal energy dominant.

For any $f_1, \dots, f_N \in \mathcal{F}$, the matrix of energy densities is the Gram matrix of their gradients. Thus, almost everywhere,

$$\begin{aligned} \operatorname{rank} \left(\frac{d\Gamma(f_i, f_j)}{d\nu}(x) \right)_{i,j=1}^N &= \operatorname{rank} (\nabla f_i(x) \cdot \nabla f_j(x))_{i,j=1}^N \\ &= \dim \operatorname{span} \{ \nabla f_1(x), \dots, \nabla f_N(x) \} \leq n. \end{aligned}$$

It follows that $p_H \leq n$ almost everywhere. Conversely, the cutoff coordinate functions above have the identity matrix as their matrix of energy densities on B , so $p_H \geq n$ almost everywhere on B . A countable covering by balls therefore gives

$$p_H(x) = n \quad \text{for } \mathcal{L}_n\text{-almost every } x \in \mathbb{R}^n.$$

Hence the martingale dimension is n , recovering the dimension of the Euclidean tangent space.

The Heisenberg group provides an example in which the martingale dimension differs from the dimension of the underlying smooth manifold. We first describe its group structure and the associated Brownian motion.

The *Heisenberg group* $\mathbb{H}$ is $\mathbb{R}^3$ equipped with the multiplication

$$(x, y, z) \odot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx') \right).$$

It is a non-commutative group whose identity is $(0, 0, 0)$, and

$$(x, y, z)^{-1} = (-x, -y, -z).$$

Lebesgue measure $\mathcal{L}_3$ is invariant under both left and right translations and hence is a Haar measure on $\mathbb{H}$.

The left-invariant vector fields obtained by translating the coordinate directions ∂_x and ∂_y at the identity are

$$V_1 = \partial_x - \frac{y}{2}\partial_z, \quad V_2 = \partial_y + \frac{x}{2}\partial_z.$$

Their span at each point is called the *horizontal tangent space*. We equip these two-dimensional spaces with the inner product for which V_1, V_2 are orthonormal. Although they span only two directions, their Lie bracket supplies the third:

$$[V_1, V_2] = \partial_z.$$

Let $(B_t^{(1)}, B_t^{(2)})_{t \geq 0}$ be standard planar Brownian motion started at the origin, and define

$$A_t = \frac{1}{2} \int_0^t \left(B_s^{(1)} dB_s^{(2)} - B_s^{(2)} dB_s^{(1)} \right), \quad X_t = (B_t^{(1)}, B_t^{(2)}, A_t).$$

The process A_t is called *Lévy's stochastic area*. To explain the terminology, consider the one-form

$$\alpha = \frac{1}{2}(x dy - y dx), \quad d\alpha = dx \wedge dy.$$

For a smooth closed planar curve, Stokes' theorem identifies the integral of α with its signed enclosed area. The stochastic integral defining A_t is the corresponding area functional for a

Brownian path. Its Itô and Stratonovich interpretations agree, since $[B^{(1)}, B^{(2)}]_t = 0$. The process X_t solves the Stratonovich equation

$$dX_t = V_1(X_t) \circ dB_t^{(1)} + V_2(X_t) \circ dB_t^{(2)}, \quad X_0 = (0, 0, 0).$$

Thus it is driven by independent Brownian noises in the two orthonormal horizontal directions. It is called *horizontal Brownian motion on the Heisenberg group*, or simply Brownian motion on $\mathbb{H}$ in what follows.

The group law also gives a direct analogue of the increment properties of Euclidean Brownian motion. For $0 \leq s < t$,

$$X_s^{-1} \odot X_t = \left(B_t^{(1)} - B_s^{(1)}, B_t^{(2)} - B_s^{(2)}, A_{s,t} \right),$$

where

$$A_{s,t} = \frac{1}{2} \int_s^t \left((B_u^{(1)} - B_s^{(1)}) dB_u^{(2)} - (B_u^{(2)} - B_s^{(2)}) dB_u^{(1)} \right).$$

This increment depends only on the Brownian increments after time s . Consequently, it is independent of the past and has the same law as X_{t-s} . Thus X_t has continuous paths and stationary, independent increments with respect to the group operation. These properties parallel those of Euclidean Brownian motion and justify the terminology “Brownian motion on the Heisenberg group.”

By Itô’s formula, the generator is

$$L = \frac{1}{2} (V_1^2 + V_2^2).$$

More explicitly, for every $f \in C_c^\infty(\mathbb{R}^3)$,

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds$$

is a martingale. The associated strongly local regular Dirichlet form is the closure on $L^2(\mathbb{H}, \mathcal{L}_3)$ of

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{H}} (V_1 f V_1 g + V_2 f V_2 g) d\mathcal{L}_3, \quad f, g \in C_c^\infty(\mathbb{R}^3).$$

Denote its domain by $\mathcal{F}$ and write

$$\nabla_H f = (V_1 f, V_2 f)$$

for the horizontal gradient. For $f \in \mathcal{F}$, these derivatives are understood in the weak sense. The energy measures satisfy

$$\Gamma(f, g) = \frac{1}{2} \nabla_H f \cdot \nabla_H g d\mathcal{L}_3, \quad f, g \in \mathcal{F}.$$

In particular, $\mathcal{L}_3$ is energy dominant. As in the Euclidean example, cutoff coordinate functions show that it is minimal. Indeed, fix a bounded Euclidean ball B and choose $\chi \in C_c^\infty(\mathbb{R}^3)$ equal to one on a neighbourhood of $\overline{B}$. The functions

$$f_1(x, y, z) = \chi(x, y, z)x, \quad f_2(x, y, z) = \chi(x, y, z)y$$

satisfy

$$\nabla_H f_1 = (1, 0), \quad \nabla_H f_2 = (0, 1) \quad \text{on } B.$$

In particular, $\Gamma(f_1, f_1)|_B = \frac{1}{2}\mathcal{L}_3|_B$. A countable covering by such balls shows that every energy dominant measure dominates $\mathcal{L}_3$.

Taking $\nu = \mathcal{L}_3$, for any $f_1, \dots, f_N \in \mathcal{F}$ we obtain, almost everywhere,

$$\begin{aligned} \text{rank} \left(\frac{d\Gamma(f_i, f_j)}{d\nu}(q) \right)_{i,j=1}^N &= \text{rank} \left(\frac{1}{2} \nabla_H f_i(q) \cdot \nabla_H f_j(q) \right)_{i,j=1}^N \\ &= \dim \text{span} \{ \nabla_H f_i(q) : 1 \leq i \leq N \} \leq 2. \end{aligned}$$

Hence $p_H \leq 2$ almost everywhere. For the two cutoff coordinate functions above, the matrix of energy densities on B is $\frac{1}{2}I_2$, so $p_H \geq 2$ almost everywhere on B . Therefore

$$p_H(q) = 2 \quad \text{for } \mathcal{L}_3\text{-almost every } q \in \mathbb{H},$$

and the martingale dimension is two. It thus recovers the dimension of the horizontal tangent space, even though $\mathbb{H}$ is a three-dimensional smooth manifold.

1.4 Lipschitz differentiability spaces

Rademacher's theorem states that every Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at Lebesgue-almost every point. Lipschitz differentiability spaces abstract this phenomenon to metric measure spaces: up to a null set, the space is covered by countably many measurable sets, each equipped with a finite-dimensional Lipschitz coordinate map with respect to which every Lipschitz function is differentiable almost everywhere. We recall the definition below and refer to [18, 8, 9, 11, 37, 10, 23] for background and further discussion. Here metric measure spaces are complete, locally compact and separable with a Borel measure of full support that is finite and positive on balls.

Let (X, d, μ) be a metric measure space. Let $U \subset X$ be a Borel set and let $\varphi : X \rightarrow \mathbb{R}^n$ be Lipschitz, where $n \geq 1$. We say that a Lipschitz function $f : X \rightarrow \mathbb{R}$ is *differentiable with respect to (U, φ) at $x \in U$* if there exists a unique linear map $df_x : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\lim_{\substack{y \rightarrow x \\ y \neq x}} \frac{|f(y) - f(x) - df_x(\varphi(y) - \varphi(x))|}{d(x, y)} = 0.$$

The linear map df_x is called the *differential* of f at x with respect to (U, φ) .

A pair (U, φ) is called a *Lipschitz differentiability chart of dimension n* if $\mu(U) > 0$ and every Lipschitz function $f : X \rightarrow \mathbb{R}$ is differentiable with respect to (U, φ) at μ -almost every $x \in U$. The exceptional μ -null set may depend on f .

The space (X, d, μ) is called a *Lipschitz differentiability space* if it admits a countable collection of Lipschitz differentiability charts $\{(U_i, \varphi_i)\}_i$, of dimensions $n_i \geq 1$, such that

$$\mu \left(X \setminus \bigcup_i U_i \right) = 0.$$

Such a collection is called a *measurable differentiable structure*.

The dimensions of overlapping charts agree μ -almost everywhere on their overlap. Consequently, the chart dimensions determine a measurable function $d_C : X \rightarrow \mathbb{N}$, uniquely up to μ -null sets, such that

$$d_C(x) = n_i \quad \text{for } \mu\text{-almost every } x \in U_i, \text{ for each } i.$$

Moreover, d_C is independent, up to μ -null sets, of the chosen measurable differentiable structure. We call d_C the *pointwise analytic dimension*. The *analytic dimension* (or *Lipschitz differentiability dimension*) of (X, d, μ) is

$$\mu\text{-esssup}_{x \in X} d_C(x).$$

Equivalently, it is the smallest $N \in \mathbb{N}$ such that every Lipschitz differentiability chart has dimension at most N , with value ∞ if no such finite bound exists.

1.5 Examples of Lipschitz differentiability spaces

If $X = \mathbb{R}^n$ is equipped with the Euclidean distance and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the identity map, then the above notion coincides with classical differentiability. At a point x where f is differentiable, its differential is

$$df_x(v) = \nabla f(x) \cdot v, \quad v \in \mathbb{R}^n.$$

Thus Rademacher's theorem implies that $(\mathbb{R}^n, \text{id})$ is a Lipschitz differentiability chart for $\mathbb{R}^n$ equipped with Lebesgue measure. The identity chart has dimension n and covers all of $\mathbb{R}^n$. Consequently, the pointwise analytic dimension satisfies

$$d_C(x) = n \quad \text{for } \mathcal{L}_n\text{-almost every } x \in \mathbb{R}^n,$$

and its essential supremum, the analytic dimension, is n .

Recall that the Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ equipped with the group operation

$$(x_1, y_1, z_1) \odot (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right).$$

Its Carnot–Carathéodory distance d_{CC} is left-invariant. The distance from the identity $(0, 0, 0)$ to (p, q, r) is the infimum of

$$\int_0^T \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt$$

over all absolutely continuous curves $(x(t), y(t))$ in $\mathbb{R}^2$ that start at $(0, 0)$, end at (p, q) , and satisfy

$$\frac{1}{2} \int_0^T (x(t)\dot{y}(t) - y(t)\dot{x}(t)) dt = r.$$

Although $(\mathbb{H}, d_{CC})$ is homeomorphic to $\mathbb{R}^3$ and has Hausdorff dimension 4, it has analytic dimension 2. Indeed, the map

$$\phi : \mathbb{H} \longrightarrow \mathbb{R}^2, \quad \phi(x, y, z) = (x, y),$$

is a global differentiability chart: for every Lipschitz function $f : \mathbb{H} \rightarrow \mathbb{R}$, at almost every $g \in \mathbb{H}$ there is a unique linear map $df_g : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$f(h) - f(g) = df_g(\phi(h) - \phi(g)) + o(d_{CC}(g, h)) \quad \text{as } h \rightarrow g.$$

This follows from a version of Rademacher's theorem due to Pansu [52]. Thus the pointwise dimension of the chart is 2 almost everywhere, and the analytic dimension of $\mathbb{H}$ is 2.

1.6 PI spaces and Cheeger's theorem

A large class of Lipschitz differentiability spaces arises from Poincaré inequalities. Let (X, d, μ) be a metric measure space such that every ball has positive, finite measure. We say that it supports a $(1, p)$ -Poincaré inequality if there are constants $C > 0$ and $\Lambda \geq 1$ such that, for every Lipschitz function $f : X \rightarrow \mathbb{R}$ and every ball $B(x, r)$,

$$\int_{B(x, r)} |f - f_{B(x, r)}| d\mu \leq Cr \left(\int_{B(x, \Lambda r)} \text{Lip}(f)^p d\mu \right)^{1/p},$$

where

$$\text{Lip}(f)(y) = \limsup_{\substack{z \rightarrow y \\ z \neq y}} \frac{|f(z) - f(y)|}{d(z, y)}, \quad f_{B(x, r)} = \int_{B(x, r)} f d\mu.$$

If, in addition, μ is doubling, meaning that

$$\mu(B(x, 2r)) \leq C_D \mu(B(x, r)) \quad \text{for all } x \in X, r > 0,$$

we call (X, d, μ) a p -PI space.

Cheeger's theorem [18] states that a complete p -PI space, with $1 \leq p < \infty$, is a Lipschitz differentiability space. Moreover, the dimensions of its charts are uniformly bounded in terms of the doubling and Poincaré inequality data. Thus Poincaré inequalities provide a way to establish differentiability of Lipschitz functions without first constructing coordinates explicitly.

The resulting class of spaces is quite broad. Laakso constructed, for every $Q > 1$, a 1-PI space of Hausdorff dimension Q [43]. The Heisenberg group $\mathbb{H}$, equipped with its Carnot–Carathéodory metric and three-dimensional Lebesgue measure, is another PI space; its Poincaré inequality follows from Jerison's work [34].

Our understanding of PI spaces remains incomplete even when the underlying topological space is $\mathbb{R}^n$. By analogy with the study of smooth structures on a fixed topological manifold, one may ask which measurable differentiable structures can arise on $\mathbb{R}^n$ when the metric and measure are allowed to vary. Here the metric is required to induce the Euclidean topology, and the metric measure space is required to satisfy the PI condition. A first step toward such a classification is to determine the possible analytic dimensions. Unlike a smooth structure, whose dimension is fixed by the underlying topology, a measurable differentiable structure need not have chart dimensions equal to the topological dimension, as the Heisenberg group illustrates. The following two questions are due to Kleiner and Schioppa [39].

Question 1.1. For each $n \geq 2$, does there exist a complete PI space $(\mathbb{R}^n, d, \mu)$ such that d induces the Euclidean topology and the analytic dimension is one?

Even the cases $n = 2$ and $n = 3$ are open. One can also ask which Hausdorff dimensions are compatible with the PI condition and the prescribed topology.

Question 1.2. For each $n \geq 2$ and $Q > n$, does there exist a complete PI space $(\mathbb{R}^n, d, \mu)$ such that d induces the Euclidean topology and μ is Ahlfors Q -regular? That is, do there exist such a space and a constant $C \geq 1$ satisfying

$$C^{-1}r^Q \leq \mu(B_d(x, r)) \leq Cr^Q \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0?$$

The case $Q = n$ is provided by Euclidean space with Lebesgue measure. When $n = 3$, the case $Q = 4$ is provided by the Heisenberg group. These questions ask how far the analytic and Hausdorff dimensions can vary while the underlying topology remains fixed. There is a corresponding question for martingale dimension, in which Gaussian heat kernel estimates provide a link between the diffusion and the metric geometry as we will see below.

2 Lecture 2: Equality under Gaussian heat kernel estimates

Before comparing martingale and analytic dimensions, we highlight a basic distinction between their definitions. Analytic dimension depends on the metric through the notion of Lipschitz differentiability, whereas martingale dimension is defined in terms of the Dirichlet form and its energy measures, without reference to a particular compatible metric. To relate these dimensions, it is therefore natural to impose conditions that connect the Dirichlet form to the metric. Heat kernel estimates provide such a connection by describing the transition densities of the associated diffusion in terms of distance and volume. We refer the reader to [55] for an introduction to this widely studied subject.

