

# Martingale and Analytic dimensions (Lecture 1)

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# Overview

- ▶ **Lecture 1:** Introduce the notions of martingale and analytic dimensions with some illustrative examples.
- ▶ **Lecture 2:** Highlight some similarities and differences between these two dimensions. Relation to other notions of dimensions such as Hausdorff and spectral dimensions.
- ▶ **Lecture 3:** The energy image density property and its role in obtaining estimates on martingale and analytic dimensions (joint work with Sylvester Eriksson-Bique).
- ▶ A number of fundamental questions concerning these dimensions remain open and will be surveyed throughout the course.

## Dirichlet forms

Let  $(\mathcal{E}, \mathcal{F})$  be a **Dirichlet form** on  $L^2(X, m)$ .

- ▶  $\mathcal{F}$  is a dense linear subspace of  $L^2(X, m)$ .
- ▶  $\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$  is a non-negative definite ( $\mathcal{E}(f, f) \geq 0$  for all  $f \in \mathcal{F}$ ), symmetric ( $\mathcal{E}(f, g) = \mathcal{E}(g, f)$  for all  $f, g \in \mathcal{F}$ ), bilinear form.
- ▶ **Closed:**  $(\mathcal{E}, \mathcal{F})$  is a closed form; that is,  $\mathcal{F}$  is a Hilbert space under the inner product  $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(X, m)}$ .
- ▶ **Markovian:**  $(\mathcal{E}, \mathcal{F})$  is a Markovian form; that is,  $f^+ \wedge 1 \in \mathcal{F}$  and  $\mathcal{E}(f^+ \wedge 1, f^+ \wedge 1) \leq \mathcal{E}(f, f)$  for any  $f \in \mathcal{F}$ .

## Regular Dirichlet forms and strong locality

Let  $(X, d, m)$  be a locally compact, separable metric measure space and  $m$  is a Radon measure with full support.

Let  $(\mathcal{E}, \mathcal{F})$  be a **Dirichlet form** on  $L^2(X, m)$ .

1.  $(\mathcal{E}, \mathcal{F})$  is called **regular** if  $\mathcal{F} \cap C_c(X)$  is dense both in  $(\mathcal{F}, \mathcal{E}_1)$  and in  $(C_c(X), \|\cdot\|_{\text{sup}})$ .
2.  $(\mathcal{E}, \mathcal{F})$  is called **strongly local** if  $\mathcal{E}(f, g) = 0$  for any  $f, g \in \mathcal{F}$  with  $\text{supp}_m[f] = \text{supp}[f \cdot m]$ ,  $\text{supp}_m[g]$  compact and  $\text{supp}_m[f - a\mathbb{1}_X] \cap \text{supp}_m[g] = \emptyset$  for some  $a \in \mathbb{R}$ .

## Semigroup associated to a Dirichlet form

Let  $(\mathcal{E}, \mathcal{F})$  be a Dirichlet form on  $L^2(X, m)$ . A family of linear operators  $\{P_t : L^2(X, m) \rightarrow L^2(X, m) \mid t > 0\}$  called the *semigroup* associated to  $(\mathcal{E}, \mathcal{F})$  such that

- (1) **Symmetry:**  $\langle P_t(f), g \rangle = \langle f, P_t(g) \rangle$  for all  $t > 0$  and  $f, g \in L^2(X, m)$ .
- (2) **Semigroup property:**  $P_{t+s} = P_t P_s, \forall t, s > 0$ .
- (3) **Contraction:**  $\|P_t(u)\| \leq \|u\|$  for all  $t > 0, u \in L^2(X, m)$ .
- (4) **Strong continuity:**  $\lim_{t \downarrow 0} \|P_t(u) - u\| = 0, \forall u \in L^2(X, m)$ .
- (5) **Markovian:** If  $f \in L^2(X, m)$  such that  $0 \leq f \leq 1$   $m$ -almost everywhere, then we have  $0 \leq P_t(f) \leq 1$   $m$ -a.e. for all  $t > 0$ .
- (6) Relation to  $(\mathcal{E}, \mathcal{F})$ :

$$\mathcal{F} = \left\{ f \in L^2(X, m) \mid \lim_{t \downarrow 0} \frac{1}{t} \langle (I - P_t)f, f \rangle < \infty \right\},$$

and

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{t} \langle (I - P_t)f, f \rangle, \quad \text{for all } f \in \mathcal{F}.$$

