Martingale and Analytic dimensions (Lecture 3)

Mathav Murugan

The University of British Columbia

Based on joint work with Sylvester Eriksson-Bique (University of Jyväskylä).

The 18th MSJ-SI:
Analysis, Geometry and Probability on Metric Measure Spaces
February 2026, Fukuoka, Japan

Outline

- Hörmander's theorem on hypoelliptic operators and Malliavin's proof of Hörmander's theorem.
- Bouleau–Hirsch formulation of the energy image density conjecture (1986) and its solution.
- Application of energy image density property to finiteness of martingale dimension.

Hypoelliptic operators

A linear differential operator L with smooth coefficients is hypoelliptic if, for every open set $U \subset \mathbb{R}^n$,

$$Lu \in C^\infty(U) \text{ implies that } u \in C^\infty(U).$$

Examples: Laplace operator Δ (Weyl's lemma), and Heat operator $\partial_t - \Delta$ on $\mathbb{R}^{n+1}$ are hypoelliptic.
The wave operator $\partial_{tt} - \Delta$ is not hypoelliptic.
Kolmogorov operator (1934) is hypoelliptic. It is given by

$$f \mapsto \partial_{vv} f + v\partial_x f - \partial_t f,$$

where f is a function of time t, position x, and velocity v and corresponds to motion with random velocity

$$dX_t = V_t\, dt, \quad dV_t = \sqrt{2} dB_t,$$

where B_t is the standard Brownian motion on $\mathbb{R}$.

Hörmander's Theorem on Hypoellipticity

Consider a second-order operator of the form

$$L = \sum_{j=1}^{d} V_j^2 + V_0,$$

where $V_0, V_1, \ldots, V_d$ are smooth real vector fields on an open set $U \subset \mathbb{R}^n$.

Hörmander's Bracket Condition. The Lie algebra generated by $\{V_0, V_1, \ldots, V_d\}$ spans the tangent space at every point.

Theorem (Hörmander, 1967). If the bracket condition holds, then L is hypoelliptic.

Example. The Kolmogorov operator $-\partial_t + v\partial_x + \partial_{vv} = V_1^2 + V_0$ corresponds to

$$V_0 = -\partial_t + v\partial_x, V_1 = \partial_v, \quad [V_0, V_1] = -\partial_x.$$

is hypoelliptic by Hörmander's theorem.

Hörmander's Theorem (probabilistic form)

Consider the Stratonovich stochastic differential equation (SDE)

$$dX_t = \sqrt{2} \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i + V_0(X_t)\, dt,$$

where $V_0, \ldots, V_d$ are smooth vector fields on $\mathbb{R}^n$ and B_t^i are i.i.d. Brownian motions.

Hörmander's Bracket condition: The Lie algebra generated by the vector fields $\{\partial_t - V_0, V_1, \ldots, V_d\}$ spans the tangent space of $\mathbb{R}^{n+1}$.

Hörmander's Theorem (1967): If Hörmander's condition holds, then for all $t > 0$, X_t has a smooth density w.r.t. Lebesgue measure.

Brownian motion on Heisenberg group and Lévy area

Let $(B_t^{(1)}, B_t^{(2)})$ denote the standard 2-D Brownian motion. We consider the $\mathbb{R}^3$-valued process

$$X_t = (B_t^{(1)}, B_t^{(2)}, \frac{1}{2}\int_0^t \left(B_s^{(1)}\, dB_s^{(2)} - B_s^{(2)}\, dB_s^{(1)}\right)).$$

The third component is called the Lévy stochastic area (1951).

If $\alpha = \frac{1}{2}(x\, dy - y\, dx)$, then $d\alpha = dx \wedge dy$, so by Stokes' theorem $\int_\gamma \frac{1}{2}(x\, dy - y\, dx)$ is the signed area swept by a smooth curve γ.

The transition probability density of the process X_t is the fundamental solution of $\partial_t u = Lu$, where

$$L = \frac{1}{2}(V_1^2 + V_2^2), \quad V_1 = \partial_x - \frac{y}{2}\partial_z, \quad V_2 = \partial_y + \frac{x}{2}\partial_z.$$

Since $[V_1, V_2] = \partial_z$, the vector fields ∂_t, V_1, V_2 satisfy Hörmander's condition on $\mathbb{R}^4$.