In this lecture, we compare martingale and analytic dimensions, highlighting their similarities and differences, and discuss their relationship with other notions of dimension, including Hausdorff and spectral dimensions. Throughout Lecture 2, we assume that (X, d) is complete, separable, and has at least two points, that the reference measure m is Radon with full support, and that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ is strongly local and regular. Furthermore, we assume that m is a doubling measure on (X, d) .

2.1 Heat kernels and Gaussian estimates

Let $\{P_t\}_{t>0}$ denote the strongly continuous, symmetric Markov semigroup associated with $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$. The Dirichlet form can be recovered from the semigroup through

$$\mathcal{F} = \left\{ f \in L^2(X, m) : \lim_{t \downarrow 0} \frac{1}{t} \langle (I - P_t)f, f \rangle_{L^2(X, m)} < \infty \right\},$$

and

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{t} \langle (I - P_t)f, f \rangle_{L^2(X, m)}, \quad f \in \mathcal{F}.$$

Thus, the Dirichlet form describes the infinitesimal behaviour of the semigroup.

A *heat kernel* for $\{P_t\}_{t>0}$ is a non-negative measurable function

$$(t, x, y) \longmapsto p_t(x, y) \quad \text{on } (0, \infty) \times X \times X$$

such that, for every $t > 0$ and $f \in L^2(X, m)$,

$$P_t f(x) = \int_X p_t(x, y) f(y) m(dy) \quad \text{for } m\text{-almost every } x \in X.$$

It satisfies the symmetry and semigroup identities

$$p_t(x, y) = p_t(y, x)$$

and

$$p_{t+s}(x, z) = \int_X p_t(x, y) p_s(y, z) m(dy),$$

for each $t, s > 0$ and almost every pair of spatial variables with respect to $m \otimes m$. Probabilistically, $p_t(x, y)$ is the transition density of the associated diffusion with respect to m . The existence of a heat kernel is an additional property; it does not follow from the Dirichlet form assumptions alone.

Definition 2.1 (Gaussian heat kernel estimates). We say that $(\mathcal{E}, \mathcal{F})$ satisfies *two-sided Gaussian heat kernel estimates* with respect to (d, m) if it admits a heat kernel and there exist constants $c_1, c_2, C_1, C_2 > 0$ such that, for every $t > 0$,

$$\begin{aligned} p_t(x, y) &\geq \frac{c_1}{m(B(x, \sqrt{t}))} \exp\left(-C_1 \frac{d(x, y)^2}{t}\right), \\ p_t(x, y) &\leq \frac{C_2}{m(B(x, \sqrt{t}))} \exp\left(-c_2 \frac{d(x, y)^2}{t}\right) \end{aligned}$$

for $m \otimes m$ -almost every $(x, y) \in X \times X$.

These estimates connect the diffusion to the metric and measure. The factor $m(B(x, \sqrt{t}))^{-1}$ gives the volume scale of the transition density, while the exponential factor describes its decay when $d(x, y)$ is large compared with $\sqrt{t}$. The term ‘‘Gaussian’’ refers to this dependence on $d(x, y)^2/t$; the volume factor need not be a power of t corresponding to an integer dimension.

Example 2.2 (Euclidean space). For the Dirichlet form

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, dx, \quad \mathcal{F} = W^{1,2}(\mathbb{R}^n),$$

the heat kernel is

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right).$$

Since $\mathcal{L}_n(B(x, \sqrt{t})) = \omega_n t^{n/2}$, where ω_n is the volume of the Euclidean unit ball, this kernel satisfies Gaussian estimates.

Example 2.3 (The Heisenberg group). The diffusion on $\mathbb{H}$ introduced in Lecture 1, with generator $\frac{1}{2}(V_1^2 + V_2^2)$, satisfies Gaussian estimates with respect to the Carnot–Carathéodory distance d_{CC} and three-dimensional Lebesgue measure $\mathcal{L}_3$. The Heisenberg dilations imply that

$$\mathcal{L}_3(B_{CC}(x, r)) = cr^4$$

for a constant $c > 0$. Consequently, the volume factor in the Gaussian estimates is proportional to t^{-2} . Thus, although the underlying space has topological dimension three and the diffusion has martingale dimension two, its heat kernel reflects volume growth of dimension four.

Gaussian estimates hold in many other settings; see [55] for background and references. Geometric examples include Brownian motion on connected Lie groups of polynomial volume growth, equipped with a left-invariant Riemannian metric and its Riemannian volume measure, as well as Brownian motion on complete connected Riemannian manifolds without boundary and with non-negative Ricci curvature, by the work of Li and Yau. Another classical class

of examples comes from symmetric uniformly elliptic divergence-form operators on $\mathbb{R}^n$. More precisely, the form

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \langle A(x) \nabla f(x), \nabla g(x) \rangle dx, \quad \mathcal{F} = W^{1,2}(\mathbb{R}^n),$$

satisfies Gaussian estimates if A is measurable and symmetric and, for some $0 < \lambda \leq \Lambda < \infty$,

$$\lambda |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq \Lambda |\xi|^2$$

for almost every x and every $\xi \in \mathbb{R}^n$ [3].

More generally, consider a complete connected weighted Riemannian manifold without boundary, with a smooth positive weight, equipped with its Riemannian distance and weighted volume measure m . If volume doubling and a scale-invariant L^2 -Poincaré inequality hold, then the form

$$\mathcal{E}(f, g) = \int_M \langle \nabla f, \nabla g \rangle dm,$$

with its natural Sobolev domain, satisfies Gaussian estimates. This is a celebrated result of Grigor'yan and Saloff-Coste [26, 54]; see also [56] for an extension to strongly local Dirichlet spaces. Examples beyond the manifold setting include Laakso spaces equipped with their natural metrics, measures, and canonical Dirichlet forms.

2.2 Equality of the pointwise dimensions

Under Gaussian heat kernel bounds the pointwise versions of martingale and analytic dimensions agree [49]. In this case, a result of Koskela and Zhou implies that the reference measure m is a minimal energy dominant measure [40] and that the underlying metric measure space is a PI space. The following result is from [49].

Theorem 2.4. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ satisfying two-sided Gaussian heat kernel estimates with respect to the metric d . Let p_H be the pointwise index of the form and d_C the pointwise dimension of Cheeger's measurable differentiable structure of the underlying metric measure space (X, d, m) . Then*

$$p_H(x) = d_C(x) \quad \text{for } m\text{-almost every } x \in X.$$

In particular, the martingale dimension of the diffusion equals the analytic dimension of (X, d, m) .

The key bridge between the different dimensions is the following estimate, essentially due to Koskela and Zhou [40] together with the comparison of the underlying metric with the intrinsic metric established in [36].

Proposition 2.5 (Energy–slope comparison). *Assume two-sided Gaussian heat kernel estimates. Then every Lipschitz function belongs locally to the form domain, the measure m is minimal energy dominant, and there is $C \geq 1$ such that*

$$C^{-1} \left(\frac{d\Gamma(f, f)}{dm}(x) \right)^{1/2} \leq \text{Lip } f(x) \leq C \left(\frac{d\Gamma(f, f)}{dm}(x) \right)^{1/2}$$

for every Lipschitz function $f : X \rightarrow \mathbb{R}$ and for m -almost every $x \in X$.

This proposition says that the metric and energy notions of the size of a first-order derivative are quantitatively equivalent. In particular, for Lipschitz functions $f_1, \dots, f_N$, the matrix

$$\left(\frac{d\Gamma(f_i, f_j)}{dm}(x) \right)_{i,j=1}^N$$

has full rank if and only if $\text{Lip}(\sum_{i=1}^N a_i f_i)(x) > 0$ for every non-zero $a \in \mathbb{R}^N$, at m -almost every x . Thus, independence can be expressed either through the rank of the energy matrix or through the pointwise slopes of linear combinations. This observation is behind the proof of Theorem 2.4.

2.3 Estimates on martingale dimension

Cheeger conjectured that the analytic dimension of a PI space is bounded above by its Hausdorff dimension [18]. De Philippis, Marchese, and Rindler [21] proved the stronger statement that, for every n -dimensional chart (U, ϕ) of a Lipschitz differentiability space (X, d, m) ,

$$\phi_*(\mathbf{1}_U m) \ll \mathcal{L}_n.$$

Since $m(U) > 0$, this implies that $\mathcal{L}_n(\phi(U)) > 0$, and hence $\dim_H \phi(U) = n$. As Lipschitz maps do not increase Hausdorff dimension,

$$n = \dim_H \phi(U) \leq \dim_H U \leq \dim_H X.$$

Taking the supremum over the chart dimensions gives the desired bound on analytic dimension. Combining this bound with Theorem 2.4, we obtain that, under two-sided Gaussian heat kernel estimates, the martingale dimension of a strongly local regular Dirichlet space is bounded above by the Hausdorff dimension of the underlying metric space.

2.4 Sub-Gaussian heat kernel estimates

Gaussian estimates are a special case of a broader class of heat kernel estimates that arise naturally in the study of diffusions on fractals [4].

Definition 2.6 (Sub-Gaussian heat kernel estimates). Let $\beta \in [2, \infty)$. We say that $(\mathcal{E}, \mathcal{F})$ satisfies *two-sided sub-Gaussian heat kernel estimates with walk dimension β* if it admits a heat kernel and there exist constants $c_1, c_2, C_1, C_2 > 0$ such that, for every $t > 0$,

$$\begin{aligned} p_t(x, y) &\geq \frac{c_1}{m(B(x, t^{1/\beta}))} \exp \left(-C_1 \left(\frac{d(x, y)^\beta}{t} \right)^{1/(\beta-1)} \right), \\ p_t(x, y) &\leq \frac{C_2}{m(B(x, t^{1/\beta}))} \exp \left(-c_2 \left(\frac{d(x, y)^\beta}{t} \right)^{1/(\beta-1)} \right) \end{aligned}$$

for $m \otimes m$ -almost every $(x, y) \in X \times X$.

When $\beta = 2$, these are precisely the Gaussian estimates. For general β , the characteristic spatial scale at time t is $t^{1/\beta}$, corresponding to a time scale r^β for motion over distance r .

Suppose additionally that m is *Ahlfors α -regular*, meaning that

$$m(B(x, r)) \asymp r^\alpha \quad \text{for all } x \in X \text{ and } 0 < r < \text{diam}(X, d),$$

with comparison constants independent of x and r . We call α the *volume growth exponent*; in this case, $\alpha = \dim_H(X, d)$. The *spectral dimension* is

$$d_s := \frac{2\alpha}{\beta}.$$

This terminology reflects the on-diagonal behaviour of the heat kernel: using its continuous version, the estimates give

$$p_t(x, x) \asymp t^{-\alpha/\beta} = t^{-d_s/2}, \quad 0 < t < \text{diam}(X, d)^\beta.$$

Thus, under Gaussian estimates, $d_s = \dim_H(X, d)$, whereas for $\beta > 2$ the spectral dimension is strictly smaller than the Hausdorff dimension.

A classical example is Brownian motion on the standard Sierpiński gasket. Barlow and Perkins [7] established sub-Gaussian estimates with

$$\alpha = \frac{\log 3}{\log 2}, \quad \beta = \frac{\log 5}{\log 2}, \quad d_s = \frac{2 \log 3}{\log 5} \in (1, 2).$$

The work of Barlow and Bass also established sub-Gaussian estimates for Brownian motion on Sierpiński carpet and higher dimensional generalizations called generalized Sierpiński carpets [5]. For generalized Sierpiński carpets, the walk dimension is strictly greater than two, although its exact value is generally unknown [35].

We now turn to martingale dimension in this broader setting. Recall that the martingale dimension d_m is the essential supremum of Hino's pointwise index with respect to a minimal energy dominant measure. If $\mathcal{E} \not\equiv 0$, then

$$d_m \geq 1.$$

Obtaining stronger lower bounds for specific fractal diffusions is much more difficult. In particular, the following question remains open.

Question 2.7. Is there a generalized Sierpiński carpet whose canonical self-similar Dirichlet form has martingale dimension strictly greater than one?

Hino [31] proved the bound $d_m \leq d_s$ for a class of self-similar spaces and Dirichlet forms. We expect this bound to hold without any self-similarity assumption; see also Hino's recent work [32] for progress toward determining martingale dimension beyond the self-similar setting. The following conjecture may be viewed as a counterpart of the bound on analytic dimension by Hausdorff dimension for PI spaces [21].

Conjecture 2.8. Let (X, d, m) be an Ahlfors α -regular metric measure space, and let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X, m)$ satisfying two-sided sub-Gaussian heat kernel estimates with walk dimension $\beta \geq 2$. Then

$$d_m \leq d_s = \frac{2\alpha}{\beta}.$$

When $\beta = 2$, this bound follows from Theorem 2.4 and the bound on analytic dimension by Hausdorff dimension. The conjecture asks whether spectral dimension provides the corresponding upper bound in the sub-Gaussian setting. The following recent result of the authors [25] gives the weaker qualitative conclusion that the martingale dimension is finite.

Theorem 2.9. *Let (X, d, m) be a metric measure space with m doubling, and let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X, m)$ satisfying two-sided sub-Gaussian heat kernel estimates with walk dimension β . Then the martingale dimension of $(\mathcal{E}, \mathcal{F})$ is finite.*

Hino's bound and known constructions further suggest the following conjectural characterization of the possible triples (α, β, d_m) .

Conjecture 2.10. Let $\alpha \geq 1$, $\beta \geq 2$, and $d_m \in \mathbb{N}$. There exists an Ahlfors α -regular metric measure space carrying a strongly local, regular Dirichlet form with two-sided sub-Gaussian heat kernel estimates of walk dimension β and martingale dimension d_m if and only if

$$2 \leq \beta \leq 1 + \alpha \quad \text{and} \quad 1 \leq d_m \leq \frac{2\alpha}{\beta}.$$

For $d_m = 1$, the existence assertion is established by the Laakso-type constructions in [50], which realize every pair $\alpha \geq 1$ and $2 \leq \beta \leq 1 + \alpha$ with martingale dimension one.

There are also examples of spaces satisfying sub-Gaussian heat kernel estimates for which the form domain contains no non-constant Lipschitz functions. This indicates that Lipschitz functions need not capture the first-order structure of a general strongly local Dirichlet form. In particular, analytic dimension is not generally suitable for describing the martingale dimension of fractal diffusions with $\beta > 2$.

For example, the standard Sierpiński gasket and Sierpiński carpet, equipped with their Euclidean metrics and Hausdorff measures, are not Lipschitz differentiability spaces. Indeed, they are Ahlfors regular of non-integer Hausdorff dimension and are bi-Lipschitz embedded in Euclidean space, whereas a theorem of Kell and Mondino [38] asserts that a Lipschitz differentiability space that admits such an embedding must be rectifiable.

More generally, Theorem 4.1 shows that if a metric measure space carries a strongly local, regular Dirichlet form satisfying two-sided sub-Gaussian heat kernel estimates with walk dimension $\beta > 2$, then it is not a Lipschitz differentiability space. This is in sharp contrast with the Gaussian case $\beta = 2$ considered in Theorem 2.4.

3 Lecture 3: The energy image density property

A common mechanism behind several of the dimension bounds discussed in Lecture 2 is an image-density principle: the presence of sufficiently many independent first-order directions forces an associated image measure on $\mathbb{R}^n$ to be absolutely continuous with respect to Lebesgue measure. In the setting of Dirichlet forms, this principle takes the form of the energy image density (EID) property. Versions of this idea play an important role in the bounds for analytic dimension in [21] and for martingale dimension in [31, 25].

In this lecture, we trace the origins of the EID property to Malliavin's proof of Hörmander's hypoellipticity theorem [33, 45]. We then introduce the EID conjecture formulated by Bouleau and Hirsch [14], explain its recent resolution in [25], and discuss how the EID property yields upper bounds on martingale and analytic dimensions.

3.1 Hörmander's theorem and Malliavin's criterion

A linear differential operator L with smooth coefficients is called *hypoelliptic* if, for every open set $U \subset \mathbb{R}^n$ and every distribution $u \in \mathcal{D}'(U)$,

$$Lu \in C^\infty(U) \implies u \in C^\infty(U).$$

Elliptic operators, such as the Laplacian, are hypoelliptic; Weyl's lemma is a classical manifestation of this fact. The heat operator $\partial_t - \Delta$ is also hypoelliptic, whereas the wave operator $\partial_{tt} - \Delta$ is not.