## Example: Heat semigroup

The heat semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy, \quad t > 0, x \in \mathbb{R}^n, f \in L^2(\mathbb{R}^n)$$

corresponds the Dirichlet form  $(\mathcal{E}, \mathcal{F})$

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla g(x) dx, \quad \text{for all } f, g \in \mathcal{F} = W^{1,2}(\mathbb{R}^n).$$

Equivalently,  $P_t$  can be described in terms of the standard Brownian motion  $(B_t)_{t \geq 0}$  on  $\mathbb{R}^n$  as

$$P_t f(x) = \mathbb{E}_x[f(B_{2t})].$$

## Brownian motion on Heisenberg group and Lévy area

Let  $(B_t^{(1)}, B_t^{(2)})$  denote the standard 2-D Brownian motion. We consider the  $\mathbb{R}^3$ -valued process

$$X_t = (B_t^{(1)}, B_t^{(2)}, \frac{1}{2} \int_0^t (B_s^{(1)} dB_s^{(2)} - B_s^{(2)} dB_s^{(1)})).$$

The third component is called the **Lévy stochastic area** (1951).

If  $\alpha = \frac{1}{2}(x dy - y dx)$ , then  $d\alpha = dx \wedge dy$ , so by Stokes' theorem  $\int_\gamma \frac{1}{2}(x dy - y dx)$  is the signed area swept by a smooth curve  $\gamma$ .

By Itô's formula, for any  $f \in C_c^\infty(\mathbb{R}^3)$ , we have  $u(X_t) - Lu(X_t)$  is martingale, where

$$L = \frac{1}{2}(V_1^2 + V_2^2), \quad V_1 = \partial_x - \frac{y}{2}\partial_z, \quad V_2 = \partial_y + \frac{x}{2}\partial_z.$$

The Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(\mathbb{R}^3)$  corresponding to the semigroup  $P_t f(x) = \mathbb{E}_x[f(X_t)]$  is given by the smallest closed extension of

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^3} (V_1 f \cdot V_1 g + V_2 f \cdot V_2 g) d\mathcal{L}_3, \quad \text{for all } f, g \in C_c^\infty(\mathbb{R}^3).$$

## Brownian motion on the Heisenberg group

By considering concatenation of curves (see picture), one is led to the group operation of the Heisenberg group:  $\mathbb{R}^3$  equipped with the operation

$$(x_1, y_1, z_1) \odot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

The process  $X_t$  has continuous paths with stationary, independent increments.

**Stationary:** For all  $0 \leq s < t$ , the distribution of the increment  $X_s^{-1} \odot X_t$  between times  $s, t$  is same as that of  $X_{t-s}$ .

**Independence:** For all  $0 < t_1 < t_2 \dots < t_n$ , the increments  $X_{t_1}^{-1} \odot X_{t_2}, X_{t_2}^{-1} \odot X_{t_3}, \dots, X_{t_{n-1}}^{-1} \odot X_{t_n}$  are independent.

## Energy measure

The **energy measure**  $\Gamma(f, f)$  of  $f \in \mathcal{F}$  associated with  $(X, d, m, \mathcal{E}, \mathcal{F})$  is defined, first for  $f \in \mathcal{F} \cap L^\infty(X, m)$  as the unique  $([0, \infty]$ -valued) Borel measure on  $X$  such that

$$\int_X g \, d\Gamma(f, f) = \mathcal{E}(f, fg) - \frac{1}{2}\mathcal{E}(f^2, g) \quad \text{for all } g \in \mathcal{F} \cap C_c(X), \quad (1)$$

and then by

$\Gamma(f, f)(A) := \lim_{n \rightarrow \infty} \Gamma((-n) \vee (f \wedge n), (-n) \vee (f \wedge n))(A)$  for each Borel subset  $A$  of  $X$  for general  $f \in \mathcal{F}$ . The signed measure  $\Gamma(f, g)$  for  $f, g \in \mathcal{F}$  is defined by polarization.