Malliavin's approach (1978)

For each realization of $B = (B^1, \ldots, B^d)$, the solution to the SDE at time t can be viewed as a deterministic function

$$X_t = F(B),$$

where F is a map from Wiener space $C([0,t], \mathbb{R}^d)$ to $\mathbb{R}^n$.

Malliavin's idea: study how X_t changes when the driving Brownian path changes (Malliavin calculus).

Malliavin's criterion: If F is a submersion (differential of F is surjective), then the law of $X_t = F(B)$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^n$.

Terminology: law of $X_t = F(B)$ is the pushforward measure $F_*(\mu)$ where μ is the probability measure on $C([0,t], \mathbb{R}^d)$ corresponding to the Brownian motion on $\mathbb{R}^d$.

Finite dimensions: law of the image of a submersion

Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth submersion (so $m \geq n$) and let X be a random variable on $\mathbb{R}^m$ with density $\rho : \mathbb{R}^m \to [0, \infty)$, then the law of $f(X)$ is absolutely continuous with respect to the Lebesgue measure; $f_*(\rho(x)\,\mathcal{L}_m(dx)) = \sigma(y)\,\mathcal{L}_n(dy)$.

The density $\sigma : \mathbb{R}^n \to [0, \infty)$ of the law of $f(X)$ is given by

$$\sigma(y) = \int_{f^{-1}(y)} \frac{\rho(x)}{J_f(x)}\, d\mathcal{H}_{m-n}(x),$$

where

$$J_f(x) = \sqrt{\det(Df(x)Df(x)^T)}, \quad \mathcal{H}_{m-n} = \text{Hausdorff measure.}$$

This follows from the coarea formula: For any non-negative measurable function $g : \mathbb{R}^m \to [0, \infty)$

$$\int_{\mathbb{R}^m} g(x)\, J_f(x)\, dx = \int_{\mathbb{R}^n} \left(\int_{f^{-1}(y)} g(x)\, d\mathcal{H}_{m-n}(x) \right) dy,$$

Dirichlet structure

A **Dirichlet structure** $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ consists of a probability space $(X, \mathcal{X}, \mu)$ and a quadratic form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ satisfying:

- $(\mathcal{E}, \mathcal{F})$ is a **Dirichlet form** on $L^2(X, \mu)$.
- There exists a **carré du champ** operator: a bilinear continuous map $\gamma : \mathcal{F} \times \mathcal{F} \to L^1(\mu)$ such that for all $f, h \in \mathcal{F} \cap L^\infty(\mu)$,

$$\mathcal{E}(fh, f) - \frac{1}{2}\mathcal{E}(h, f^2) = \int h\gamma(f, f)\, d\mu.$$

- **Strong locality:** if $(f + a)g = 0$, then $\mathcal{E}(f, g) = 0$.

Energy image density property

For $f = (f_1, \dots, f_n) \in \mathcal{F}^n$, define the carré du champ matrix

$$\gamma(f) := [\,\gamma(f_i, f_j)\,]_{1 \le i,j \le n}, \quad \gamma(f) : X \to \mathbb{R}^{n \times n}.$$

The Dirichlet structure $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ satisfies the energy image density property (EID) if for every $n \in \mathbb{N}$ and $f \in \mathcal{F}^n$,

$$f_*\big(\mathbb{1}_{\{\det(\gamma(f))>0\}} \cdot \mu\big) \ll \mathcal{L}_n,$$

where $\mathcal{L}_n$ is the Lebesgue measure on $\mathbb{R}^n$.

Example: for $X = \mathbb{R}^m$ with standard Gaussian measure μ,

$$\gamma(f, g) = \nabla f \cdot \nabla g,$$

corresponds to the Dirichlet structure of the Ornstein–Uhlenbeck process on $\mathbb{R}^m$.

In this case, $\det(\gamma(f)) > 0$ means the differential $Df(x)$ is surjective, i.e. f is a submersion at x.