An important non-elliptic example is the Kolmogorov operator

$$K = \partial_{vv} + v\partial_x - \partial_t,$$

introduced by Kolmogorov in 1934. Here t , x , and v represent time, position, and velocity, respectively. This operator is associated with the diffusion

$$dX_s = V_s ds, \quad dV_s = \sqrt{2} dB_s,$$

where B is a one-dimensional Brownian motion. Randomness acts directly only in the velocity variable, but is transmitted to the position variable through the equation $dX_s = V_s ds$. Nevertheless, K is hypoelliptic. Hörmander's theorem explains this phenomenon. Consider an operator

$$L = \sum_{j=1}^d V_j^2 + V_0,$$

where $V_0, V_1, \dots, V_d$ are smooth real vector fields on an open set $U \subset \mathbb{R}^n$. Recall that the Lie bracket of two vector fields is defined by

$$[V, W] = VW - WV.$$

Hörmander's bracket condition requires that the vector fields $V_0, V_1, \dots, V_d$, together with all of their iterated Lie brackets, span the tangent space at every point of U .

Theorem 3.1 (Hörmander [33]). *If Hörmander's bracket condition holds, then*

$$L = \sum_{j=1}^d V_j^2 + V_0$$

is hypoelliptic.

For the Kolmogorov operator, take

$$V_0 = v\partial_x - \partial_t, \quad V_1 = \partial_v.$$

Since

$$[V_0, V_1] = -\partial_x,$$

the vector fields V_0 , V_1 , and $[V_0, V_1]$ span the t , v , and x directions. Thus Hörmander's theorem implies that the Kolmogorov operator is hypoelliptic.

There is a corresponding probabilistic formulation. Consider the Stratonovich stochastic differential equation

$$dX_t = \sqrt{2} \sum_{i=1}^d V_i(X_t) \circ dB_t^i + V_0(X_t) dt,$$

where $B^1, \dots, B^d$ are independent Brownian motions. The generator of this diffusion is

$$L = \sum_{i=1}^d V_i^2 + V_0.$$

The relevant parabolic Hörmander condition is that the Lie algebra generated on $\mathbb{R} \times \mathbb{R}^n$ by

$$\partial_t - V_0, V_1, \dots, V_d$$

spans the tangent space at every point. This condition implies that, for every $t > 0$, the law of X_t has a smooth density with respect to Lebesgue measure.

The Brownian motion on the Heisenberg group, discussed in Lecture 1, provides a useful example. Let $(B_t^{(1)}, B_t^{(2)})$ be a two-dimensional Brownian motion and set

$$X_t = \left(B_t^{(1)}, B_t^{(2)}, \frac{1}{2} \int_0^t \left(B_s^{(1)} dB_s^{(2)} - B_s^{(2)} dB_s^{(1)} \right) \right).$$

As explained in Lecture 1, the generator of X_t is

$$L = \frac{1}{2}(V_1^2 + V_2^2), \quad V_1 = \partial_x - \frac{y}{2}\partial_z, \quad V_2 = \partial_y + \frac{x}{2}\partial_z.$$

Although the process is driven by only two independent Brownian motions, the bracket

$$[V_1, V_2] = \partial_z$$

produces the third spatial direction. Hence Hörmander's condition holds, and X_t has a smooth density on $\mathbb{R}^3$ for every $t > 0$.

Malliavin's probabilistic proof of Hörmander's theorem begins by viewing the solution at time t as a functional of the driving Brownian path:

$$X_t = F_t(B).$$

Thus,

$$F_t : C_0([0, t], \mathbb{R}^d) \longrightarrow \mathbb{R}^n$$

is a map defined on Wiener space. Malliavin calculus studies how $F_t(B)$ changes when the Brownian path is perturbed along directions in the Cameron–Martin space H . Suppose

$$F = (F_1, \dots, F_n)$$

has components in a suitable Malliavin Sobolev space. Its Malliavin matrix is

$$\gamma_F = (\langle DF_i, DF_j \rangle_H)_{i,j=1}^n,$$

where DF_i denotes the Malliavin derivative of F_i . The matrix γ_F is invertible precisely when

$$DF : H \longrightarrow \mathbb{R}^n$$

is surjective. The one-derivative Malliavin criterion states that

$$F_* (\mathbf{1}_{\{\det \gamma_F > 0\}} \mathbb{P}) \ll \mathcal{L}_n.$$

In particular, if $\det \gamma_F > 0$ almost surely, then the law $F_* \mathbb{P}$ is absolutely continuous with respect to Lebesgue measure. In Malliavin's proof of Hörmander's theorem, the bracket condition is used to establish the non-degeneracy of the Malliavin matrix. Stronger estimates on its inverse, combined with Malliavin's integration-by-parts formula, then give smoothness of the density.

This criterion is the prototype for the energy image density property. On Wiener space, the Malliavin matrix is the carré du champ matrix of the Ornstein–Uhlenbeck Dirichlet form. The EID property replaces the Wiener-space Malliavin matrix by the carré du champ matrix of a general Dirichlet structure while retaining the same image-measure conclusion.

3.2 EID conjecture of Bouleau and Hirsch

The finite-dimensional analogue of Malliavin's criterion follows from the coarea formula. Let

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n, \quad m \geq n,$$

be a smooth map, and define its n -dimensional Jacobian by

$$J_f(x) := \sqrt{\det(Df(x)Df(x)^\top)}.$$

The matrix $Df(x)Df(x)^\top$ is the Gram matrix of the component gradients of f . Thus,

$$J_f(x) > 0 \iff Df(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n \text{ is surjective.}$$

The coarea formula states that, for every non-negative measurable function $g : \mathbb{R}^m \rightarrow [0, \infty]$,

$$\int_{\mathbb{R}^m} g(x) J_f(x) dx = \int_{\mathbb{R}^n} \left(\int_{f^{-1}(y)} g(x) d\mathcal{H}_{m-n}(x) \right) dy.$$

Now let X be an $\mathbb{R}^m$ -valued random variable with density ρ with respect to $\mathcal{L}_m$. Applying the coarea formula with $g = \rho/J_f$ on the set $\{J_f > 0\}$ shows that

$$f_* (\mathbf{1}_{\{J_f > 0\}} \rho \mathcal{L}_m) \ll \mathcal{L}_n.$$

Its density is given by

$$\sigma(y) = \int_{f^{-1}(y) \cap \{J_f > 0\}} \frac{\rho(x)}{J_f(x)} d\mathcal{H}_{m-n}(x).$$

In particular, if f is a submersion, so that $J_f > 0$ everywhere, then the law of $f(X)$ is absolutely continuous and

$$f_*(\rho \mathcal{L}_m) = \sigma \mathcal{L}_n.$$

This is the finite-dimensional prototype of Malliavin's criterion. The matrix $DfDf^\top$ is replaced on Wiener space by the Malliavin matrix, and its non-degeneracy again forces the corresponding image measure to be absolutely continuous. The EID property extends this principle further by replacing the Malliavin matrix with the carré du champ matrix of an abstract Dirichlet structure.

The preceding finite-dimensional discussion suggests an abstraction in which the carré du champ takes the place of the Gram matrix $DfDf^\top$.

Definition 3.2 (Dirichlet structure). A *Dirichlet structure* $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ consists of a probability space $(X, \mathcal{X}, \mu)$ and a symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ satisfying the following additional properties:

- (i) The form is strongly local: if $f, g \in \mathcal{F}$ and $a \in \mathbb{R}$ satisfy $(f + a)g = 0$, then $\mathcal{E}(f, g) = 0$.
- (ii) The constant function $\mathbf{1}$ belongs to $\mathcal{F}$ and $\mathcal{E}(\mathbf{1}, \mathbf{1}) = 0$.
- (iii) The form admits a *carré du champ*; that is, there exists a positive symmetric bilinear map

$$\gamma : \mathcal{F} \times \mathcal{F} \longrightarrow L^1(\mu)$$

such that, for every $f, h \in \mathcal{F} \cap L^\infty(\mu)$,

$$\mathcal{E}(fh, f) - \frac{1}{2}\mathcal{E}(h, f^2) = \int_X h \gamma(f, f) d\mu.$$

For an $\mathbb{R}^n$ -valued function $f = (f_1, \dots, f_n) \in \mathcal{F}^n$, its *carré du champ matrix* is

$$\gamma(f) := (\gamma(f_i, f_j))_{1 \leq i, j \leq n}.$$

This is a measurable, symmetric, positive semidefinite $n \times n$ matrix. The condition

$$\det \gamma(f)(x) > 0$$

therefore expresses the infinitesimal independence of the components $f_1, \dots, f_n$ at x .

Definition 3.3 (Energy image density property). A Dirichlet structure satisfies the *energy image density property*, abbreviated EID, if, for every $n \in \mathbb{N}$ and every $f \in \mathcal{F}^n$,

$$f_* (\mathbf{1}_{\{\det \gamma(f) > 0\}} \mu) \ll \mathcal{L}_n.$$

Thus, after restricting μ to the set on which the components of f are infinitesimally independent, its image under f is absolutely continuous with respect to Lebesgue measure.

As a finite-dimensional example, let μ be the standard Gaussian measure on $\mathbb{R}^m$ and consider the Ornstein–Uhlenbeck Dirichlet structure

$$\mathcal{F} = W^{1,2}(\mathbb{R}^m, \mu), \quad \mathcal{E}(u, v) = \int_{\mathbb{R}^m} \nabla u \cdot \nabla v d\mu.$$

Its carré du champ is

$$\gamma(u, v) = \nabla u \cdot \nabla v.$$

Consequently, for a smooth map $f = (f_1, \dots, f_n) : \mathbb{R}^m \rightarrow \mathbb{R}^n$,

$$\gamma(f) = Df Df^\top.$$

Hence $\det \gamma(f)(x) > 0$ precisely when $Df(x)$ is surjective. In this case, the EID property recovers the absolute-continuity statement obtained from the coarea formula. For the Ornstein–Uhlenbeck Dirichlet structure on Wiener space, the matrix $\gamma(f)$ is the classical Malliavin matrix. Motivated by Malliavin’s criterion, Bouleau and Hirsch formulated the following conjecture.

Conjecture 3.4 (Bouleau–Hirsch, 1986 [14]). Every Dirichlet structure satisfies the energy image density property.

This conjecture was recently resolved affirmatively [25].

Theorem 3.5. *Every Dirichlet structure satisfies the energy image density property.*

The result also extends beyond the original framework of Dirichlet structures. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form, let ν be a minimal energy dominant measure, and define

$$\gamma_\nu(f) = \left(\frac{d\Gamma(f_i, f_j)}{d\nu} \right)_{1 \leq i, j \leq n}.$$

Then, using quasicontinuous representatives of the components of f ,

$$f_* (\mathbf{1}_{\{\det \gamma_\nu(f) > 0\}} \nu) \ll \mathcal{L}_n.$$

Versions of the EID property are also established for p -Dirichlet spaces with $1 < p < \infty$. These include Sobolev spaces defined using upper gradients and self-similar p -energies on fractals obtained as rescaled limits of discrete energies.

Several important special cases of the EID conjecture were known previously. The scalar case $n = 1$ was established by Bouleau and by Bouleau–Hirsch [13, 14]. Bouleau and Hirsch also proved the EID property for the Ornstein–Uhlenbeck Dirichlet structure on Wiener space [15]. In that setting, the carré du champ matrix is the Malliavin matrix, and the resulting absolute-continuity statement is commonly called the Bouleau–Hirsch criterion.

An important feature of the Bouleau–Hirsch criterion is that it requires only first-order Malliavin regularity: if the components of f have first-order Malliavin regularity and $\det \gamma(f) > 0$ almost surely, then the law of f is absolutely continuous. By contrast, Malliavin’s original integration-by-parts approach required additional regularity. The two classical approaches to image-density results may be summarized as follows.

Malliavin’s approach uses an integration-by-parts formula to show that suitable distributional derivatives of the image measure are Radon measures. In the setting of general Dirichlet forms, this method typically requires additional regularity of f and of the entries of its carré du champ matrix. It therefore does not apply to arbitrary $f \in \mathcal{F}^n$. The approach of Bouleau and Hirsch instead uses Federer’s coarea formula. Its advantage is that it requires only first-order regularity and consequently yields the one-derivative criterion on Wiener space. Its limitation is that a coarea formula is not available for an arbitrary Dirichlet structure. Extensions of this approach therefore require additional geometric or structural assumptions.

A further partial result was obtained by Malicet and Poly [44]. They proved that if $f \in \mathcal{F}^n$ satisfies

$$\det \gamma(f) > 0 \quad \mu\text{-almost surely,}$$

then its law $f_*\mu$ is a Rajchman measure; that is, its Fourier transform tends to zero at infinity. This conclusion is weaker than absolute continuity, since a Rajchman measure may still be singular with respect to Lebesgue measure.

Thus, the central difficulty in the EID conjecture was to show, using only the abstract first-order structure of a Dirichlet form, that non-degeneracy of the carré du champ matrix forces absolute continuity of the corresponding image measure, without assuming the additional regularity needed for integration by parts or the geometric structure needed for a coarea formula.

Next we discuss some ideas behind the proof of Theorem 3.5. The proof of the EID conjecture draws on tools developed in connection with a converse to Rademacher's theorem. Recall that Rademacher's theorem implies that, if ν is a Radon measure on $\mathbb{R}^n$ satisfying $\nu \ll \mathcal{L}_n$, then every Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at ν -almost every point. De Philippis and Rindler [22] proved the converse: if every Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable ν -almost everywhere, then

$$\nu \ll \mathcal{L}_n.$$

Our proof of the EID conjecture uses a criterion for absolute continuity arising from the same paper.

We first recall briefly the language of currents. A k -dimensional current T in $\mathbb{R}^n$ is a continuous linear functional on the space $\mathcal{D}^k(\mathbb{R}^n)$ of smooth, compactly supported k -forms. Its boundary is the $(k-1)$ -current defined by

$$\partial T(\omega) := T(d\omega), \quad \omega \in \mathcal{D}^{k-1}(\mathbb{R}^n),$$

and its mass is

$$\mathbf{M}(T) := \sup \left\{ T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^n), |\omega| \leq 1 \right\},$$

where $|\omega|$ denotes the pointwise comass norm. For example, if T_Ω is the current of integration over a smoothly bounded, oriented domain $\Omega \subset \mathbb{R}^n$, then ∂T_Ω is the current of integration over $\partial\Omega$, with the induced orientation, and

$$\mathbf{M}(T_\Omega) = \mathcal{L}_n(\Omega), \quad \mathbf{M}(\partial T_\Omega) = \mathcal{H}_{n-1}(\partial\Omega).$$

A current T is called *normal* if both T and ∂T have finite mass. We shall only use one-dimensional currents. Every 1-current of finite mass can be represented as

$$T = \vec{T} \|T\|,$$

where $\|T\|$ is a finite positive measure and $\vec{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is measurable with $|\vec{T}| = 1$ for $\|T\|$ -almost every point. More explicitly,

$$T(\omega) = \int_{\mathbb{R}^n} \langle \omega(y), \vec{T}(y) \rangle d\|T\|(y).$$

The following consequence of the structure theorem of De Philippis and Rindler is the main geometric ingredient.