**Heat semigroup example:** If  $\mathcal{E}(f, g) = \int \nabla f \cdot \nabla g \, dx$  for all  $f, g \in W^{1,2}(\mathbb{R}^n)$ , then for all  $f, g \in W^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ , we have

$$\mathcal{E}(f, fg) - \frac{1}{2}\mathcal{E}(f^2, g) = \int (\nabla f \cdot \nabla(fg) - \frac{1}{2}\nabla(f^2) \cdot \nabla g) \, dx = \int g |\nabla f|^2 \, dx.$$

So  $\Gamma(f, g)(dx) = \nabla f(x) \cdot \nabla g(x) \, dx$ .

## Minimal energy dominant measure

Let  $(\mathcal{E}, \mathcal{F})$  be a strongly local, regular, Dirichlet form on  $L^2(X, m)$ .

We say  $\nu$  is a **minimal energy dominant measure**, that is  $\nu$  satisfies the following

- (a) (Energy dominance)  $\Gamma(f, f) \ll \nu$  for every  $f \in \mathcal{F}$ , where  $\Gamma(f, f)$  is the energy measure of  $f$ .
- (b) (Minimality) If  $\tilde{\nu}$  satisfies (a), then  $\nu \ll \tilde{\nu}$ .

Remark: any two minimal energy dominant measures are mutually absolutely continuous.

**Existence Theorem:**(Hino '10, Nakao '85) Every strongly local, regular, Dirichlet form has a minimal energy dominant measure.

## Martingale dimension

Let  $(\mathcal{E}, \mathcal{F})$  be a strongly local, regular, Dirichlet form on  $L^2(X, m)$ .

Let  $\Gamma(\cdot, \cdot)$  denote the corresponding energy measure and let  $\nu$  be a minimal energy dominant measure.

(i) (Hino '10) The **pointwise index** is a measurable function  $\rho_H : X \rightarrow \mathbb{N} \cup \{0, \infty\}$  such that the following hold:

(a) For any  $N \in \mathbb{N}$ ,  $f_1, \dots, f_N \in \mathcal{F}$ , we have

$$\text{rank} \left( \frac{d\Gamma(f_i, f_j)}{d\nu}(x) \right)_{1 \leq i, j \leq N} \leq \rho_H(x) \quad \text{for } \nu\text{-almost every } x \in X.$$

(b) For any other function  $\rho'_H : X \rightarrow \mathbb{N} \cup \{0, \infty\}$  that satisfies (a) with  $\rho'_H$  instead of  $\rho_H$ , then  $\rho_H(x) \leq \rho'_H(x)$  for  $\nu$ -almost every  $x \in X$ .

(ii) The **martingale dimension** of the strongly local regular Dirichlet form  $(\mathcal{E}, \mathcal{F})$  is defined as  $\nu\text{-esssup}_{x \in X} \rho_H(x)$ , where  $\rho_H$  is a pointwise index.

## Martingale dimension: Historical remarks

This concept was originally developed by Motoo and Watanabe ('64) for **martingale additive functionals** corresponding to a Markov process: roughly speaking, the martingale dimension of a Markov process is the minimal number  $k$  such that every martingale additive functional satisfying suitable integrability conditions can be expressed as a sum of  $k$  stochastic integrals with respect to a chosen set of  $k$  martingale additive functionals.

The equivalence between the probabilistic definition and the analytic definition based on pointwise index is due to Hino ('10). This builds upon Kusuoka's works on martingale dimension of p.c.f. self-similar fractals ('89, '93).

Later generalized to a more abstract setting of filtered probability spaces by Davis and Varaiya ('74) under the name **multiplicity of a filtration**.

## Example: Martingale dimension of heat semigroup

Consider  $(\mathcal{E}, \mathcal{F})$  on  $L^2(\mathbb{R}^n, \mathcal{L}_n)$  given by

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla g(x) \mathcal{L}_n(dx), \quad \text{for all } f, g \in \mathcal{F} = W^{1,2}(\mathbb{R}^n).$$

Then  $\nu = \mathcal{L}_n$  is a minimal energy dominant measure.