Energy image density conjecture

Energy Image Density Conjecture (Bouleau–Hirsch, 1986): Every Dirichlet structure $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ satisfies the energy image density (EID) property.

Theorem (Eriksson-Bique, M., 2025 +): The Bouleau–Hirsch conjecture is true.

We also establish the EID property:

- for all strongly local regular Dirichlet forms, and
- for analogues of $W^{1,p}$-Sobolev spaces for $1 < p < \infty$, including spaces built from upper gradients and self-similar energies on fractals obtained as rescaled limits of discrete energies. More on this in Sylvester's talk next week.

Past works

The scalar case ($n = 1$) was solved by Bouleau–Hirsch (1986).

For the Ornstein–Uhlenbeck Dirichlet structure on Wiener space, the carré du champ matrix is also called the Malliavin matrix, and the EID property was established by Bouleau–Hirsch (1986).

In this setting, the EID property is called the Bouleau–Hirsch criterion, which generalizes Malliavin's criterion as Malliavin needed additional assumptions beyond $f \in \mathcal{F}^n$.

Both these methods were extended beyond the Wiener space setting:

- Malliavin: first-order distributional derivatives are Radon measures. This requires an integration by parts formula.
- Bouleau–Hirsch: approach based on Federer's coarea formula.

Malicet and Poly (2013) show that if $\det(\gamma(f)) > 0$ μ-a.s., then $f_*(\mu)$ (law of f) is a Rajchman measure.

Difficulty in extending to the general case

The integration by parts approach does not work for all $f \in \mathcal{F}^n$.

The coarea formula approach applies only to a special class of Dirichlet structures, since a coarea formula is not available in general.

Even in the classical setting of Ornstein–Uhlenbeck Dirichlet form on the Wiener space there is no alternative to coarea formula based approach. Malliavin emphasizes (1997):

> *"There does not exist currently an alternative approach to the regularity of laws for $\mathbb{R}^d$-valued functionals with one derivative."*

Theorems of Rademacher and De Philippis–Rindler

Rademacher's theorem (1919): If $\nu \ll \mathcal{L}_n$, then every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable ν-almost everywhere.

Converse theorem (De Philippis–Rindler, 2016): Let ν be a Radon measure on $\mathbb{R}^n$ such that every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable ν-almost everywhere. Then $\nu \ll \mathcal{L}_n$.

Our proof of the energy image density conjecture draws on tools developed for the converse of Rademacher's theorem.

Currents in $\mathbb{R}^n$

A *k-dimensional current* T in $\mathbb{R}^n$ is a continuous linear functional on the space of smooth, compactly supported k-forms $\mathcal{D}^k(\mathbb{R}^n)$.

The boundary of T is the $(k-1)$-current defined by

$$(\partial T)(\omega) := T(d\omega), \quad \omega \in \mathcal{D}^{k-1}(\mathbb{R}^n),$$

where $d\omega$ is the exterior derivative of ω.

The mass of T is

$$\mathbf{M}(T) := \sup\{\, T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^n),\ |\omega| \leq 1 \,\}.$$

Example: If T the oriented volume form (n-form) of a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, then ∂T the volume form ($n-1$-form) of the smooth surface $\partial\Omega$ with induced orientation.

$$\mathbf{M}(T) = \mathcal{L}_n(\Omega), \quad \mathbf{M}(\partial T) = \mathcal{H}_{n-1}(\partial\Omega).$$

Normal currents (Federer and Fleming 1960)

A current T is normal if both T and its boundary ∂T have finite mass.

We only need the cases $k = 0$ and $k = 1$.

- $k = 0$: a 0-current with finite mass is a finite signed measure on $\mathbb{R}^n$.
- $k = 1$: a 1-current with finite mass is an $\mathbb{R}^n$-valued finite signed measure.

Every 1-current with finite mass admits a canonical decomposition

$$T = \vec{T}\,\|T\|,$$

where $\|T\|$ is a finite positive measure and $\vec{T} : \mathbb{R}^n \to \mathbb{R}^n$ is measurable with $|\vec{T}| = 1$ $\|T\|$-a.e.