Theorem 3.6 (De Philippis–Rindler [22]). *Let*

$$T_i = \vec{T}_i \|T_i\|, \quad i = 1, \dots, n,$$

be one-dimensional normal currents in $\mathbb{R}^n$, and let ν be a positive Radon measure such that

$$\nu \ll \|T_i\| \quad \text{for every } i = 1, \dots, n.$$

If

$$\text{span}\{\vec{T}_1(y), \dots, \vec{T}_n(y)\} = \mathbb{R}^n \quad \text{for } \nu\text{-almost every } y,$$

then

$$\nu \ll \mathcal{L}_n.$$

We now explain how the Dirichlet structure produces the normal currents required by this theorem. Let $A : D(A) \rightarrow L^2(X, \mu)$ denote the generator of the Dirichlet form, with our sign convention

$$\mathcal{E}(g, h) = -\langle A(g), h \rangle_{L^2(\mu)}, \quad g \in D(A), \quad h \in \mathcal{F}.$$

Equivalently,

$$D(A) = \{g \in \mathcal{F} : |\mathcal{E}(g, h)| \leq C \|h\|_{L^2(\mu)} \text{ for some } C > 0 \text{ and every } h \in \mathcal{F}\}.$$

The energy measures of the Dirichlet structure are

$$\Gamma(u, v) := \gamma(u, v) \mu.$$

Given $f = (f_1, \dots, f_n) \in \mathcal{F}^n$ and $g \in \mathcal{F}$, define a one-dimensional current $T_{f,g}$ in $\mathbb{R}^n$ by

$$T_{f,g} \left(\sum_{j=1}^n h_j dx_j \right) := \sum_{j=1}^n \int_X h_j(f(x)) d\Gamma(f_j, g)(x).$$

Equivalently, $T_{f,g}$ is the $\mathbb{R}^n$ -valued measure

$$T_{f,g} = f_*((\Gamma(f_j, g))_{j=1}^n).$$

The Cauchy–Schwarz inequality for energy measures shows that this current has finite mass.

The crucial observation is that $T_{f,g}$ is normal whenever $g \in D(A)$. Indeed, if $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\phi_0 = \phi - \phi(0)$, then the chain rule gives

$$\begin{aligned} \partial T_{f,g}(\phi) &= T_{f,g}(d\phi) \\ &= \sum_{j=1}^n \int_X \frac{\partial \phi}{\partial x_j}(f(x)) d\Gamma(f_j, g)(x) \\ &= \mathcal{E}(\phi_0 \circ f, g) \\ &= - \int_X A(g) \phi_0(f) d\mu. \end{aligned}$$

Moreover,

$$\int_X A(g) d\mu = -\mathcal{E}(g, \mathbf{1}) = 0.$$

It follows that

$$\partial T_{f,g}(\phi) = - \int_X A(g) \phi(f) d\mu,$$

or equivalently,

$$\partial T_{f,g} = -f_*(A(g) \mu).$$

Thus $\partial T_{f,g}$ has finite mass and $T_{f,g}$ is a normal current.

To apply the De Philippis–Rindler theorem, fix $f = (f_1, \dots, f_n) \in \mathcal{F}^n$ and set

$$I_f := \{x \in X : \det \gamma(f)(x) > 0\}, \quad \nu_f := f_*(\mathbf{1}_{I_f} \mu).$$

For each $i = 1, \dots, n$, consider the current

$$T_i := T_{f, f_i}.$$

The invertibility of the carré du champ matrix implies

$$\nu_f \ll \|T_i\|, \quad i = 1, \dots, n,$$

and

$$\text{span}\{\vec{T}_1(y), \dots, \vec{T}_n(y)\} = \mathbb{R}^n \quad \text{for } \nu_f\text{-almost every } y.$$

Thus, if $f \in D(A)^n$, the currents $T_1, \dots, T_n$ are normal and the De Philippis–Rindler theorem yields

$$\nu_f = f_*(\mathbf{1}_{\{\det \gamma(f) > 0\}} \mu) \ll \mathcal{L}_n.$$

This is precisely the EID property for f .

For a general $f \in \mathcal{F}^n$, the currents T_{f, f_i} need not be normal because the components f_i need not belong to $D(A)$. The full proof uses an additional approximation argument based on the density of $D(A)$ in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$, together with continuity properties of energy measures. This extends the conclusion from $D(A)^n$ to all of $\mathcal{F}^n$.

The preceding construction is intrinsically quadratic, since it uses bilinear energy measures and the Hilbert-space generator, and therefore does not extend directly to nonlinear p -energies. In the nonlinear setting, the De Philippis–Rindler theorem is instead used indirectly to detect a missing infinitesimal direction for a singular image measure; suitable Lipschitz approximations in this direction, together with the chain rule and weak lower semicontinuity of the energy measures, then contradict the assumed independence of the coordinate functions [25].

3.3 Extension to regular Dirichlet forms and applications

The EID property also extends to all strongly local, regular Dirichlet forms. There is, however, an important difference from the setting of Dirichlet structures. If $(\mathcal{E}, \mathcal{F})$ is a strongly local, regular Dirichlet form on $L^2(X, m)$, its energy measures need not be absolutely continuous with respect to m ; indeed, on many fractals they are singular with respect to m . Consequently, a carré du champ density relative to the reference measure need not exist.

To overcome this issue, let ν be a minimal energy dominant measure and, for $f = (f_1, \dots, f_n) \in \mathcal{F}^n$, define

$$\gamma_\nu(f) := \left(\frac{d\Gamma(f_i, f_j)}{d\nu} \right)_{1 \leq i, j \leq n}.$$

Since functions in $\mathcal{F}$ are initially defined only up to m -null sets, while ν need not be absolutely continuous with respect to m , we use quasicontinuous representatives of the components of f . Such representatives are uniquely defined outside a set of zero capacity, and energy dominant measures do not charge such sets. The resulting image measure is therefore well defined.

Any two minimal energy dominant measures are mutually absolutely continuous. Moreover, changing the minimal energy dominant measure multiplies the matrix $\gamma_\nu(f)$ by a strictly positive scalar function. Thus the set on which $\det \gamma_\nu(f) > 0$, as well as the measure class of the corresponding image measure, is independent of the choice of ν . A modification of the proof of Theorem 3.5 gives the following extension [25].

Theorem 3.7. *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X, m)$, and let ν be a minimal energy dominant measure. For every $n \in \mathbb{N}$ and $f = (f_1, \dots, f_n) \in \mathcal{F}^n$, using quasicontinuous representatives of the components of f , we have*

$$f_*(\mathbf{1}_{\{\det \gamma_\nu(f) > 0\}} \nu) \ll \mathcal{L}_n.$$

This statement is reminiscent of the theorem of De Philippis, Marchese, and Rindler discussed in Lecture 2. For an n -dimensional differentiability chart (U, ϕ) , their result gives

$$\phi_*(\mathbf{1}_U m) \ll \mathcal{L}_n.$$

Since ϕ is Lipschitz and the image has positive n -dimensional Lebesgue measure, one obtains

$$n \leq \dim_H(X, d).$$

Taking the supremum over the chart dimensions yields the bound of analytic dimension by Hausdorff dimension.

The same argument cannot be applied directly to martingale dimension in the sub-Gaussian setting, because the functions in the domain of the Dirichlet form need not be Lipschitz. In fact, recent work of Anttila, Eriksson-Bique, and Shimizu provides examples satisfying sub-Gaussian heat kernel estimates with walk dimension $\beta > 2$ for which $\mathcal{F}$ contains no non-constant Lipschitz functions [2, Theorem 1.9].

Heat kernel regularity nevertheless supplies a sufficiently large family of Hölder continuous functions. A consequence of the results of Barlow, Grigor'yan, and Kumagai [6] is the following.

Proposition 3.8. *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on a metric measure space (X, d, m) , where m is doubling. Suppose that its heat kernel satisfies two-sided sub-Gaussian estimates. Then there exists $\theta \in (0, 1]$, depending only on the constants in the heat kernel estimates, such that $\mathcal{F}$ contains an $\mathcal{E}_1$ -dense collection of functions admitting θ -Hölder continuous versions.*

More concretely, heat kernel regularity implies that functions of the form

$$P_t h, \quad t > 0, \quad h \in \mathcal{F} \cap L^\infty(X, m),$$

admit θ -Hölder continuous versions. Such functions form an $\mathcal{E}_1$ -dense subset of $\mathcal{F}$.

We can now combine this density with the EID property. Suppose that the martingale dimension is at least n . The density of the Hölder functions and the continuity of energy measures under $\mathcal{E}_1$ -convergence allow us to choose a θ -Hölder map

$$f = (f_1, \dots, f_n) : X \longrightarrow \mathbb{R}^n$$

whose modified carré du champ matrix has rank n on a set of positive ν -measure. The EID property then implies that

$$f_*(\mathbf{1}_{\{\det \gamma_\nu(f) > 0\}} \nu)$$

is a nonzero measure absolutely continuous with respect to $\mathcal{L}_n$. Consequently, $f(X)$ has positive n -dimensional Lebesgue measure and hence

$$\dim_H f(X) = n.$$

A θ -Hölder map can increase Hausdorff dimension by at most a factor of $1/\theta$. Therefore,

$$n = \dim_H f(X) \leq \frac{\dim_H(X, d)}{\theta}.$$

Taking the supremum over all such n gives

$$d_m \leq \frac{\dim_H(X, d)}{\theta} < \infty.$$

This proves Theorem 2.9. In the Gaussian case $\beta = 2$, the results of Koskela and Zhou [40] provide an $\mathcal{E}_1$ -dense collection of Lipschitz functions. Thus one may take $\theta = 1$ in the dimension argument, recovering

$$d_m \leq \dim_H(X, d).$$

4 Non-existence of measurable differentiable structures

The goal of this section is to prove a new result showing that a large class of metric measure spaces arising in the study of diffusions on fractals are not Lipschitz differentiability spaces. In particular, the result applies to spaces carrying strongly local, regular Dirichlet forms that satisfy two-sided sub-Gaussian heat kernel estimates with walk dimension strictly greater than two.

To state this result, we recall two analytic conditions adapted to a general space–time scaling function. A *scale function* is a continuous and strictly increasing function

$$\Psi: [0, \infty) \longrightarrow [0, \infty), \quad \Psi(0) = 0,$$

for which there exist constants $C_\Psi \geq 1$ and $1 < \beta_1 \leq \beta_2 < \infty$ such that

$$C_\Psi^{-1} \left(\frac{R}{r} \right)^{\beta_1} \leq \frac{\Psi(R)}{\Psi(r)} \leq C_\Psi \left(\frac{R}{r} \right)^{\beta_2}$$

whenever $0 < r \leq R < \text{diam}(X, d)$. The basic example is

$$\Psi(r) = r^\beta.$$

Here $\Psi(r)$ represents the characteristic time scale associated with the spatial scale r . The Gaussian case corresponds to $\beta = 2$, while $\beta > 2$ is the strictly sub-Gaussian regime.

Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X, m)$, and let Γ denote its energy measure. We say that the Dirichlet form satisfies the Ψ -Poincaré inequality, denoted by $\text{PI}(\Psi)$, if there exist constants $C_{\text{PI}} > 0$ and $A \geq 1$ such that

$$\int_{B(x, r)} |f - f_{B(x, r)}|^2 dm \leq C_{\text{PI}} \Psi(r) \Gamma(f, f)(B(x, Ar))$$

for every $f \in \mathcal{F}$, $x \in X$, and $r > 0$, where

$$f_{B(x, r)} = \frac{1}{m(B(x, r))} \int_{B(x, r)} f dm.$$

Thus $\text{PI}(\Psi)$ controls the oscillation of a function on a ball by its energy on a slightly larger ball, with the factor $\Psi(r)$ accounting for the relevant space–time scaling.

We next recall the corresponding capacity upper bound. If $K \subset U$, where U is open, define the relative capacity by

$$\text{Cap}_U(K) := \inf \{ \mathcal{E}(u, u) : u \in \mathcal{F}_U, \tilde{u} \geq 1 \text{ quasi-everywhere on } K \},$$

where $\tilde{u}$ denotes a quasicontinuous representative and

$$\mathcal{F}_U = \{ u \in \mathcal{F} : \tilde{u} = 0 \text{ quasi-everywhere on } X \setminus U \}.$$

We say that $\text{Cap}(\Psi)_\leq$ holds if there exist $A > 1$ and $C_{\text{Cap}} > 0$ such that

$$\text{Cap}_{B(x, Ar)}(B(x, r)) \leq C_{\text{Cap}} \frac{m(B(x, r))}{\Psi(r)}$$

for every $x \in X$ and every relevant radius $r > 0$. Equivalently, up to changing the constants, one can require the existence of a cutoff function $u \in \mathcal{F}$ satisfying

$$0 \leq u \leq 1, \quad u = 1 \text{ on } B(x, r), \quad u = 0 \text{ outside } B(x, Ar),$$

and

$$\mathcal{E}(u, u) \leq C_{\text{Cap}} \frac{m(B(x, r))}{\Psi(r)}.$$

The particular value of the enlargement factor $A > 1$ is immaterial: if the capacity upper bound holds for one such A , then a standard covering argument, together with volume doubling and the scaling properties of Ψ , shows that it holds for every $A > 1$, after suitably adjusting C_{Cap} .

These conditions arise naturally from heat kernel estimates. In the setting of strongly local regular Dirichlet forms, two-sided heat kernel estimates with scale function Ψ imply $\text{PI}(\Psi)$ and the capacity upper bound $\text{Cap}(\Psi)_{\leq}$; see [6]. Under the standing assumptions, the converse was a long-standing open problem and was recently established by the first-named author [24]. The sub-Gaussian estimates with walk dimension β considered earlier correspond to the scale function

$$\Psi(r) = r^{\beta}.$$

For this choice,

$$\frac{\Psi(\epsilon r)}{\Psi(r)\epsilon^2} = \epsilon^{\beta-2}.$$

Consequently,

$$\lim_{\epsilon \rightarrow 0} \limsup_{r \downarrow 0} \frac{\Psi(\epsilon r)}{\Psi(r)\epsilon^2} = 0$$

whenever $\beta > 2$, whereas the expression is identically equal to 1 in the Gaussian case $\beta = 2$. The following theorem therefore gives a sharp contrast with the Gaussian situation discussed in Lecture 2.

Theorem 4.1. *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X, m)$, where (X, d, m) is a metric measure space and m is doubling. Suppose that $(\mathcal{E}, \mathcal{F})$ satisfies $\text{PI}(\Psi)$ and $\text{Cap}(\Psi)_{\leq}$, and that*

$$\lim_{\epsilon \rightarrow 0} \limsup_{r \downarrow 0} \frac{\Psi(\epsilon r)}{\Psi(r)\epsilon^2} = 0.$$

Then (X, d, m) is not a Lipschitz differentiability space.

In particular, if the Dirichlet form satisfies two-sided sub-Gaussian heat kernel estimates with walk dimension $\beta > 2$, then the associated metric measure space does not admit a measurable differentiable structure. Thus the Gaussian case $\beta = 2$, in which martingale and analytic dimensions can be compared, is fundamentally different from the strictly sub-Gaussian case.

4.1 A consequence of the Poincaré inequality

Throughout, we assume that (X, d) is a complete, separable, locally compact metric space and that m is a Radon measure on it that satisfies $0 < m(B(x, r)) < \infty$ for all $x \in X, r > 0$. We also assume that (X, d, m) satisfies the doubling property: that is there exists $D > 0$ such that

$$m(B(x, 2r)) \leq Dm(B(x, r)), \quad \text{for all } x \in X, r > 0.$$

We refer to D as a *doubling constant* of m . Since (X, d, m) is a doubling metric measure space, the Lebesgue differentiation theorem [28, Theorem 1.8] implies that, for every $f \in L^1_{\text{loc}}(X, m)$,

$$\tilde{f}(x) := \begin{cases} \lim_{r \downarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dm(y), & \text{if the limit exists,} \\ 0, & \text{otherwise,} \end{cases}$$

defines a measurable function satisfying $\tilde{f} = f$ m -almost everywhere. We call $\tilde{f}$ the *precise representative* of f , and throughout we identify $f \in L^1_{\text{loc}}(X, m)$ with its precise representative.

We say that $x \in X$ is a *Lebesgue point* of f if

$$\lim_{r \downarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) = 0.$$

For a Radon measure ν on X , let $M_r \nu(x) = \sup_{s \in (0, r)} \frac{\nu(B(x, s))}{m(B(x, s))}$. The proof of the following lemma is based on the maximal-function telescoping argument of Hajlasz and Koskela [27], adapted here to the Poincaré inequality with scale function Ψ .

Lemma 4.2. *Assume that the metric measure Dirichlet space satisfies $\text{PI}(\Psi)$ and volume doubling. Then there exist constants $A \geq 1$ and $C_{PI} > 0$ such that the following holds for every $f \in \mathcal{F}$, represented by its precise representative.*

If, for some $x \in X$ and $r > 0$,

$$M_r \Gamma(f, f)(x) < \infty,$$

then x is a Lebesgue point of f .

