Martingale and Analytic dimensions (Lecture 2)

Mathav Murugan

The University of British Columbia

The 18th MSJ-SI:
Analysis, Geometry and Probability on Metric Measure Spaces
February 2026, Fukuoka, Japan

Summary of Lecture 1: martingale and analytic dimensions

Martingale dimension Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(X, m)$ with a minimal energy dominant measure μ. Then the martingale dimension is the smallest number $N \in \mathbb{N} \cup \{0, \infty\}$ such that for any $k \in \mathbb{N}$, and for any $f_1, \dots, f_k \in \mathcal{F}$, we have

$$\operatorname{rank} \left(\frac{d\Gamma(f_i, f_j)}{d\nu}(x) \right)_{1 \le i,j \le k} \le N, \quad \text{for } \mu\text{-a.e. } x \in X.$$

Analytic dimension Let (X, d, m) be a metric measure space that admits a measurable differentiable structure. Then the analytic dimension is the smallest integer N such that the dimension of each chart is at most N.

Pointwise versions: Hino's pointwise index p_H (defined μ-a.e.) and Cheeger's pointwise dimension of charts d_C (defined m-a.e.).

Examples: Brownian motion on $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}$.

Outline

Equality between martingale and analytic dimensions under Gaussian heat kernel estimate

Estimates on analytic dimension

Estimates on martingale dimension under sub-Gaussian heat kernel estimates

Heat kernel associated with a Dirichlet form

Let $\{P_t : t > 0\}$ denote a Markov semigroup associated to a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$.

This means that $(\mathcal{E}, \mathcal{F})$ can be expressed in terms of the semigroup as

$$\mathcal{F} = \left\{ f \in L^2(X, m) | \lim_{t \downarrow 0} \frac{1}{t} \langle (I - P_t) f, f \rangle < \infty \right\},$$

and

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{t} \langle (I - P_t) f, f \rangle, \quad \text{for all } f \in \mathcal{F}.$$

The **heat kernel** associated with the Markov semigroup $\{P_t\}$ (if it exists) is a family of measurable functions $p_t(\cdot, \cdot) : X \times X \mapsto [0, \infty)$ for every $t > 0$, such that

$$P_t f(x) = \int p_t(x, y) f(y)\, m(dy), \quad \forall f \in L^2, t > 0,\ m\text{-a.e. } x \in X,$$

$$p_t(x, y) = p_t(y, x), \quad \text{for all } x, y \in X \text{ and } t > 0,$$

$$p_{t+s}(x, y) = \int p_s(x, y) p_t(y, z)\, m(dy), \quad \text{for all } t, s > 0,\ x, y \in X.$$

Gaussian heat kernel bounds

Let (X, d, m) be a metric measure space. We say that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ satisfies **Gaussian heat kernel estimates**, if its heat kernel $\{p_t\}_{t>0}$ exists and there exist $C_1, c_1, C_2, c_2 \in (0, \infty)$ such that for each $t > 0$,

$$p_t(x, y) \geq \frac{c_2}{m\big(B(x, \sqrt{t})\big)} \exp\left(-C_2 \frac{d(x, y)^2}{t}\right),$$
$$p_t(x, y) \leq \frac{C_1}{m\big(B(x, \sqrt{t})\big)} \exp\left(-c_1 \frac{d(x, y)^2}{t}\right)$$

for m-a.e. $x, y \in X$.

Examples: Gaussian estimates

- Brownian motion on $\mathbb{R}^n$ and Heisenberg group $\mathbb{H}$.
- Brownian motion on Lie groups of polynomial growth (Varopoulos '80s).
- Brownian motion on manifolds with non-negative Ricci curvature (Li, Yau '86).
- Uniformly elliptic operators in $\mathbb{R}^n$ (Aronson '67).
- Weighted manifolds (Grigor'yan, Saloff-Coste '06).
- Laakso spaces.

Equality between martingale and analytic dimensions

Theorem Let (X, d, m) be a metric measure space with a strongly local, regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ that satisfies Gaussian heat kernel estimates. Then we have the following:

1. (Koskela–Zhou '12) Then m is a minimal energy dominant measure.
2. (Koskela–Zhou '12, Kajino–M., '20) The metric measure space (X, d, m) is a PI space and hence admits a measurable differetiable structure.
3. (M. '25) The Hino's pointwise index $p_H(\cdot)$ of the Dirichlet form agrees m-a.e. with the pointwise dimension $d_C(\cdot)$ of a measurable differentiable structure on (X, d, m); that is

$$p_H(x) = d_C(x) \quad \text{for } m\text{-almost every } x \in X.$$

In particular, the martingale dimension of $(\mathcal{E}, \mathcal{F})$ coincides with the analytic dimension of (X, d, m).