De Philippis–Rindler criterion for absolute continuity

Theorem (De Philippis, Rindler 2016)
Let

$$T_1 = \vec{T}_1 \, \| T_1 \|, \ \ldots, \ T_n = \vec{T}_n \, \| T_n \|$$

be one-dimensional normal currents on $\mathbb{R}^n$, and let ν be a positive Radon measure on $\mathbb{R}^n$ such that:

(i) $\nu \ll \| T_i \|$ for $i = 1, \ldots, n$;

(ii) For ν-a.e. $x \in \mathbb{R}^n$,

$$\operatorname{span}\{ \vec{T}_1(x), \ldots, \vec{T}_n(x) \} = \mathbb{R}^n.$$

Then

$$\nu \ll \mathcal{L}_n.$$

Generator and energy measures

Let $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet structure.
Generator. The generator $A : D(A) \to L^2(X, \mu)$ is defined by

$$D(A) := \left\{ f \in \mathcal{F} : \exists\, C > 0 \text{ s.t. } |\mathcal{E}(f, g)| \leq C\|g\|_{L^2(\mu)} \text{ for all } g \in \mathcal{F} \right\},$$

and for $f \in D(A)$,

$$\mathcal{E}(f, g) = -\langle A(f), g\rangle_{L^2(\mu)}.$$

Energy measure. For $f, g \in \mathcal{F}$, the *energy measure* is

$$\Gamma(f, g) := \gamma(f, g)\, \mu,$$

where γ is the carré du champ operator.

A construction of normal currents from energy measures

For $f = (f_1, \ldots, f_n) \in \mathcal{F}^n$ and $g \in \mathcal{F}$, define the 1-dimensional current $T_{f,g}$ on $\mathbb{R}^n$ by

$$T_{f,g}\Big(\sum_{j=1}^{n} h_j\, dx_j\Big) := \sum_{j=1}^{n} \int h_j(f(x))\,\Gamma(f_j, g)(dx),$$

for smooth, compactly supported 1-forms $\sum_j h_j\, dx_j$.

Equivalently, $T_{f,g}$ is the $\mathbb{R}^n$-valued measure

$$T_{f,g} = f_*\big((\Gamma(f_i, g))_{1 \leq i \leq n}\big),$$

which has finite mass, hence defines a 1-dimensional current.

Lemma (Eriksson-Bique–M.): If $f \in \mathcal{F}^n$ and $g \in D(A)$, then $T_{f,g}$ is a *normal current* with

$$\partial T_{f,g} = -\, f_*(A(g) \cdot \mu).$$

Proof of $\partial T_{f,g} = - f_*(A(g) \cdot \mu)$

For all $\phi \in C_c^\infty(\mathbb{R}^n)$, we have (denote $\phi_0(\cdot) = \phi(\cdot) - \phi(0)$)

$$\partial T_{f,g}(\phi) = T_{f,g}\left(\sum_{j=1}^{n} \frac{\partial \phi}{\partial x_j} \, dx_j \right)$$

$$= \sum_{j=1}^{n} \int \frac{\partial \phi}{\partial x_j}(f(x)) \, \Gamma(f_j, g)(dx)$$

$$= \mathcal{E}(\phi_0(f), g) \quad \text{(by the chain rule for energy measures)}$$

$$= -\int_X A(g) \, \phi(f) \, d\mu = -\int_{\mathbb{R}^n} \phi_0(y) \, f_*(A(g) \cdot \mu)(dy).$$

Hence $T_{f,g}$ is a one-dimensional normal current for all $f \in \mathcal{F}^n$, $g \in D(A)$ whose boundary is

$$\partial T_{f,g} = -f_*(A(g) \cdot \mu) + \left(\int_X A(g) \, d\mu \right) \delta_0.$$

Linear independence properties of currents

Lemma (Eriksson-Bique–M.)