Moreover, if $x, y \in X$ satisfy

$$M_{Ad(x, y)} \Gamma(f, f)(x) < \infty \quad \text{and} \quad M_{Ad(x, y)} \Gamma(f, f)(y) < \infty,$$

then

$$\begin{aligned} & |f(x) - f(y)| \\ & \leq C_{PI} \Psi(d(x, y))^{1/2} \left((M_{Ad(x, y)} \Gamma(f, f)(x))^{1/2} \right. \\ & \quad \left. + (M_{Ad(x, y)} \Gamma(f, f)(y))^{1/2} \right). \end{aligned}$$

Proof. We first prove the assertion concerning Lebesgue points. Suppose that $M_r \Gamma(f, f)(x) < \infty$, and choose $R > 0$ sufficiently small that $AR < r$. Set $B_i := B(x, 2^{-i}R)$, $i \geq 0$. By the Poincaré inequality and volume doubling,

$$|f_{B_i} - f_{B_{i+1}}| \leq C \Psi(2^{-i}R)^{1/2} (M_r \Gamma(f, f)(x))^{1/2}.$$

The scaling properties of Ψ imply that

$$\sum_{i=0}^{\infty} \Psi(2^{-i}R)^{1/2} \leq C \Psi(R)^{1/2},$$

and hence $(f_{B_i})_{i \geq 0}$ is Cauchy.

If $2^{-i-1}R < s \leq 2^{-i}R$, another application of the Poincaré inequality and volume doubling gives

$$|f_{B(x,s)} - f_{B_i}| \leq C\Psi(2^{-i}R)^{1/2} (M_r\Gamma(f, f)(x))^{1/2}.$$

Therefore $\lim_{s \downarrow 0} f_{B(x,s)}$ exists. Since f is represented by its precise representative, this limit equals $f(x)$. Furthermore,

$$\frac{1}{m(B(x,s))} \int_{B(x,s)} |f - f_{B(x,s)}| dm \leq C\Psi(s)^{1/2} (M_r\Gamma(f, f)(x))^{1/2} \longrightarrow 0.$$

Together with $f_{B(x,s)} \rightarrow f(x)$, this yields

$$\frac{1}{m(B(x,s))} \int_{B(x,s)} |f - f(x)| dm \longrightarrow 0,$$

so x is a Lebesgue point of f .

We next prove the pointwise estimate. Set $R = d(x, y)$. By the preceding telescoping argument,

$$|f(x) - f_{B(x,R)}| \leq C\Psi(R)^{1/2} (M_{AR}\Gamma(f, f)(x))^{1/2},$$

and similarly,

$$|f(y) - f_{B(y,R)}| \leq C\Psi(R)^{1/2} (M_{AR}\Gamma(f, f)(y))^{1/2}.$$

After increasing A by a fixed factor if necessary, the Poincaré inequality and volume doubling also give

$$|f_{B(x,R)} - f_{B(y,R)}| \leq C\Psi(R)^{1/2} (M_{AR}\Gamma(f, f)(x))^{1/2}.$$

Combining the preceding three estimates and increasing the constant gives

$$|f(x) - f(y)| \leq C_{PI}\Psi(R)^{1/2} \left((M_{AR}\Gamma(f, f)(x))^{1/2} + (M_{AR}\Gamma(f, f)(y))^{1/2} \right).$$

□

4.2 Minimal *-upper gradients

The proof of Theorem 4.1 uses some of the main tools of the theory of differentiability spaces: curve fragments and *-upper gradients, which we now explain.

A *curve fragment* in X is a bi-Lipschitz map $\gamma : \text{dom}(\gamma) \rightarrow X$, where $\text{dom}(\gamma)$ is a compact subset of $\mathbb{R}$. Let $\Gamma(X)$ denote the collection of all curve fragments in X . For $\gamma \in \Gamma(X)$, set

$$K := \text{dom}(\gamma), \quad a := \min K, \quad b := \max K,$$

and write

$$[a, b] \setminus K = \bigcup_{i \in I} (a_i, b_i),$$

where the intervals (a_i, b_i) are the connected components of $[a, b] \setminus K$. These intervals are called the *gaps* of γ , and we define

$$\text{gap}(\gamma) := \sum_{i \in I} (b_i - a_i).$$

For disjoint sets $E, F \subset X$, let $\Gamma(E, F)$ denote the collection of all $\gamma \in \Gamma(X)$ such that

$$\gamma(\min \operatorname{dom}(\gamma)) \in E \quad \text{and} \quad \gamma(\max \operatorname{dom}(\gamma)) \in F.$$

For a curve fragment $\gamma : K \rightarrow X$, we define the *metric derivative* as

$$d_\gamma(t) := \lim_{\substack{s \in K \\ s \rightarrow t}} \frac{d(\gamma(s), \gamma(t))}{|s - t|},$$

which exists at a.e. nonisolated point $t \in \operatorname{dom}(\gamma)$. If $g : X \rightarrow [0, \infty]$ is a Borel function, we define

$$\int_\gamma g \, ds := \int_{\operatorname{dom}(\gamma)} g(\gamma(t)) d_\gamma(t) \, dt.$$

We say that a curve fragment is *parameterized by unit speed* if $d_\gamma(t) = 1$ for a.e. $t \in \operatorname{dom}(\gamma)$ and

$$d(\gamma(a_i), \gamma(b_i)) = |a_i - b_i|$$

for every gap (a_i, b_i) , $i \in I$. A unit-speed parametrization can be obtained by embedding X isometrically into $\ell^\infty(X)$ via the Kuratowski embedding [28, Exercise 12.4], extending the curve piecewise linearly, and applying classical results for curves in [1, Lemma 1.1.4]; see [23, Lemma 2.15] for details. More precisely, it is shown in [23, Lemma 2.15] that there exists a surjective Lipschitz reparametrization

$$\sigma : \operatorname{dom}(\gamma) \rightarrow \operatorname{dom}(\tilde{\gamma})$$

such that $\tilde{\gamma} \circ \sigma(t) = \gamma(t)$ for all $t \in \operatorname{dom}(\gamma)$ and $d_{\tilde{\gamma}}(\sigma(t)) = 1$ for a.e. $t \in \operatorname{dom}(\gamma)$. The surjectivity of σ is established in the proof. Since σ is Lipschitz, it maps sets of Lebesgue measure zero to sets of Lebesgue measure zero, and hence $d_{\tilde{\gamma}}(s) = 1$ for a.e. $s \in \operatorname{dom}(\tilde{\gamma})$. We also recall the standard fact that the above integrals are independent of the choice of parametrization [23, Lemma 2.15-(5)]. Also, let

$$\operatorname{len}(\gamma) := \sup_{\substack{n \in \mathbb{N}, t_1, \dots, t_n \in K \\ t_1 \leq \dots \leq t_n}} \sum_{i=1}^{n-1} d(\gamma(t_{i+1}), \gamma(t_i))$$

be the length of a curve fragment. In [23, Lemma 2.15-(2),(3)] it is shown that $\operatorname{len}(\gamma) = \operatorname{len}(\tilde{\gamma})$ and $\operatorname{gap}(\tilde{\gamma}) = \operatorname{gap}(\gamma)$. It follows from [23, Proof of Lemma 2.15] that $\operatorname{len}(\tilde{\gamma}) = \operatorname{gap}(\tilde{\gamma}) + \int_{\tilde{\gamma}} 1 \, ds$. Thus, for any curve fragment $\gamma \in \Gamma(X)$,

$$\operatorname{len}(\gamma) = \operatorname{gap}(\gamma) + \int_\gamma 1 \, ds. \tag{4.1}$$

For any curve fragment $\gamma \in \Gamma(X)$, we get from this

$$d(\gamma(\max(\operatorname{dom}(\gamma))), \gamma(\min(\operatorname{dom}(\gamma)))) \leq \operatorname{len}(\gamma) \leq \operatorname{gap}(\gamma) + \int_\gamma 1 \, ds. \tag{4.2}$$

We need the following lemma for covering a curve fragment parameterized by unit speed.

Lemma 4.3. *Let $\delta \in (0, 1)$, let $\gamma \in \Gamma(X)$ be parameterized by unit speed, and let $g : X \rightarrow [c, M]$ be a bounded lower semicontinuous function, where $M > c > 0$. For each $n \in \mathbb{N}$, there exists a collection of intervals $\mathcal{I}_n$ in $\mathbb{R}$ with pairwise disjoint interiors such that the following hold.*

1. The endpoints of each $I \in \mathcal{I}_n$ lie in $\text{dom}(\gamma)$.

2. For each $I = [a, b] \in \mathcal{I}_n$, we have

$$2^{-n-1} \leq d(\gamma(b), \gamma(a)) \leq |b - a| \leq 2^{-n}.$$

3. For each $I \in \mathcal{I}_n$ we have $\text{gap}(\gamma|_{I \cap \text{dom}(\gamma)}) \leq \delta 2^{-n-4}$.

4. For each $I = [a, b] \in \mathcal{I}_n$

$$\int_{\gamma|_{I \cap \text{dom}(\gamma)}} g ds \leq 2^{1-n} \inf_{p \in B(\gamma(a), 2^{-n})} g(p).$$

5.

$$\lim_{n \rightarrow \infty} \lambda(\text{dom}(\gamma) \setminus \bigcup_{I \in \mathcal{I}_n} I) = 0.$$

Proof. Define $K := \text{dom}(\gamma)$. Let $\tilde{K}_n$ be the set of $t \in K$ such that the following hold:

(a)

$$\frac{\lambda([t, t + 2^{-n}] \setminus K)}{2^{-n}} \leq 2^{-4}\delta. \quad (4.3)$$

(b) For all $s \in K \cap (t - 2^{-n}, t + 2^{-n}) \setminus \{t\}$

$$\frac{d(\gamma(s), \gamma(t))}{|t - s|} \geq (1 - 2^{-2}). \quad (4.4)$$

(c)

$$\int_{\gamma|_{[t, t+2^{-n}] \cap \text{dom}(\gamma)}} g ds \leq 2^{1-n} \inf_{p \in B(\gamma(t), 2^{-n})} g(p). \quad (4.5)$$

We claim that the set $\tilde{K}_n$ is closed. To see this, assume that $t_i \in \tilde{K}_n$ are such that $\lim_{i \rightarrow \infty} t_i = t$. By the dominated convergence theorem

$$\lim_{i \rightarrow \infty} \frac{\lambda([t_i, t_i + 2^{-n}] \setminus K)}{2^{-n}} = \frac{\lambda([t, t + 2^{-n}] \setminus K)}{2^{-n}}$$

and hence t satisfies (a). Now, assume that t fails (b). Then there exists $s \in K \cap (t - 2^{-n}, t + 2^{-n}) \setminus \{t\}$ for which the opposite inequality would hold strictly. Since $\lim_{i \rightarrow \infty} t_i = t$, $s \in K \cap (t_i - 2^{-n}, t_i + 2^{-n})$ for all large enough i and

$$\lim_{i \rightarrow \infty} \frac{d(\gamma(s), \gamma(t_i))}{|t_i - s|} = \frac{d(\gamma(s), \gamma(t))}{|t - s|} < (1 - 2^{-2})$$

implies that t_i doesn't satisfy (b) for large enough i , which is a contradiction. Thus t satisfies (b). Finally, for (c), notice that the function

$$t \mapsto \int_{\gamma|_{[t, t+2^{-n}] \cap \text{dom}(\gamma)}} g ds$$

is continuous in t by the estimate

$$\begin{aligned} & \left| \int_{\gamma|_{[t, t+2^{-n}] \cap \text{dom}(\gamma)}} g \, ds - \int_{\gamma|_{[t', t'+2^{-n}] \cap \text{dom}(\gamma)}} g \, ds \right| \\ & \leq 2 \sup |g| |t - t'| \leq 2M |t - t'|. \end{aligned}$$

On the other hand, the function $t \mapsto \inf_{p \in B(\gamma(t), 2^{-n})} g(p)$ is upper-semicontinuous in t , since the function $x \mapsto \inf_{p \in B(x, 2^{-n})} g(p)$ is upper-semicontinuous due to $B(x, 2^{-n})$ being open. This implies that the set $t \in K$ satisfying (4.5) is a closed set, which concludes the proof of the claim that the set $\tilde{K}_n$ is closed.

Next, we claim that

$$\lim_{n \rightarrow \infty} \lambda(K \setminus \tilde{K}_n) = 0. \quad (4.6)$$

Lebesgue differentiation implies that

$$\lim_{n \rightarrow \infty} \frac{\lambda([t, t+2^{-n}] \setminus K)}{2^{-n}} = 0, \quad \text{for a.e. } t \in K.$$

The choice of unit speed parametrization for γ imply that

$$\lim_{s \rightarrow t} \frac{d(\gamma(s), \gamma(t))}{|t - s|} = 1, \quad \text{for a.e. } t \in K.$$

Lebesgue differentiation and the lower semicontinuity of g implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} 2^n \int_{\gamma|_{[t, t+2^{-n}] \cap \text{dom}(\gamma)}} g \, ds &= g(\gamma(t)) \\ &= \lim_{n \rightarrow \infty} \inf_{p \in B(\gamma(t), 2^{-n})} g(p), \end{aligned}$$

for a.e. $t \in K$. Hence for a.e. $t \in K$, there exists $n \in \mathbb{N}$ such that $t \in \bigcap_{l \geq n} \tilde{K}_l$. Thus (4.6) follows from

$$\begin{aligned} \limsup_{n \rightarrow \infty} \lambda(K \setminus \tilde{K}_n) &\leq \lim_{n \rightarrow \infty} \lambda \left(K \setminus \bigcap_{l \geq n} \tilde{K}_l \right) \\ &= \lambda \left(\bigcap_{n=1}^{\infty} \left(K \setminus \bigcap_{l \geq n} \tilde{K}_l \right) \right) \\ &= \lambda \left(K \setminus \bigcup_{n=1}^{\infty} \bigcap_{l \geq n} \tilde{K}_l \right) = 0. \end{aligned}$$

If $\tilde{K}_n = \emptyset$, set $\mathcal{I}_n = \emptyset$, and no further construction is needed. For the remainder, assume that $\tilde{K}_n \neq \emptyset$ and let $a_1 = \min(\tilde{K}_n)$. Since $\tilde{K}_n$ is closed, $a_1 \in \tilde{K}_n$. Hence by (4.3), there exists a $b_1 \in K \cap [a_1 + 3 \cdot 2^{-n-2}, a_1 + 2^{-n}]$. Set $I_1 = [a_1, b_1]$. We next describe the recursive construction of $a_{i+1}, b_{i+1}, I_{i+1}$ assuming $a_i, b_i, I_i = [a_i, b_i]$ have been defined such that $a_i \in \tilde{K}_n$ and $b_i \in K$. If $\max(\tilde{K}_n) < b_i$, we stop the recursive construction and set

$$\mathcal{I}_n := \{[a_j, b_j] : j = 1, \dots, i\}.$$

Next, we consider the case $\max(\tilde{K}_n) \geq b_i$. In this case, let $a_{i+1} = \inf(\tilde{K}_n \cap [b_i, \infty))$. Since $\tilde{K}_n$ is closed, $a_{i+1} \in \tilde{K}_n$. By (4.3), there exists a $b_{i+1} \in K \cap [a_{i+1} + 3 \cdot 2^{-n-2}, a_{i+1} + 2^{-n})$. Set $I_{i+1} = [a_{i+1}, b_{i+1}]$.

By compactness of K , and since each interval I_i has length at least $3 \cdot 2^{-n-2}$, this recursion stops after finitely many steps and yields a collection $\mathcal{I}_n = \{I_1, \dots, I_N\}$. Since the intervals were constructed with increasing end points, the intervals have pairwise disjoint interiors. We now show the following properties for $i = 1, \dots, N$:

- (i) $a_i \in \tilde{K}_n$, $b_i > a_i$ and $b_i \in K$.
- (ii) $I_i = [a_i, b_i]$ satisfies all the properties (1)-(5) in the statement of the Lemma.