Ingredients behind the proof

- Characterization of Gaussian bounds (Grigor'yan, Saloff-Coste '92, Sturm '96)
- Comparision of Dirichlet energy to Lipschitz energy (Koskela, Zhou '12)
- Comparison of given metric with intrinsic metric (Kajino, M. '20).

Bi-Lipschitz equivalence to the intrinsic metric

Theorem (Kajino, M. '20): Let (X, d, m) be a metric measure space with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ that satisfies Gaussian heat kernel estimates.

Then the metric d is bi-Lipschitz equivalent to the **intrinsic metric** $d_{\mathrm{int}} : X \times X \to [0, \infty]$ by

$$d_{\mathrm{int}}(x, y) := \sup\bigl\{ f(x) - f(y) \bigm| f \in \mathcal{F}_{\mathrm{loc}} \cap C(X),\, \Gamma(f, f) \leq m \bigr\},$$

where

$$\mathcal{F}_{\mathrm{loc}} := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable} \\ \text{functions on } X \text{ such that } f\mathbb{1}_V = f^{\#}\mathbb{1}_V \ m\text{-a.e. for some} \\ f^{\#} \in \mathcal{F} \text{ for each relatively compact open subset } V \text{ of } X \end{array} \right\}$$

and the energy measure $\Gamma(f, f)$ of $f \in \mathcal{F}_{\mathrm{loc}}$ associated with $(\mathcal{E}, \mathcal{F})$ is defined as the unique Borel measure on X such that $\Gamma(f, f)(A) = \Gamma(f^{\#}, f^{\#})(A)$ for any relatively compact Borel subset A of X and any $V, f^{\#}$ as above with $A \subset V$.

Characterization of Gaussian heat kernel estimates

Theorem (Grigor'yan '91, Saloff-Coste '92, Sturm '96): Let (X, d, m) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X, m)$ with energy measure $\Gamma(\cdot, \cdot)$. Then the following are equivalent:

(a) The Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies **Gaussian heat kernel estimates**.

(b) The metric measure space satisfies the **volume doubling property**: there exists $C_D \in (1, \infty)$ such that

$$m(B(x, 2r)) \leq C_D m(B(x, r)), \quad \text{for all } x \in X, r > 0;$$

and the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the **Poincaré inequality**: there exist $C_P \in (0, \infty), A \in [1, \infty)$ such that

$$\inf_{\alpha \in \mathbb{R}} \int_{B(x,r)} |f - \alpha|^2 \, dm \leq C_P r^2 \int_{B(x,Ar)} \Gamma(f, f),$$

for all $x \in X, r > 0, f \in \mathcal{F}$.

Comparison of Dirichlet and Lipschitz energies

Theorem (Koskela, Zhou '12): Let (X, d, m) be a metric measure space with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ that satisfies Gaussian heat kernel estimates.

We have $\mathrm{Lip}(X) \cap C_c(X) \subset \mathcal{F}$, and $\mathrm{Lip}(X) \subset \mathcal{F}_{\mathrm{loc}}$.

Moreover, $\mathrm{Lip}(X) \cap C_c(X)$ is dense subspace of the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$.

Moreover, the metric measure space (X, d, m) supports a $(1, 2)$-Poincaré inequality and there exists $C \in [1, \infty)$ such that

$$C^{-1}\sqrt{\frac{d\Gamma(f,f)}{dm}(x)} \leq \mathrm{Lip}\, f(x) \leq C\sqrt{\frac{d\Gamma(f,f)}{dm}(x)}$$

for all $f \in \mathrm{Lip}(X)$, and for m-almost every $x \in X$.

Cheeger's conjecture and its resolution

Cheeger's theorem (1999): Let (X, d, μ) be a p-PI space for some $p \in [1, \infty)$. Then there exists a measurable differentiable structure with the dimension of the charts uniformly bounded by a constant that depends only on the constants involved in the volume doubling property and $(1, p)$-Poincaré inequality.

Cheeger's conjecture (1999): If (U, ϕ) is an n-dimensional chart, then $\mathcal{H}_n(\phi(U)) > 0$. In particular, the analytic dimension is bounded from above by the Hausdorff dimension of (X, d).