Suppose  $f_1, \dots, f_N \in C^1(\mathbb{R}^n) \cap \mathcal{F}$ , so that the gradients  $\nabla f_i$  are pointwise well-defined in the classical sense.

Then

$$\begin{aligned} \text{rank} \left( \frac{d\Gamma(f_i, f_j)}{d\nu}(x) \right)_{1 \leq i, j \leq N} &= \text{rank} (\nabla f_i(x) \cdot \nabla f_j(x))_{1 \leq i, j \leq N} \\ &= \dim (\text{span} \{ \nabla f_i(x) : 1 \leq i \leq N \}) \leq n. \end{aligned}$$

So  $p_H(x)$  is analogous to the dimension of the tangent space at  $x$ .

In this case,  $p_H \equiv n$  and martingale dimension is  $n$ .

## Martingale dimension for BM on Heisenberg group

Consider the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(\mathbb{R}^3)$  given by the smallest closed extension of

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^3} (V_1 f \cdot V_1 g + V_2 f \cdot V_2 g) d\mathcal{L}_3, \quad \text{for all } f, g \in C_c^1(\mathbb{R}^3),$$

where

$$V_1 = \partial_x - \frac{y}{2} \partial_z, \quad V_2 = \partial_y + \frac{x}{2} \partial_z.$$

$\nu = \mathcal{L}_3$  is a minimal energy dominant measure.

For any  $f_1, \dots, f_N \in C_c^1(\mathbb{R}^3) \cap \mathcal{F}$ , we have

$$\begin{aligned} \text{rank} \left( \frac{d\Gamma(f_i, f_j)}{d\nu}(x) \right)_{1 \leq i, j \leq N} &= \text{rank} (V_1 f_i(x) V_1 f_j(x) + V_2 f_i(x) V_2 f_j(x))_{i, j} \\ &= \dim \text{span} \{ (V_1 f_i)_{i=1}^N, (V_2 f_i)_{i=1}^N \} \leq 2. \end{aligned}$$

In this case  $p_H \equiv 2$  and so the martingale dimension is 2.

## Radamacher's theorem (1999)

Every Lipschitz function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable  $\mathcal{L}_n$ -almost everywhere, where  $\mathcal{L}_n$  denotes the Lebesgue measure on  $\mathbb{R}^n$ .

## Differentiability in metric spaces: Cheeger (1999)

**Differentiability in a metric space  $(X, d)$ :** Suppose  $f : X \rightarrow \mathbb{R}$  and  $\phi = (\phi_1, \dots, \phi_N) : X \rightarrow \mathbb{R}^N$  are Lipschitz functions on a metric space  $(X, d)$ . Then  $f$  is **differentiable with respect to  $\phi$  at  $x_0 \in X$**  if there is a unique  $a = (a_1, \dots, a_N) \in \mathbb{R}^N$  such that  $f$  and the linear combination  $a \cdot \phi = \sum_{i=1}^N a_i \phi_i$  agree to first order near  $x_0$ :

$$\limsup_{x \rightarrow x_0} \frac{|f(x) - f(x_0) - a \cdot (\phi(x) - \phi(x_0))|}{d(x, x_0)} = 0.$$

The tuple  $a \in \mathbb{R}^N$  is the *derivative of  $f$  with respect to  $\phi$*  and will be denoted by  $\partial_\phi f(x_0)$ .

If  $X = \mathbb{R}^n$ ,  $d =$  Euclidean distance, and  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the identity map, then the above notion coincides with the classical notion of differentiability and  $a = \partial_\phi f(x_0) = \nabla f(x_0)$ .

## Charts in a metric space

A *chart of dimension  $N$*  on a metric measure space  $(X, d, \mu)$  is a pair  $(U, \phi)$  where:

- (i)  $U \subset X$  is measurable and  $\phi : X \rightarrow \mathbb{R}^N$  is Lipschitz.
- (ii) Every Lipschitz function  $f : X \rightarrow \mathbb{R}$  is differentiable with respect to  $\phi$  at  $\mu$ -almost every  $x_0 \in U$  and the derivative defines a measurable function  $\partial_\phi f : U \rightarrow \mathbb{R}^N$ .