The properties (i) and (1) are immediate from the construction and the fact that $\tilde{K}_n \subset K$. For verifying (2), note that

$$d(\gamma(b_i), \gamma(a_i)) \stackrel{(4.4)}{\geq} (1 - 2^{-2})(b_i - a_i) \geq 3(1 - 2^{-2})2^{-n-2} \geq 2^{-n-1}.$$

Since $a_i \in \tilde{K}_n$ and $a_i < b_i \leq a_i + 2^{-n}$, we have

$$\frac{\lambda([a_i, b_i] \setminus K)}{2^{-n}} \leq \frac{\lambda([a_i, a_i + 2^{-n}] \setminus K)}{2^{-n}} \stackrel{(4.3)}{\leq} 2^{-4}\delta.$$

Thus we have $\text{gap}(\gamma|_{I_i \cap \text{dom}(\gamma)}) \leq \delta 2^{-n-4}$, which yields (3).

Now, (4) follows since $\int_{\gamma|_{I_i \cap \text{dom}(\gamma)}} g \, ds \leq \int_{\gamma|_{[a_i, a_i + 2^{-n}] \cap \text{dom}(\gamma)}} g \, ds$ and $a_i \in \tilde{K}_n$.

It remains to show that (5) holds. To show this, it suffices to observe that the construction ensures that $\mathcal{I}_n$ covers $\tilde{K}_n$, and to apply (4.6). $\square$

Let $f \in \text{LIP}(X)$. We call a Borel function $g: X \rightarrow [0, M]$, for some $M > 0$, a **-upper gradient* of f if there exists a constant $L > 0$ such that, for every $\gamma \in \Gamma(X)$,

$$|f(\gamma(\min K)) - f(\gamma(\max K))| \leq \int_{\gamma} g \, ds + L \text{gap}(\gamma), \quad (4.7)$$

where $K = \text{dom}(\gamma)$. We say that g is a *minimal *-upper gradient* of f if g is a *-upper gradient of f and satisfies $g' \geq g$ m -almost everywhere for every *-upper gradient g' of f . The minimal *-upper gradient is often denoted by g_f . By [9, Proposition 2.10], a minimal *-upper gradient exists and is unique up to m -almost everywhere equivalence.

Lemma 4.4. *Let $f \in \text{LIP}(X)$. If g is an *-upper gradient of f , then $\min(g, \text{LIP}[f])$ is an *-upper gradient of f and (4.7) is true with $L = \text{LIP}[f]$.*

Proof. By Lebesgue differentiation, the condition that g is a *-upper gradient of f is equivalent to $|(f \circ \gamma)'(t)| \leq g(\gamma(t))d_{\gamma}(t)$ for a.e. $t \in \text{dom}(\gamma)$ and for all $\gamma \in \Gamma(X)$. Since $|(f \circ \gamma)'(t)| \leq \text{LIP}[f]d_{\gamma}(t)$, we see that $\min(g, \text{LIP}[f])$ is always an *-upper gradient of f whenever g is one.

Let $K = \text{dom}(\gamma)$. Now, if $(a_i, b_i), i \in I$ are the gaps of γ , then

$$\begin{aligned} & f(\gamma(\max(K))) - f(\gamma(\min(K))) \\ &= \int_{\text{dom}(\gamma)} (f \circ \gamma)'(t) \, dt + \sum_{i \in I} (f(\gamma(b_i)) - f(\gamma(a_i))). \end{aligned}$$

From this, and the previous paragraph we immediately get (4.7) with $L = \text{LIP}[f]$. $\square$

Lemma 4.5. Assume that $g_i : X \rightarrow [0, L]$ is measurable for every $i \in \mathbb{N}$. Then, for every bounded Borel set $A \subset X$, there exists a measurable function $g : X \rightarrow [0, L]$, depending on A , such that

$$\int_A g \, dm \leq \liminf_{i \rightarrow \infty} \int_A g_i \, dm$$

and, for every $\gamma \in \Gamma(X)$,

$$\int_\gamma g \, ds \geq \liminf_{i \rightarrow \infty} \int_\gamma g_i \, ds.$$

Proof. Let $A \subset X$ be bounded. After passing to a subsequence, we may assume that

$$\liminf_{i \rightarrow \infty} \int_A g_i \, dm = \lim_{i \rightarrow \infty} \int_A g_i \, dm.$$

By Dunford–Pettis theorem [17, Theorem 4.30] and [17, Exercise 4.36], the set $\{g_n|_A : n \in \mathbb{N}\}$ is relatively compact in the weak topology of $L^1(A, m|_A)$. By using Mazur’s lemma [17, Corollary 3.6, Exercise 3.4-(1)] and taking convex combinations $\tilde{g}_k = \sum_{i=k}^{N_k} \alpha_{k,i} g_i$, we can ensure that $\tilde{g}_k|_A$ converges in $L^1(A, m|_A)$. By passing to a further subsequence if necessary [17, Theorem 4.9], we may assume that $\tilde{g}_k$ converge pointwise m -a.e. in A . Define $g : X \rightarrow [0, L]$ by

$$g(x) = \begin{cases} \lim_{k \rightarrow \infty} \tilde{g}_k(x), & \text{if } x \in A \text{ and the limit exists,} \\ L, & \text{otherwise.} \end{cases}$$

Since $\tilde{g}_k \leq L$ everywhere, we have

$$0 \leq L - g(x) \leq \liminf_{k \rightarrow \infty} (L - \tilde{g}_k(x)), \quad \text{for all } x \in X.$$

Therefore by Fatou’s lemma and linearity of integrals, we have

$$\int_\gamma g \, ds \geq \limsup_{k \rightarrow \infty} \int_\gamma \tilde{g}_k \, ds \geq \liminf_{i \rightarrow \infty} \int_\gamma g_i \, ds, \quad \text{for all } \gamma \in \Gamma(X).$$

Since $\tilde{g}_k$ converges in $L^1(A, m|_A)$ to g , we have

$$\begin{aligned} \int_A g \, dm &= \lim_{k \rightarrow \infty} \int_A \tilde{g}_k \, dm \\ &= \lim_{k \rightarrow \infty} \sum_{i=k}^{N_k} \alpha_{k,i} \int_A g_i \, dm \\ &= \lim_{i \rightarrow \infty} \int_A g_i \, dm = \liminf_{i \rightarrow \infty} \int_A g_i \, dm. \end{aligned}$$

□

We adapt a definition of connectivity from [23, Definition 3.1] and [10, Lemma 3.5]. In these references, such a condition arose in the context of proving Poincaré inequalities for spaces supporting a stronger form of differentiation.

Definition 4.6. Let $\delta, \epsilon \in (0, 1)$. We say that an annulus $B(x, 2r) \setminus B(x, r)$ is (δ, ϵ) -connected if for every Borel set $E \subset B(x, 2r)$ with $m(E)/m(B(x, 2r)) < \epsilon$ there exists a $\gamma \in \Gamma(B(x, r), X \setminus B(x, 2r))$ with $\gamma(\text{dom}(\gamma)) \cap E \subset \{\gamma(\min(\text{dom}(\gamma)))\}$ and

$$\text{gap}(\gamma) \leq \delta r.$$

The argument in the following lemma is inspired by the proof that positive modulus implies a lower bound for conformal dimension. The original argument is due to Pansu and Bourdon [53, 16], but the presentation here is closer to that of [57] see also [12, Theorem 4.1]. In these proofs, one can find a sequence of functions with small L^p -norm, and extract an estimate for modulus by taking a limit of such functions. Here, the role of modulus is replaced by the $*$ -upper-gradient property, while poorly connected annuli provide the deformations needed in the argument. These deformations are used to modify the minimal $*$ -upper gradient and construct a sequence of $*$ -upper gradients that ultimately contradicts its minimality.

Lemma 4.7. *Let (X, d) be a complete separable metric space, and let m be a doubling measure on X with doubling constant D . Set $C_D := 2^9 D^3$. Let $f \in \text{LIP}(X)$, and suppose that its minimal $*$ -upper gradient g_f satisfies*

$$m(\{x \in X : g_f(x) > 0\}) > 0. \quad (4.8)$$

Then, for every $\delta \in (0, 1)$ and $r_0 > 0$, there exist $x \in X$ and $r \in (0, r_0)$ such that the annulus $B(x, 2r) \setminus B(x, r)$ is $(\delta, \delta/C_D)$ -connected.

Proof. For each $n \in \mathbb{N}$, let N_n be a 2^{-n} -net of X ; that is, a maximal subset of X such that any two distinct points $x, y \in N_n$ satisfy $d(x, y) \geq 2^{-n}$.

Suppose, to the contrary, that there exist $\delta \in (0, 1)$ and $r_0 > 0$ such that every annulus $B(x, 2r) \setminus B(x, r)$, with $x \in X$ and $r \in (0, r_0)$, is not $(\delta, \delta/C_D)$ -connected. We will obtain a contradiction by showing that $g_f = 0$ m -a.e. By replacing the metric d with αd , where $\alpha := 2r_0^{-1} \in (0, \infty)$, we may assume that $r_0 = 2$. Note that this rescaling preserves the doubling constant D of m , while the minimal $*$ -upper gradient g_f is replaced by $\alpha^{-1} g_f$. By (4.8), $\text{Lip}(f) \neq 0$. Replacing f by $\text{Lip}(f)^{-1} f$, we may assume that f is 1-Lipschitz. Note that the rescaled function still satisfies (4.8). By Lemma 4.4, and after possibly modifying g_f on a set of measure zero, we may and will assume that $g_f \leq 1$.

Since (X, d) is separable, (4.8) implies that there exists $x_0 \in X$ such that $\int_{B(x_0, r)} g_f dm > 0$ for all $r > 0$. Since m is finite on balls, $m(\partial B(x_0, r)) > 0$ for at most countably many $r > 0$. We may therefore choose $r \in (0, r_0)$ such that $m(\partial B(x_0, r)) = 0$. Since m is a Radon measure that is finite on all balls, m is outer regular, see e.g. the proof in [46, Theorem 1.10]. Since X is separable and m is outer regular, we may apply the Vitali–Carathéodory theorem [29, p. 108, Chapter 4] to the function $g_f|_{\overline{B(x_0, r)}}$ on the metric measure space $(\overline{B(x_0, r)}, m|_{\overline{B(x_0, r)}})$. Thus, after truncating and choosing $\epsilon > 0$ sufficiently small, there exists a lower semicontinuous function $\tilde{g} : \overline{B(x_0, r)} \rightarrow [\epsilon, 1]$ such that $\tilde{g} \geq g_f$ on $\overline{B(x_0, r)}$ and

$$\int_{B(x_0, r)} \tilde{g} dm < \frac{3}{2} \int_{B(x_0, r)} g_f dm.$$

Define $g : X \rightarrow [\epsilon, 1]$ by

$$g(x) := \begin{cases} \tilde{g}(x), & x \in \overline{B(x_0, r)}, \\ 1, & x \in X \setminus \overline{B(x_0, r)}. \end{cases}$$

Then g is lower semicontinuous and satisfies $g_f \leq g$ everywhere. Since $m(\partial B(x_0, r)) = 0$, we have $\lim_{n \rightarrow \infty} \int_{B(x_0, r+2^{-n})} g dm = \int_{B(x_0, r)} g dm$. Consequently, for all sufficiently large n ,

$$\int_{B(x_0, r+2^{-n})} g dm < 2 \int_{B(x_0, r)} g_f dm. \quad (4.9)$$

For each $n \in \mathbb{N}$ and $p \in N_n$, the failure of the $(\delta, \delta/C_D)$ -connectivity condition of the annulus $B(p, 2^{1-n}) \setminus B(p, 2^{-n})$ yields a set $E_p \subset B(p, 2^{1-n})$ satisfying

$$m(E_p)/m(B(p, 2^{1-n})) < \delta/C_D, \quad (4.10)$$

such that, for every $\gamma \in \Gamma(B(p, 2^{-n}), X \setminus B(p, 2^{1-n}))$ satisfying $\gamma(\text{dom}(\gamma)) \cap E_p \subset \{\gamma(\min(\text{dom}(\gamma)))\}$, we have

$$\text{gap}(\gamma) > \delta 2^{-n}. \quad (4.11)$$

By outer regularity and the strict inequality in (4.10), and after possibly enlarging E_p slightly, we may moreover assume that E_p is open and $E_p \subset B(p, 2^{1-n})$. Note that the condition in (4.11) is preserved under such an enlargement.

Define

$$\psi_p(x) := \frac{2^6}{\delta} 1_{E_p}(x), \quad p \in N_n,$$

and, for each $n \in \mathbb{N}$, let

$$g_n(x) := \sum_{p \in N_n} \psi_p(x) \inf_{y \in B(p, 2^{-n})} g(y).$$

We claim that for each $\gamma \in \Gamma(X)$

$$\int_{\gamma} g \, ds \leq \liminf_{n \rightarrow \infty} \int_{\gamma} g_n \, ds. \quad (4.12)$$

We parametrize γ by unit speed. Lemma 4.3 yields for every $n \in \mathbb{N}$ a collection $\mathcal{I}_n$ of intervals in $\mathbb{R}$ with pairwise disjoint interiors such that

1. The endpoints of each $I \in \mathcal{I}_n$ lie in $\text{dom}(\gamma)$.

2. For each $I = [a, b] \in \mathcal{I}_n$, we have

$$2^{-n-1} \leq d(\gamma(b), \gamma(a)) \leq |b - a| \leq 2^{-n}.$$

3. For each $I \in \mathcal{I}_n$ we have $\text{gap}(\gamma|_{I \cap \text{dom}(\gamma)}) \leq \delta 2^{-n-4}$.

4. For each $I = [a, b] \in \mathcal{I}_n$

$$\int_{\gamma|_{I \cap \text{dom}(\gamma)}} g \, ds \leq 2^{1-n} \inf_{y \in B(\gamma(a), 2^{-n})} g(y).$$

5.

$$\lim_{n \rightarrow \infty} \lambda(\text{dom}(\gamma) \setminus \bigcup_{I \in \mathcal{I}_n} I) = 0$$

For each $n \in \mathbb{N}$ and for each $I = [a, b] \in \mathcal{I}_n$ let $p_I \in N_{n+3}$ be such that $\gamma(a) \in B(p_I, 2^{-n-3})$. We slightly abbreviate the restrictions as follows $\gamma_I := \gamma|_{\text{dom}(\gamma) \cap I}$. By condition (2), $\gamma_I \in \Gamma(B(p_I, 2^{-n-3}), X \setminus B(p_I, 2^{-n-2}))$ since $d(p_I, \gamma(a)) < 2^{-n-3}$ and $d(p_I, \gamma(b)) \geq d(\gamma(a), \gamma(b)) - d(p_I, \gamma(a)) > 2^{-n-1} - 2^{-n-3} > 2^{-n-2}$. We also have $B(p_I, 2^{-n-3}) \subset B(\gamma(a), 2^{-n})$ for every $I = [a, b] \in \mathcal{I}_n$, since $d(\gamma(a), p_I) < 2^{-n-3}$. Thus, by condition (4) above

$$2^{1-n} \inf_{y \in B(p_I, 2^{-n-3})} g(y) \geq 2^{1-n} \inf_{y \in B(\gamma(a), 2^{-n})} g(y) \geq \int_{\gamma_I} g \, ds. \quad (4.13)$$

We next show a lower bound for $\int_{\gamma_I} 1_{E_{p_I}} ds$. Consider the curve fragment $\hat{\gamma}_I = \gamma|_{\{a,b\} \cup (\text{dom}(\gamma_I) \setminus \gamma_I^{-1}(E_{p_I}))}$, whose domain is closed since E_{p_I} is open. We have $\hat{\gamma}_I(a) \in B(p_I, 2^{-n-3})$ and $d(\hat{\gamma}_I(b), p_I) \geq d(\hat{\gamma}_I(b), \hat{\gamma}_I(a)) - d(\hat{\gamma}_I(a), p_I) \geq 2^{-n-2}$. Thus $\hat{\gamma}_I(b) \notin B(p_I, 2^{-n-2})$ and hence

$$\hat{\gamma}_I \in \Gamma(B(p_I, 2^{-n-3}), X \setminus B(p_I, 2^{-n-2})) \quad \text{and} \\ \hat{\gamma}_I(\text{dom}(\hat{\gamma}_I)) \cap E_{p_I} \subset \{\gamma(a)\}.$$