Proof of the last claim: Since ϕ is Lipschitz, we have $\dim_H(X, d) \geq \dim_H(U, d) \geq \dim_H(\phi(U))$. Since $\mathcal{H}_n(\phi(U)) > 0$, we have $\dim_H(\phi(U)) = n$ and hence the dimension of every chart is bounded from above by the Hausdorff dimension $\dim_H(X, d)$.

Theorem (de Philippis, Marchese, Rindler 2017): If (U, ϕ) is an n-dimensional chart, then $\phi_*(\mathbb{1}_U \cdot \mu) \ll \mathcal{L}_n$. In particular, Cheeger's conjecture holds.

Proof of the last claim: Since $\phi_*(\mathbb{1}_U \cdot \mu)$ is a non-zero measure on

Hino's estimate

Corollary (M. '25): Under Gaussian heat kernel bounds, the martingale dimension is bounded from above by the Hausdorff dimension.

Hino (2013) obtained a similar bound as above under more general assumptions on heat kernel estimate but more restrictive assumption on the underlying space (self-similar space) that is motivated by diffusion on fractals.

Hino's estimate states that the **martingale dimension** is bounded from above by the **spectral dimension** d_s; $d_m \leq d_s$.

Sub-Gaussian heat kernel bounds

Let (X, d, m) be a metric measure space and $\beta \in [2, \infty)$. We say that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ satisfies **sub-Gaussian heat kernel estimates** with **walk dimension** β, if its heat kernel $\{p_t\}_{t>0}$ exists and there exist $C_1, c_1, C_2, c_2 \in (0, \infty)$ such that for each $t > 0$,

$$p_t(x, y) \geq \frac{c_2}{m\big(B(x, t^{1/\beta})\big)} \exp\left(-C_2 \left(\frac{d(x, y)^{\beta}}{t}\right)^{1/(\beta-1)}\right),$$

$$p_t(x, y) \leq \frac{C_1}{m\big(B(x, t^{1/\beta})\big)} \exp\left(-c_1 \left(\frac{d(x, y)^{\beta}}{t}\right)^{1/(\beta-1)}\right)$$

for m-a.e. $x, y \in X$.

Remark: While these estimates can be defined for all $\beta \in (1, \infty)$, Hino ('05) showed that $\beta \geq 2$ must necessarily hold.

Volume growth exponent and spectral dimension

Suppose (X, d, m) be a metric measure space with a strongly local, regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ that satisfies sub-Gaussian heat kernel estimates with walk dimension β, where $\beta \in [2, \infty)$.

If m is α-Ahlfors regular; that is $m(B(x, r)) \asymp r^\alpha$ for all $x \in X, 0 < r < \mathrm{diam}(X, d)$, we say that α is the **volume growth exponent** of (X, d, m).

In this case, we have $\alpha = \dim_H(X, d)$ and define the **spectral dimension** as $d_s = 2\alpha/\beta$. If $\beta = 2$, $d_s = \dim_H(X, d)$.

Note that $p_t(x, x) \asymp t^{-d_s/2}$ for $t < \mathrm{diam}(X, d)^\beta$.

Example: BM on the Sierpiński gasket

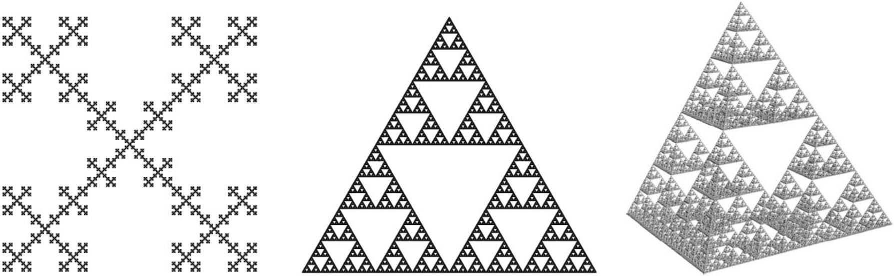

Figure: Vicsek set, N-dimensional Sierpiński gasket for $N = 2, 3$.

Barlow, Perkins '88: Sub-Gaussian heat kernel estimates for Brownian motion on Sierpiński gasket with walk dimension $\beta = \log 5/\log 2$. Note that $\alpha = \dim_H(X, d) = \log 3/\log 2$. $d_s = 2\log 3/\log 5 \in (1, 2)$.