A **measurable differentiable structure** on  $(X, d, \mu)$  is a countable collection  $\{(U_\alpha, \phi_\alpha)\}$  of charts with uniformly bounded dimension such that  $X = \bigcup_\alpha U_\alpha$  and  $\mu(U_\alpha) > 0$  for all  $\alpha$ .

A metric measure space that admits a measurable differentiable structure is called a **Lipschitz differentiability space**.

This turns Rademacher's theorem into a definition.

## PI spaces: Heinonen and Koskela (1998)

$(X, d, \mu)$  satisfies **(1,  $p$ )-Poincaré inequality** if there exist  $C \in (0, \infty), \Lambda \in [1, \infty)$  such that for any compactly supported Lipschitz function  $f : X \rightarrow \mathbb{R}$ , and for all  $x \in X, r > 0$ , we have

$$\int_{B(x,r)} |f - f_{B(x,r)}| d\mu \leq Cr \left( \int_{B(x,\Lambda r)} \text{Lip}(f)^p d\mu \right)^{1/p},$$

where

$$\text{Lip}(f)(y) = \limsup_{\substack{z \rightarrow y, \\ z \neq y}} \frac{|f(y) - f(z)|}{d(y,z)}, \quad f_{B(x,r)} := \mu(B(x,r))^{-1} \int_{B(x,r)} f d\mu.$$

$(X, d, \mu)$  is a  **$p$ -PI space**, if it satisfies the **(1,  $p$ )-Poincaré inequality** and the **volume doubling property**: there exists  $C > 0$  such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \quad \text{for all } x \in X, r > 0.$$

**Theorem** (Laakso) (2000): For any  $Q \in [1, \infty)$ , there is a 1-PI space of Hausdorff dimension  $Q$ .

## Heisenberg group with Carnot–Carathéodory metric

Recall the Heisenberg group  $\mathbb{H} = \mathbb{R}^3$  with group operation

$$(x_1, y_1, z_1) \odot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

There is a natural distance  $d_{CC} : \mathbb{H} \times \mathbb{H} \rightarrow [0, \infty)$  on  $\mathbb{H}$  called the **Carnot–Carathéodory metric** is defined by the following properties:

Left-invariant:  $d_{CC}(g_1, g_2) = d_{CC}(h \odot g_1, h \odot g_2)$  for all  $h, g_1, g_2 \in \mathbb{H}$ .

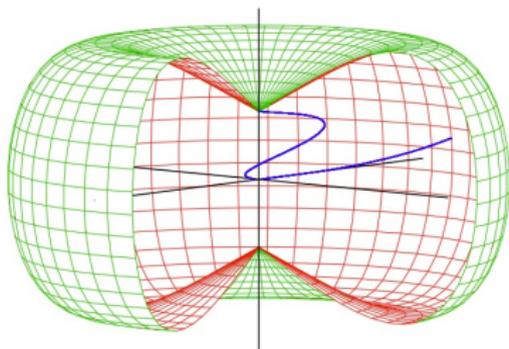
The Carnot–Carathéodory distance  $d_{CC}(0, (p, q, r))$  is the infimum of

$$\int_0^T \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt$$

over all absolutely continuous (or piecewise  $C^1$ ) curves  $\gamma(t) = (x(t), y(t))$  in  $\mathbb{R}^2$  with  $\gamma(0) = (0, 0)$ ,  $\gamma(T) = (p, q)$ , whose signed area satisfies

$$\frac{1}{2} \int_0^T (x(t)\dot{y}(t) - y(t)\dot{x}(t)) dt = r.$$

## Geometry of Heisenberg group



**Figure:** The unit ball of  $\mathbb{H}$  with respect to  $d_{CC}$ . A section has been removed to show geodesics (shortest paths, blue) from the origin. (Image by Nate Eldredge)

$(\mathbb{H}, d_{CC}, \mathcal{L}_3)$  is a PI space (Jersion 1986).