Therefore by (4.11) applied at scale $n+3$, we have

$$\text{gap}(\hat{\gamma}_I) > \delta 2^{-n-3}. \quad (4.14)$$

By the definition of $\hat{\gamma}_I$, its domain is obtained from $\text{dom}(\gamma_I)$ by removing $\gamma_I^{-1}(E_{p_I})$, apart from the two endpoints. Since γ_I is parameterized by unit speed, it follows that

$$\text{gap}(\hat{\gamma}_I) = \text{gap}(\gamma_I) + \int_{\gamma_I} 1_{E_{p_I}} ds.$$

Consequently, (4.14) and condition (3) give

$$\delta 2^{-n-3} < \text{gap}(\hat{\gamma}_I) \leq \delta 2^{-n-4} + \int_{\gamma_I} 1_{E_{p_I}} ds.$$

Therefore,

$$\delta 2^{-n-4} \leq \int_{\gamma_I} 1_{E_{p_I}} ds. \quad (4.15)$$

Combining (4.15) with (4.13) we get

$$\begin{aligned} \int_{\gamma_I} g_{n+3} ds &\geq \frac{2^6}{\delta} \inf_{y \in B(p_I, 2^{-n-3})} g(y) \int_{\gamma_I} 1_{E_{p_I}} ds \\ &\stackrel{(4.15)}{\geq} \inf_{y \in B(p_I, 2^{-n-3})} g(y) 2^{2-n} \stackrel{(4.13)}{\geq} \int_{\gamma_I} g ds. \end{aligned}$$

Summing this estimate over $I \in \mathcal{I}_n$, we obtain

$$\sum_{I \in \mathcal{I}_n} \int_{\gamma_I} g_{n+3} ds \geq \sum_{I \in \mathcal{I}_n} \int_{\gamma_I} g ds. \quad (4.16)$$

Since the intervals in $\mathcal{I}_n$ have pairwise disjoint interiors, and γ is parametrized by unit speed,

$$\sum_{I \in \mathcal{I}_n} \int_{\gamma_I} g ds = \int_{\text{dom}(\gamma) \cap \bigcup_{I \in \mathcal{I}_n} I} g(\gamma(t)) dt.$$

Thus, since $0 \leq g \leq 1$, condition (5) implies

$$0 \leq \int_{\gamma} g ds - \sum_{I \in \mathcal{I}_n} \int_{\gamma_I} g ds \leq \lambda \left(\text{dom}(\gamma) \setminus \bigcup_{I \in \mathcal{I}_n} I \right) \rightarrow 0. \quad (4.17)$$

Moreover, since $g_{n+3} \geq 0$ and the intervals in $\mathcal{I}_n$ have pairwise disjoint interiors, we have

$$\sum_{I \in \mathcal{I}_n} \int_{\gamma_I} g_{n+3} ds \leq \int_{\gamma} g_{n+3} ds.$$

Consequently, we obtain (4.12) since

$$\begin{aligned} \int_{\gamma} g \, ds &\stackrel{(4.17)}{=} \lim_{n \rightarrow \infty} \sum_{I \in \mathcal{I}_n} \int_{\gamma_I} g \, ds \stackrel{(4.16)}{\leq} \liminf_{n \rightarrow \infty} \sum_{I \in \mathcal{I}_n} \int_{\gamma_I} g_{n+3} \, ds \\ &\leq \liminf_{n \rightarrow \infty} \int_{\gamma} g_{n+3} \, ds = \liminf_{n \rightarrow \infty} \int_{\gamma} g_n \, ds. \end{aligned}$$

We next estimate the integral of the functions g_n . We have

$$\begin{aligned} &\int_{B(x_0, r)} g_n \, dm \\ &\leq \sum_{\substack{p \in N_n \\ B(p, 2^{1-n}) \cap B(x_0, r) \neq \emptyset}} \inf_{y \in B(p, 2^{-n})} g(y) \int \psi_p \, dm \\ &\leq \sum_{\substack{p \in N_n \\ B(p, 2^{1-n}) \cap B(x_0, r) \neq \emptyset}} \frac{2^6}{\delta} m(E_p) \inf_{y \in B(p, 2^{-n})} g(y) \\ &\stackrel{(4.10)}{\leq} \sum_{\substack{p \in N_n \\ B(p, 2^{1-n}) \cap B(x_0, r) \neq \emptyset}} \frac{2^6}{C_D} m(B(p, 2^{1-n})) \inf_{y \in B(p, 2^{-n})} g(y) \\ &\leq \sum_{\substack{p \in N_n \\ B(p, 2^{1-n}) \cap B(x_0, r) \neq \emptyset}} \frac{2^6 D^2}{C_D} m(B(p, 2^{-1-n})) \inf_{y \in B(p, 2^{-n-1})} g(y) \\ &\leq \sum_{\substack{p \in N_n \\ B(p, 2^{1-n}) \cap B(x_0, r) \neq \emptyset}} \frac{D^2 2^6}{C_D} \int_{B(p, 2^{-n-1})} g(y) \, dm \\ &\leq \frac{2^6 D^2}{C_D} \int_{B(x_0, r+2^{2-n})} g(y) \, dm \stackrel{(4.9)}{\leq} \frac{2^7 D^2}{C_D} \int_{B(x_0, r)} g_f(y) \, dm. \end{aligned}$$

Here, the final estimate holds for all large enough n and we used that D is the doubling constant of m , that the balls $B(p, 2^{-n-1})$ are pairwise disjoint and $m(\partial B(x_0, r)) = 0$. Thus, by our choice of C_D we get for all large enough n

$$\int_{B(x_0, r)} g_n \, dm \leq \frac{1}{2} \int_{B(x_0, r)} g_f \, dm. \quad (4.18)$$

Let h be the function provided by Lemma 4.5 with $A = B(x_0, r)$. By (4.12) and the defining property of h ,

$$\int_{\gamma} h \, ds \geq \liminf_{n \rightarrow \infty} \int_{\gamma} g_n \, ds \geq \int_{\gamma} g \, ds$$

for every $\gamma \in \Gamma(X)$. Since g is a $*$ -upper gradient of f , it follows that h is also a $*$ -upper gradient of f . Moreover, Lemma 4.5 and (4.18) give

$$\int_{B(x_0, r)} h \, dm \leq \frac{1}{2} \int_{B(x_0, r)} g_f \, dm.$$

On the other hand, the minimality of g_f implies that $g_f \leq h$ m -almost everywhere. This contradicts

$$\int_{B(x_0, r)} g_f dm > 0.$$

□

The following is a weak-type inequality for the maximal function.

Lemma 4.8. *Let (X, d) be a complete separable metric space and let m be a doubling measure with doubling constant D . Let ν be a positive Radon measure on X . Fix $x \in X$, $r > 0$, and $L > 0$, and define*

$$E := \{y \in B(x, r) : M_r \nu(y) \geq L\},$$

where

$$M_r \nu(y) := \sup_{s \in (0, r)} \frac{\nu(B(y, s))}{m(B(y, s))}.$$

Then E is Borel and

$$m(E) \leq \frac{D^3}{L} \nu(B(x, 2r)).$$

Proof. We first note that $M_r \nu$ is lower semicontinuous. Let $y_i \rightarrow y$ and suppose that $M_r \nu(y) > a$. Choose $s \in (0, r)$ such that

$$\frac{\nu(B(y, s))}{m(B(y, s))} > a.$$

Set $s_i := s - d(y_i, y)$. For all sufficiently large i , we have $s_i \in (0, r)$ and

$$B(y_i, s_i) \subset B(y, s).$$

Moreover,

$$1_{B(y_i, s_i)} \longrightarrow 1_{B(y, s)}$$

pointwise. Hence, by dominated convergence,

$$\nu(B(y_i, s_i)) \longrightarrow \nu(B(y, s)) \quad \text{and} \quad m(B(y_i, s_i)) \longrightarrow m(B(y, s)).$$

Therefore, for all sufficiently large i ,

$$M_r \nu(y_i) \geq \frac{\nu(B(y_i, s_i))}{m(B(y_i, s_i))} > a.$$

Thus $M_r \nu$ is lower semicontinuous, and in particular E is Borel.

Fix $L' \in (0, L)$. For every $y \in E$, since $M_r \nu(y) \geq L > L'$, there exists $s_y \in (0, r)$ such that

$$\nu(B(y, s_y)) > L' m(B(y, s_y)).$$

Apply the $5B$ -covering lemma [28, Theorem 1.2] to the family

$$\mathcal{B} := \{B(y, s_y) : y \in E\}.$$

There exists a pairwise disjoint subcollection $\{B_i\}_{i \in I}$ such that

$$E \subset \bigcup_{i \in I} 5B_i.$$

Since X is separable, the family $\{B_i\}_{i \in I}$ is at most countable. By volume doubling,

$$m(5B_i) \leq m(8B_i) \leq D^3 m(B_i),$$

and hence

$$m(E) \leq \sum_i m(5B_i) \leq D^3 \sum_i m(B_i) < \frac{D^3}{L'} \sum_i \nu(B_i).$$

Since the balls B_i are pairwise disjoint and each B_i is contained in $B(x, 2r)$,

$$\sum_i \nu(B_i) \leq \nu(B(x, 2r)).$$

Thus

$$m(E) \leq \frac{D^3}{L'} \nu(B(x, 2r)).$$

Letting $L' \uparrow L$ gives

$$m(E) \leq \frac{D^3}{L} \nu(B(x, 2r)).$$

□

Lemma 4.9. *Let $\Psi : [0, \infty) \rightarrow [0, \infty)$ be a scale function such that*

$$\lim_{\epsilon \rightarrow 0} \limsup_{r \downarrow 0} \frac{\Psi(\epsilon r)}{\Psi(r) \epsilon^2} = 0.$$

Then $\lim_{r \downarrow 0} \frac{\Psi(r)}{r^2} = 0$.

Proof. Set

$$h(r) := \frac{\Psi(r)}{r^2}.$$

By assumption, we may choose $\epsilon \in (0, 1)$ and $r_0 > 0$ such that

$$h(\epsilon r) \leq \frac{1}{2} h(r)$$

for all $r \in (0, r_0)$. Iterating, for every $n \in \mathbb{N}$,

$$h(\epsilon^n r_0) \leq 2^{-n} h(r_0).$$

Now let $r \in (0, r_0)$. Choose $n \in \mathbb{N}$ such that

$$\epsilon^{n+1} r_0 < r \leq \epsilon^n r_0.$$

Since Ψ is a scale function,

$$\frac{\Psi(r)}{r^2} \lesssim \frac{\Psi(\epsilon^n r_0)}{\epsilon^{2n+2} r_0^2} = \epsilon^{-2} h(\epsilon^n r_0) \leq \epsilon^{-2} 2^{-n} h(r_0).$$

As $r \downarrow 0$, we have $n \rightarrow \infty$, and hence

$$\lim_{r \downarrow 0} \frac{\Psi(r)}{r^2} = 0.$$

□

Theorem 4.10. *Assume the hypotheses of Theorem 4.1. Then every $f \in \text{LIP}(X)$ has minimal $*$ -upper gradient $g_f = 0$ m -almost everywhere.*

Proof. Let D denote the doubling constant of m , and let $A, C_{PI} \geq 1$ be the constants in Lemma 4.2. Let $C_D > 0$ be the constant in Lemma 4.7.

By [48, Corollary 1.10], there exists $C_\Psi \geq 1$ such that

$$\frac{\Psi(s)}{s^2} \leq C_\Psi \frac{\Psi(R)}{R^2}, \quad 0 < s \leq R \leq \text{diam}(X, d). \quad (4.19)$$

By $\text{Cap}(\Psi)_\leq$ and regularity, after changing the structural constant if necessary, there exists $C_U > 0$ such that for every $p \in X$ and $r > 0$ there exists $\phi \in \mathcal{F} \cap C(X)$, with $0 \leq \phi \leq 1$, satisfying

$$\phi = 1 \quad \text{on } B(p, 5r/4), \quad \phi = 0 \quad \text{on } X \setminus B(p, 3r/2),$$

and

$$\mathcal{E}(\phi, \phi) \leq C_U \frac{m(B(p, r))}{\Psi(r)}. \quad (4.20)$$

Choose $\delta \in (0, 1)$ sufficiently small that

$$\delta < \min \left\{ \frac{1}{4}, \frac{2}{A} \right\}, \quad C_{PI} (2C_U C_D D^3 C_\Psi \delta)^{1/2} \leq \frac{1}{2}. \quad (4.21)$$

Using the hypothesis on Ψ , we may decrease δ further so that

$$\limsup_{r \downarrow 0} \frac{\Psi(\delta r)}{\Psi(r) \delta^2} < 1.$$

Hence there exists $r_0 > 0$ such that

$$\frac{\Psi(\delta r)}{\Psi(r) \delta^2} \leq 1 \quad (4.22)$$

for all $r \in (0, r_0)$.

Suppose, to the contrary, that there exists $f \in \text{LIP}(X)$ such that $g_f \neq 0$ on a set of positive measure. By Lemma 4.7, there exist $p \in X$ and $r \in (0, r_0)$ such that the annulus

$$B(p, 2r) \setminus B(p, r)$$

is $(\delta, \delta/C_D)$ -connected. Let $\phi \in \mathcal{F}$ be the cutoff from (4.20), represented by its precise representative, and set

$$\Lambda := 2C_U C_D D^3 \delta^{-1}.$$

Define

$$E := \left\{ x \in B(p, 2r) : M_{2r} \Gamma(\phi, \phi)(x) \geq \frac{\Lambda}{\Psi(r)} \right\}.$$

By Lemma 4.8, E is Borel and

$$\begin{aligned} \frac{m(E)}{m(B(p, 2r))} &\leq \frac{D^3 \Psi(r)}{\Lambda} \frac{\Gamma(\phi, \phi)(B(p, 4r))}{m(B(p, 2r))} \\ &\leq \frac{D^3 \Psi(r)}{\Lambda} \frac{\mathcal{E}(\phi, \phi)}{m(B(p, 2r))} \end{aligned}$$

$$\leq \frac{C_U D^3}{\Lambda} \leq \frac{\delta}{2C_D}.$$

In particular,

$$\frac{m(E)}{m(B(p, 2r))} < \frac{\delta}{C_D}.$$

By the $(\delta, \delta/C_D)$ -connectivity of the annulus, there exists a curve fragment

$$\gamma \in \Gamma(B(p, r), X \setminus B(p, 2r))$$

such that

$$\text{gap}(\gamma) \leq \delta r$$

and

$$\gamma(\text{dom}(\gamma)) \cap E \subset \{\gamma(\min(\text{dom}(\gamma)))\}.$$

By truncating γ at its first exit from $B(p, 2r)$, if necessary, we may assume that

$$\gamma(t) \in B(p, 2r)$$

for every $t \in \text{dom}(\gamma) \setminus \{\max(\text{dom}(\gamma))\}$. We parametrize γ by unit speed. Thus, for every $t \in \text{dom}(\gamma) \setminus \{\min(\text{dom}(\gamma)), \max(\text{dom}(\gamma))\}$,

$$M_{2r}\Gamma(\phi, \phi)(\gamma(t)) < \frac{\Lambda}{\Psi(r)}.$$

In particular, Lemma 4.2 implies that every such $\gamma(t)$ is a Lebesgue point of ϕ .