Generalized Sierpiński carpets

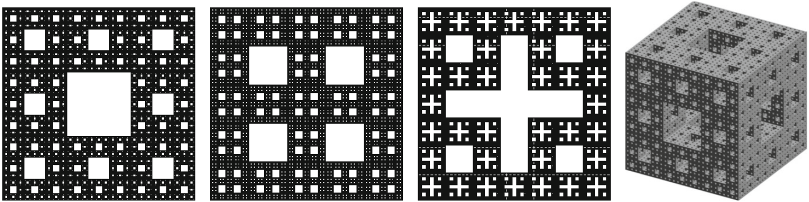

Figure: Sierpiński carpet and some generalized Sierpiński carpets

Barlow, Bass '89, '92,'99: Sub-Gaussian heat kernel estimates for Brownian motion on Sierpiński carpet and generalized Sierpiński carpets with walk dimension $\beta > 2$ (exact value of β unknown).

Lower bounds on martingale dimension

Recall that we defined the martingale dimension is an essential supremum of Hino's pointwise index with respect to a minimal energy dominant measure.

We have the trivial lower bound $d_m \geq 1$ under the assumption $\mathcal{E} \not\equiv 0$.

Despite this, obtaining non-trivial lower even on fractals is challenging. For instance, the following question is open:

Question: Is there a generalized Sierpiński carpet whose associated self-similar Dirichlet form has martingale dimension strictly larger than one?

Remark: The spectral dimension of this class of Dirichlet forms is unbounded and the walk dimension is always strictly larger than two.

A conjecture based on Hino's estimate

Hino ('13) obtained the estimate $d_m \leq d_s$ between martingale dimension d_m and spectral dimension d_s under additional assumption of self-similarity of the underlying space and of the Dirichlet form.

Conjecture: Let (X, d, m) be a metric measure space with volume growth exponent α and let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ that satisfies sub-Gaussian heat kernel estimates with and walk dimension β. Then the martingale dimension d_m satisfies $d_m \leq d_s = \frac{2\alpha}{\beta}$.

Since $\beta \geq 2$, we have the following weaker conjecture.

Conjecture: Let (X, d, m) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ that satisfies sub-Gaussian heat kernel estimates with and walk dimension β. Then the martingale dimension d_m satisfies $d_m \leq d_H(X, d)$.

Joint behavior of α, β, d_m

Theorem (M. '25): Let $\alpha \in [1, \infty), \beta \in (1, \infty)$. Then there exists a metric measure space (X, d, m) with volume growth α and a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ that satisfies sub-Gaussian heat kernel estimates with walk dimension β if and only if

$$2 \leq \beta \leq 1 + \alpha.$$

Remarks: For random walks of graphs a similar result is due to Barlow '04. The 'only if' part is well-known and follows by the same argument in Barlow's proof of a similar result for random walks on graphs.

The examples constructed to show the 'if part' all have martingale dimension one and is based on a variant of Laakso's construction. Hino's result for self-similar Dirichlet forms along with some examples suggest the following conjecture regarding the joint behavior of α, β, d_m.

Conjecture: There is a metric measure space with volume growth exponent α along with an associated Dirichlet form satisfying

Non-existence of measurable differentiable structure

The notion of analytic dimension typically fails to be useful for understanding martingale dimension of diffusion on fractals.

For example, the standard Sierpiński gasket and carpet with Euclidean metric and Hausdorff measure do not admit a measurable differentiable structure and hence are not PI spaces.

Such non-existence follows from a theorem of Kell and Mondino ('18) on the structure of Lipschitz differentiability spaces that are locally bi-Lipschitz embeddedable in Euclidean spaces.

Finiteness theorem for martingale dimension

Theorem (Eriksson-Bique, M. '25+): Suppose (X, d, m) be a metric measure space with a strongly local, regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ that satisfies sub-Gaussian heat kernel estimates with walk dimension β, then the martingale dimension of $(\mathcal{E}, \mathcal{F})$ is finite.

Summary: Bounds on martingale and analytic dimensions

	Analytic dimension	**Martingale dimension**
Finiteness	Cheeger '99	Eriksson–Bique, M. '25+
Sharper bound	Hausdorff dimension (De Philippis, Rindler, Marchese '17)	Spectral dimension (Hino '13, self-similarity)

These results with the exception of Cheeger's theorem rely on different variants of the energy image density property.
This will be the topic of the final lecture tomorrow.

Thank you for your attention!

Slides available at https://personal.math.ubc.ca/∼mathav/msj/