MATH 419/545
HW 3

Due February 14, 2020 (at the beginning of the class)

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1. (a) Assume $\{Y_i : i \in I\}$ is a uniformly integrable collection of non-negative random variables and Z is a non-negative integrable random variable. If $|X_i| \leq Y_i + Z$ for all i, prove that $\{X_i : i \in I\}$ is also uniformly integrable

   (b) Let $\{X_n\}$ be a uniformly integrable $(\mathcal{F}_n)$-submartingale and $\mathcal{T}$ be the collection of all $\mathcal{F}_n$-stopping times. Prove that $\{X_T : T \in \mathcal{T}\}$ is uniformly integrable. (Remark: This shows that the submartigale $\{X_{T_k}\}$ obtained in optional stopping theorem is uniformly integrable).
   Hint: We prove this in class if $\{X_n\}$ were a martingale. One approach is to use Doob's decomposition (Theorem 4.3.2).

2. Assume $\{X_n : n \leq 0\}$ is a backwards $(\mathcal{F}_n)_{n\leq 0}$-martingale such that $E(|X_0|^p) < \infty$ for some $p > 1$. Prove that X_n converges a.s. and in L^p.

3. Let $(X_n : n \in \mathbb{N})$ be a non-negative L^1 bounded $(\mathcal{F}_n)$-submartingale. Let $Y_n = \sup_{k\geq n} E(X_k|\mathcal{F}_n)$. Show that (Y_n) is an $(\mathcal{F}_n)$-martingale such that:

   (i) $X_n \leq Y_n$ for all n a.s.

   (ii) $\|Y\|_1 = \|X\|_1$

   (iii) If (N_n) is an $(\mathcal{F}_n)$-martingale such that $X_n \leq N_n$ for all n a.s., then $Y_n \leq N_n$ for all n a.s.

4. Exercise 4.6.7

5. (a) Prove the following modified weak L^1 inequality: Let $(X_n : n \in \mathbb{N})$ be a non-negative $(\mathcal{F}_n)$-submartingale. Then for all $\lambda > 0$

$$P(X_n^* \geq 2\lambda) \leq \frac{1}{\lambda} \int_{\{X_n \geq \lambda\}} X_n \, dP.$$

Hint: For fixed λ, n consider the martigale

$$M_k = E(X_n 1_{\{X_n \geq \lambda\}} | \mathcal{F}_k).$$

(Even if you don't do (a) you can use it in (b) and (c).)

(b) Let (X_n) be as in (a). Prove that $\|X^*\|_1 \leq \sup_n 2E(X_n \log^+ X_n) + 2$, where $\log^+ x = \log(x) \vee 0, \log(0) = -\infty$.

(c) Let $\{Y_i\}$ be iid random variables and let $S_n = \sum_{i=1}^n Y_i$. Prove that $E(|Y_1| \log^+ |Y_1|) < \infty$ implies $E\left(\sup_n \frac{|S_n|}{n}\right) < \infty$.

Practice Problems (do not hand in)

1. Read the statement and proof of Doob's decomposition theorem (Theorem 4.3.2).

2. Show that every martingale that converges in L^p is L^p bounded ($1 \leq p < \infty$).

3. We used the following fact implicitly in the proof of optional sampling theorem (prove it): Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Let X, Y be two integrable random variables such that X is $\mathcal{F}$-measurable and Y is $\mathcal{G}$-measurable. Then the following are equivalent

   (a) $Y \leq E(X|\mathcal{G})$ almost surely.

   (b) For any $A \in \mathcal{G}$, we have

   $$E(Y 1_A) \leq E(X 1_A).$$

4. Exercises 4.8.2 to 4.8.10.

5. Do other exercises stated in lectures.