For each $n \in \mathbb{N}$ and $f = (f_1, \ldots, f_n) \in \mathcal{F}^n$, we have

$$f_*\big(\mathbb{1}_{\{\det(\gamma(f))>0\}} \cdot \mu\big) \ll \| T_{f,f_i} \|,$$

$$\operatorname{span}\{ \vec{T}_{f,f_1}(y), \ldots, \vec{T}_{f,f_n}(y)\} = \mathbb{R}^n, \quad \text{for } f_*\big(\mathbb{1}_{\{\det(\gamma(f))>0\}} \cdot \mu\big)\text{-a.e. } y.$$

By the previous lemma, the currents T_{f,f_i} are normal whenever $f \in D(A)^n$.

By the De Philippis–Rindler criterion for absolute continuity, this implies the energy image density property for all $f \in D(A)^n$.

For a general $f \in \mathcal{F}^n$, the one-dimensional currents T_{f,f_i} need not be normal, and an additional approximation argument using the density of $D(A)$ in $\mathcal{F}$ is required.

Extensions to strongly local, regular, Dirichlet forms

We also prove an energy image density (EID) property for all regular strongly local Dirichlet forms.

In this setting, the reference measure μ and the energy measures $\Gamma(f, g)$, $f, g \in \mathcal{F}$, can be mutually singular, so the measure $f_*(\mathbb{1}_{\{\gamma(f)>0\}}\mu)$ is not well defined.

We select a quasicontinuous version of f and define a modified carré du champ matrix via Radon–Nikodym derivatives with respect to a minimal energy dominant measure ν:

$$\gamma_\nu(f)_{ij} = \frac{d\Gamma(f_i, f_j)}{d\nu}, \qquad f = (f_1, \dots, f_n) \in \mathcal{F}^n.$$

This ensures that $f_*(\mathbb{1}_{\{\det(\gamma_\nu(f))>0\}}\nu)$ is well-defined for a given ν and is well defined up to measure equivalence as ν varies among minimal energy dominant measures.

EID property for strongly local, regular, Dirichlet forms

Theorem (Eriksson-Bique, M. '25): Let $(\mathcal{E}, \mathcal{F})$ be a regular, strongly local Dirichlet form on $L^2(X, m)$ and let ν be a minimal energy dominant measure. Then for any $n \in \mathbb{N}, f \in \mathcal{F}^n$, we have

$$f_*(\mathbb{1}_{\{\det(\gamma_\nu(f))>0\}}\nu) \ll \mathcal{L}_n.$$

This is reminiscent of the de Philippis–Marchese–Rindler theorem (from Lecture 2) for chart ϕ on U: $f_*(\mathbb{1}_U \cdot \mu) \ll \mathcal{L}_n$.

Recall that since ϕ is Lipschitz, this was enough to show that the analytic dimension is less than or equal to Hausdorff dimension.

The same argument cannot be used for martingale dimension.

There are strongly local Dirichlet forms with sub-Gaussian heat kernel estimate with walk dimension $\beta \in (2, \infty)$ such that the **domain has no non-constant Lipschitz functions** (Anttila, Eriksson-Bique, Shimizu '25+).

Proof of finiteness of martingale dimension

Although there need not be no non-constant Lipschitz functions under sub-Gaussian heat kernel estimates, we have the following:

Theorem (Barlow, Bass, Kumagai '06): Let $(\mathcal{E}, \mathcal{F})$ be a strongly local Dirichlet form on a metric measure space (X, d, m) with sub-Gaussian heat kernel estimate with walk dimension $\beta \in (2, \infty)$. Then there exists $\alpha \in (0, 1]$ such that $\mathcal{F}$ contains a dense set of α-Hölder continuous functions, where α depends only on the constants involved in the heat kernel estimates.

Remark: If $\beta = 2$, then $\alpha = 1$ due to Koskela–Zhou '12 (Lec. 2).

Proof sketch: Using density of α-Hölder functions in $\mathcal{F}$, the continuity of (modified) carré du champ matrix $f \mapsto \det(\gamma_\nu(f))$, and the same argument in the proof of Cheeger's conjecture, we have that the martingale dimension d_m is bounded by

$$d_m \leq \dim_H(X, d)/\alpha.$$

Thank you for your attention!

Slides available at https://personal.math.ubc.ca/∼mathav/msj/