As a topological space  $(\mathbb{H}, d_{CC})$  is homeomorphic to  $\mathbb{R}^3$ . It has **Hausdorff dimension four** and has a **two dimensional analytic structure** in the sense that  $\phi : \mathbb{H} \rightarrow \mathbb{R}^2$  given by  $\phi(x, y, z) = (x, y)$  is a (global) chart on  $\mathbb{H}$  defining a measurable differentiable structure (Pansu 1989).

## Cheeger's theorem, Analytic dimension

**Cheeger's theorem** (1999): Let  $(X, d, \mu)$  be a  $p$ -PI space for some  $p \in [1, \infty)$ . Then there exists a measurable differentiable structure with the dimension of the charts uniformly bounded by a constant that depends only on the constants involved in the volume doubling property and  $(1, p)$ -Poincaré inequality.

Furthermore, there is a measurable function  $d_C : X \rightarrow \mathbb{N}$  such that  $d_C(x) = N(\alpha)$  for all  $\alpha \in I$ ,  $m$ -almost every  $x \in U_\alpha$ , where  $N(\alpha)$  is the dimension of the chart  $(U_\alpha, \phi_\alpha)$ . Furthermore, up to sets of  $m$ -measure zero this function  $d_C$  does not depend on the choice of the measurable differentiable structure.

**Analytic dimension** (or Lipschitz differentiability dimension) of a PI space (or more generally, a Lipschitz differentiability space)  $(X, d, \mu)$  is defined to be the  $\mu$ -esssup  $d_C$ . Equivalently, it is the smallest natural number  $N$  such that every chart has dimension at most  $N$ .

## Analytic dimension: Examples

Consider the metric measure space  $(\mathbb{R}^n, d, \mathcal{L}_n)$  where  $d$  is the Euclidean metric. Let  $U \subset \mathbb{R}^n$  be a measurable subset of positive measure.

Then a Lipschitz function  $\phi : U \rightarrow \mathbb{R}^N$  is a chart if and only if  $N = n$  and the differential  $D\phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}^N$  (well-defined for almost every  $x \in \mathbb{R}^n$  due to Radamacher's theorem) is a bijection at  $\mathcal{L}_n$ -a.e.  $x \in U$ . This shows that  $d_C \equiv n$  and the analytic dimension is  $n$ .

The Heisenberg group  $(\mathbb{H}, d_{CC}, \mathcal{L}_3)$  satisfies  $d_C \equiv 2$  and  $\phi : \mathbb{H} \rightarrow \mathbb{R}^3$  given by  $\phi(x, y, z) = (x, y)$  is a chart on  $\mathbb{H}$ .

Laakso's (2000) PI spaces have analytic dimension one and arbitrary Hausdorff dimension  $Q \in [1, \infty)$ .

Kleiner and Schioppa (2017) construct PI spaces with analytic dimension one and arbitrary topological dimension  $N \in \mathbb{N}$ .

## Open problems.

We do not have a good understanding of PI spaces even on  $\mathbb{R}^n$  (viewed as a topological space). The following questions are motivated by the general problem of classifying measurable differentiable structures on  $\mathbb{R}^n$  (topological space).

**Question** (Kleiner, Schioppa '17): Let  $n \geq 2$ . Is there a PI space  $(\mathbb{R}^n, d, \mu)$  such that  $d$  is compatible with the Euclidean topology with analytic dimension 1? (for instance,  $n = 2$  or 3).

**Question** (Kleiner, Schioppa '17): Let  $n \geq 2$  and  $Q \geq n$ . Is there a PI space  $(\mathbb{R}^n, d, \mu)$  such that  $d$  is compatible with the Euclidean topology and is Ahlfors  $Q$ -regular; that is,  $m(B(x, r)) \asymp r^Q$  for all  $x \in \mathbb{R}^n, r > 0$ ?

**Question:** Let  $n \geq 2$ . Is there a metric measure space  $(\mathbb{R}^n, d, \mu)$  and a strongly local, regular Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(\mathbb{R}^n, \mu)$  that satisfies Gaussian heat kernel estimates with martingale dimension one such that  $(\mathbb{R}^n, d)$  is homeomorphic to  $\mathbb{R}^n$  with the Euclidean metric?

Thank you for your attention!

Slides available at <https://personal.math.ubc.ca/~mathav/msj/>