Let t, t' be interior points of $\text{dom}(\gamma)$ such that

$$d(\gamma(t), \gamma(t')) \leq \delta r.$$

Since $A\delta r < 2r$, Lemma 4.2 gives, after increasing C_{PI} by a fixed factor if necessary,

$$|\phi(\gamma(t)) - \phi(\gamma(t'))| \leq C_{PI}\Lambda^{1/2} \frac{\Psi(d(\gamma(t), \gamma(t')))^{1/2}}{\Psi(r)^{1/2}}. \quad (4.23)$$

If $s \leq \delta r$, then by (4.19) and (4.22),

$$\frac{\Psi(s)}{\Psi(r)} \leq C_\Psi \frac{s^2}{\delta^2 r^2} \frac{\Psi(\delta r)}{\Psi(r)} \leq C_\Psi \frac{s^2}{r^2}. \quad (4.24)$$

Hence (4.23) yields

$$|\phi(\gamma(t)) - \phi(\gamma(t'))| \leq C_{PI}(C_\Psi\Lambda)^{1/2} \frac{d(\gamma(t), \gamma(t'))}{r}.$$

Together with $0 \leq \phi \leq 1$ and the fact that $\phi \circ \gamma$ is constant in a neighbourhood of each endpoint of $\text{dom}(\gamma)$, this shows that $\phi \circ \gamma$ is Lipschitz.

Moreover, for every interior $t \in \text{dom}(\gamma)$, letting $t' \rightarrow t$ through $\text{dom}(\gamma)$ in (4.23) gives

$$(\phi \circ \gamma)'(t) = 0$$

at almost every such t , since Lemma 4.9 gives

$$\lim_{s \downarrow 0} \frac{\Psi(s)^{1/2}}{s} = 0.$$

Let $\{(a_i, b_i) : i \in I\}$ denote the gaps of γ . Since $\phi \circ \gamma$ is Lipschitz and its derivative vanishes almost everywhere,

$$|\phi(\gamma(\max(\text{dom}(\gamma)))) - \phi(\gamma(\min(\text{dom}(\gamma))))| \leq \sum_{i \in I} |\phi(\gamma(b_i)) - \phi(\gamma(a_i))|. \quad (4.25)$$

If $a_i = \min(\text{dom}(\gamma))$, then

$$d(\gamma(a_i), \gamma(b_i)) \leq \text{gap}(\gamma) \leq \delta r < r/4.$$

Since $\gamma(a_i) \in B(p, r)$, both endpoints of this gap lie in $B(p, 5r/4)$, and hence

$$\phi(\gamma(a_i)) = \phi(\gamma(b_i)) = 1.$$

Similarly, if $b_i = \max(\text{dom}(\gamma))$, then

$$d(p, \gamma(a_i)) \geq 2r - \delta r > \frac{3}{2}r,$$

so

$$\phi(\gamma(a_i)) = \phi(\gamma(b_i)) = 0.$$

Thus gaps incident to the endpoints contribute nothing to (4.25).

For every remaining gap, both $\gamma(a_i)$ and $\gamma(b_i)$ avoid E , and

$$s_i := d(\gamma(a_i), \gamma(b_i)) \leq \text{gap}(\gamma) \leq \delta r.$$

Hence, by (4.23) and (4.24),

$$|\phi(\gamma(b_i)) - \phi(\gamma(a_i))| \leq C_{PI}(C_\Psi \Lambda)^{1/2} \frac{s_i}{r}.$$

Summing over the gaps and using

$$\sum_{i \in I} s_i = \text{gap}(\gamma) \leq \delta r,$$

we obtain

$$\begin{aligned} |\phi(\gamma(\max(\text{dom}(\gamma)))) - \phi(\gamma(\min(\text{dom}(\gamma))))| &\leq C_{PI}(C_\Psi \Lambda)^{1/2} \delta \\ &= C_{PI} (2C_U C_D D^3 C_\Psi \delta)^{1/2} \\ &\leq \frac{1}{2}, \end{aligned}$$

where the last inequality follows from (4.21). On the other hand, $\phi(\gamma(\min(\text{dom}(\gamma)))) = 1$ and $\phi(\gamma(\max(\text{dom}(\gamma)))) = 0$, so the left hand side above equals 1, a contradiction. Therefore $g_f = 0$ m -almost everywhere for every $f \in \text{LIP}(X)$. $\square$

The proof of Theorem 4.1 is based on the vanishing of minimal $*$ -upper gradients obtained from the above contradiction argument. We note, that roughly dual to the notion of minimal $*$ -upper gradients is the notion of Alberti representations from [8], and a proof of the following theorem could likely be obtained by using the blow-up tools of [19] with the characterization of differentiability spaces using Alberti representations in [8]. This duality is made precise in [9]. This alternative argument seems to take considerably more effort than the proof presented in this section, and is not pursued further.

Proof of Theorem 4.1. Suppose, to the contrary, that (X, d, m) is a Lipschitz differentiability space. By [9, Theorem 1.7], for every $f \in \text{LIP}(X)$ the minimal $*$ -upper gradient satisfies

$$g_f = \text{Lip } f \quad m\text{-almost everywhere,}$$

where $\text{Lip } f$ denotes the pointwise upper Lipschitz constant of f . By Theorem 4.10, $g_f = 0$ m -almost everywhere for every $f \in \text{LIP}(X)$. Hence

$$\text{Lip } f = 0 \quad m\text{-almost everywhere}$$

for every $f \in \text{LIP}(X)$.

Let (U, φ) be a Lipschitz differentiability chart of dimension $n \geq 1$, where

$$\varphi = (\varphi^1, \dots, \varphi^n).$$

Fix $j \in \{1, \dots, n\}$. Since

$$\varphi^j(y) - \varphi^j(x) = e_j^*(\varphi(y) - \varphi(x)),$$

the coordinate functional e_j^* is a differential of φ^j at every $x \in U$. On the other hand, $\text{Lip } \varphi^j = 0$ for m -almost every $x \in U$, and hence

$$|\varphi^j(y) - \varphi^j(x)| = o(d(x, y))$$

at almost every such x . Thus the zero linear map is also a differential of φ^j at x . Since $e_j^* \neq 0$, this contradicts the uniqueness of the differential. Therefore (X, d, m) is not a Lipschitz differentiability space. $\square$

Remark 4.11. Theorems 4.1 and 4.10 extend, with minor modifications, to the nonlinear setting of p -strongly local regular energy spaces introduced in [25]. We conjecture that the conclusions of Theorems 4.10 and 4.1 remain valid under the weaker assumption of Ψ given in Lemma 4.9; that is,

$$\lim_{r \downarrow 0} \frac{\Psi(r)}{r^2} = 0.$$

The assumption on the scale function Ψ in Theorem 4.1 also plays an important role in the proof of the singularity of energy measures in [36]. There is a similar conjecture to prove the singularity of energy measure under the slightly weaker assumption on Ψ [36, Conjecture 2.15]. We refer to [36, Remark 2.14] for examples of scale functions that satisfy the weaker condition on Ψ but not the stronger one assumed in the Theorem 4.1.

References

- [1] Ambrosio, L., Gigli, N. and Savaré, G. Gradient flows in metric spaces and in the space of probability measures. 2nd ed., Lectures in Math. ETH Zürich, Birkhäuser Verlag, Basel, 2008.
- [2] Anttila, R., Eriksson-Bique, S., Shimizu, R. Construction of self-similar energy forms and singularity of Sobolev spaces on Laakso-type fractal spaces, (preprint) arXiv:2503.13258 (2026).
- [3] Aronson, D. G. Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.* **73** 1967 890–896.
- [4] Barlow, M. T. Diffusions on fractals, *Lecture Notes in Math.* **1690**, 1–121, Springer, Berlin, 1998.
- [5] Barlow, M. T. and Bass, R. F. Brownian motion and harmonic analysis on Sierpiński carpets, *Canad. J. Math.* **51** (1999), no. 4, 673–744.
- [6] Barlow, M. T., Grigor’yan, A. and Kumagai, T. On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan* **64** (2012), no. 4, pp. 1091–1146.
- [7] Barlow, M. T. and Perkins, E. A. Brownian motion on the Sierpiński gasket. *Probab. Theory Related Fields* **79** (1988), no. 4, pp. 543–623.
- [8] Bate, D. Structure of measures in Lipschitz differentiability spaces. *J. Amer. Math. Soc.* **28**, 2 (2015), pp. 421–482.
- [9] Bate, D., Eriksson-Bique, S. and Soultanis, E. Fragment-wise differentiable structures. *Analysis and Geometry in Metric Spaces* (to appear) (2026).
- [10] D. Bate and S. Li Differentiability and Poincaré-type inequalities in metric measure spaces *Adv. Math.* **333** (2018), 868–930.
- [11] Bate, D. and Speight, G. Differentiability, porosity and doubling in metric measure spaces. *Proc. Amer. Math. Soc.* **141** (2013), no. 3, pp. 971–985.
- [12] Bishop, C.J., Hakobyan, H., and Williams, M. Quasisymmetric dimension distortion of Ahlfors regular subsets of a metric space *Geom. Funct. Anal.* **26** (2016) nr. 2, pp. 379–421.
- [13] Bouleau, N. Décomposition de l’énergie par niveau de potentiel. In *Théorie du potentiel* (Orsay, 1983), pp. 149–172. Lecture Notes in Math., **1096**, Springer-Verlag, Berlin, 1984.
- [14] Bouleau, N. and Hirsch, F. Formes de dirichlet générales et densité des variables aléatoires réelles sur l’espace de Wiener. *J. Funct. Anal.* **69** (1986), no. 2, pp. 229–259.
- [15] Bouleau, N. and Hirsch, F. Propriétés d’absolue continuité dans les espaces de Dirichlet et application aux équations différentielles stochastiques. *Séminaire de Probabilités XX*, 1984/85, pp. 131–161. Lecture Notes in Math., **1204**, Springer-Verlag, Berlin, 1986.
- [16] Bourdon, M. *Au bord de certains polyèdres hyperboliques*. Ann. Inst. Fourier, **45** (1995), 1, pp. 119–141.

- [17] Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. New York, NY: Springer xiii, 599 p. (2011).
- [18] Cheeger, J. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.* **9**, 3 (1999), pp. 428–517.
- [19] Cheeger, J., Kleiner, B. and Schioppa, A. Infinitesimal structure of differentiability spaces, and metric differentiation. *Anal. Geom. Metr. Spaces* **4** (2016), p. 104–159.
- [20] Davis, M. H. A. and Varaiya, P. The multiplicity of an increasing family of σ -fields. *Ann. Probab.* **2** (1974), pp. 958–963.
- [21] De Philippis, G., Marchese, A., and Rindler, F. On a conjecture of Cheeger. *Partial Differ. Equ. Meas. Theory* De Gruyter Open, Warsaw, 2017, pp. 145–155.
- [22] De Philippis, G. and Rindler, F. On the structure of $\mathcal{A}$ -free measures and applications. *Ann. of Math.* (2) **184** (2016), no. 3, pp. 1017–1039.
- [23] Eriksson-Bique, S. Characterizing spaces satisfying Poincaré inequalities and applications to differentiability *Geometric and Functional Analysis* **29.1** (2019), pp. 119–189.
- [24] Eriksson-Bique, S. On the resistance conjecture, arXiv:2602.05477, preprint (2026).
- [25] Eriksson-Bique, S. Murugan, M. On the energy image density conjecture of Bouleau and Hirsch, arXiv:2510.13659, preprint (2025).
- [26] Grigor'yan, A. The heat equation on noncompact Riemannian manifolds. (in Russian) *Matem. Sbornik*. **182** (1991), 55–87. (English transl.) *Math. USSR Sbornik* **72** (1992), 47–77.
- [27] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688.
- [28] Heinonen, J. *Lectures on Analysis on Metric Spaces*, Universitext. Springer-Verlag, New York, 2001. x+140 pp.
- [29] Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J. T. *Sobolev spaces on metric measure spaces. An approach based on upper gradients*. New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015.
- [30] Hino, M. Energy measures and indices of Dirichlet forms, with applications to derivatives on some fractals. *Proc. Lond. Math. Soc.* (3) **100**, 1 (2010), pp. 269–302.
- [31] Hino, M. Upper estimate of martingale dimension for self-similar fractals. *Probab. Theory Related Fields* **156** (2013), no. 3–4, pp. 739–793.
- [32] Hino, M. Martingale dimensions for a class of metric measure spaces, arXiv:2511.09930, preprint (2025)
- [33] Hörmander, L. Hypoelliptic second order differential equations. *Acta Math.* **119** (1967), pp. 147–171.

- [34] Jerison, D. The Poincaré inequality for vector fields satisfying Hörmander’s condition. *Duke Math. J.* **53** (1986), no. 2, pp. 503–523.
- [35] Kajino, N. An elementary proof that walk dimension is greater than two for Brownian motion on Sierpiński carpets. *Bull. London Math. Soc.* **55** (2023), no. 1, pp. 508–521.
- [36] Kajino, N. and Murugan, M. On singularity of energy measures for symmetric diffusions with full off-diagonal heat kernel estimates. *Ann. Probab.* **48** (2020), no. 6, pp. 2920–2951.
- [37] Keith, S. A differentiable structure for metric measure spaces. *Adv. Math.* **183** (2004), no. 2, pp. 271–315.
- [38] Kell, M., Mondino, A. On the volume measure of non-smooth spaces with Ricci curvature bounded below. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **18** (2018), no. 2, 593–610.
- [39] Kleiner, B., Schioppa, A. PI spaces with analytic dimension 1 and arbitrary topological dimension. *Indiana Univ. Math. J.* **66** (2017), no. 2, 495–546.
- [40] Koskela, P., Zhou, Y. Geometry and analysis of Dirichlet forms. *Adv. Math.* **231** (2012), no. 5, pp. 2755–2801.
- [41] Kusuoka, S. Dirichlet forms on fractals and products of random matrices. *Publ. Res. Inst. Math. Sci.* **25** (1989), no. 4, pp. 659–680.
- [42] Kusuoka, S. Lecture on diffusion processes on nested fractals, *Statistical mechanics and fractals*, Lecture Notes in Mathematics 1567 (Springer, Berlin, 1993), pp. 39–98.
- [43] Laakso, T. J. Ahlfors Q -regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.* **10**(2000), no.1, 111–123.
- [44] Malicet, D. and Poly, G. Properties of convergence in Dirichlet structures. *J. Funct. Anal.* **264** (2013), no. 9, pp. 2077–2096.
- [45] Malliavin, P. Stochastic calculus of variation and hypoelliptic operators. In *Proc. Internat. Sympos. Stochastic Differential Equations*, Wiley-Intersci. Publ., John Wiley & Sons, New York-Chichester-Brisbane, 1978, pp. 195–263.
- [46] Mattila, P. Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability. Cambridge Stud. Adv. Math., **44**, Cambridge University Press, Cambridge, 1995. xii+343 pp.
- [47] Motoo, M. and Watanabe, S. On a class of additive functionals of Markov processes. *J. Math. Kyoto Univ.* **4** (1964/65), pp. 429–469.
- [48] Murugan, M. On the length of chains in a metric space *Journal of Functional Analysis* **279**(6) 108627 (2020).
- [49] Murugan, M. Martingale and analytic dimensions coincide under Gaussian heat kernel bounds. *Bull. Sci. Math.* **204** (2025), Paper No. 103710, 24 pp.
- [50] Murugan, M. Diffusions and random walks with prescribed sub-Gaussian heat kernel estimates, *Ann. Probab.* (to appear) 2026+

- [51] Nakao, S. Stochastic calculus for continuous additive functionals of zero energy. *Z. Wahrsch. Verw. Gebiete* **69** (1985), no. 4, pp. 557–578.
- [52] Pansu, P. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)* **129** (1989), no. 1, pp. 1–60.
- [53] Pansu, P. Dimension conforme et sphère à l’infini des variétés à courbure négative. *Ann. Acad. Sci. Fenn. Ser. A Math.*, **14** (1989), nr. 2, p. 177–212.
- [54] Saloff-Coste, L. A note on Poincaré, Sobolev, and Harnack inequalities. *Inter. Math. Res. Notices* **2** (1992), 27–38.
- [55] Saloff-Coste, L. The heat kernel and its estimates, Probabilistic approach to geometry, 405–436, *Adv. Stud. Pure Math.*, **57**, Math. Soc. Japan, Tokyo, 2010.
- [56] K.-T. Sturm, Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality, *J. Math. Pures Appl.* **75** (9) (1996) 273–297.
- [57] Tyson, J. Sets of minimal Hausdorff dimension for quasiconformal maps. *Proc. Am. Math. Soc.* **128** (2000), No. 11, pp. 3361–3367.

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