Heat kernel estimates and Harnack inequalities

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1 Introduction

The study of heat kernels and their estimates is at the interface of analysis, geometry and probability. Let us start with some examples of heat kernel. The fundamental solution of the heat equation on $\mathbb{R}^n$

$$\partial_t u = \frac{1}{2}\Delta u, \tag{1.1}$$

is given by the classical Gauss–Weierstrass kernel

$$p_t(x,y) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t}\right). \tag{1.2}$$

That is, for any fixed $x \in \mathbb{R}^n$, the function $(t,y) \mapsto p_t(x,y)$ solves the heat equation on $(0,\infty)\times\mathbb{R}^n$ and $\lim_{t\downarrow 0} p_t(x,\cdot) = \delta_x$, where δ_x is the Dirac mass at x in the sense of distributions. In particular, for any $f \in C_c^\infty(\mathbb{R}^n)$, the function $u(t,x) = \int_{\mathbb{R}^n} p_t(x,y)f(y)\,dy$, $(t,x) \in (0,\infty)\times\mathbb{R}^n$ solves the Cauchy problem for the heat equation (1.1) with initial condition $u(0,x) = f(x)$ for all $x \in \mathbb{R}^n$.

The fundamental solution of the heat equation can alternately be viewed as the transition probability density of the standard Brownian motion as $p_t(x,\cdot)$ is the density of the multivariate normal random variable with mean x and covariance matrix $tI_{n\times n}$. Thus the measure $p_t(x,y)dy$ is the law of the Brownian motion B_t starting at $B_0 = x$. By extension, we use the term *heat kernel* to refer to the transition kernel of a variety of Markov processes.

It is known that the behavior heat kernel is closely related to the geometry of the underlying space. A classical result in this direction is the Varadhan's asymptotic formula [Var] (see also [Nor, HR]). It states that the fundamental solution of the heat equation $\partial_t u = \frac{1}{2}\Delta u$ on a Riemannian manifold satisfies

$$\lim_{t\downarrow 0} 2t \log p_t(x,y) = -d(x,y)^2,$$

where Δ is the Laplace-Beltrami operator and d denotes the Riemannian distance. Despite the universal nature of Varadhan's short time asymptotic mentioned above, the long time behavior of the heat kernel of a manifold can be significantly different from the Euclidean case. For instance, the heat kernel of the 3-dimensional hyperbolic space $\mathbb{H}^3$ is given by [DM, Theorem 2.1]

$$p_t(x,y) = \frac{1}{(2\pi t)^{3/2}} \frac{d(x,y)}{\sinh(d(x,y))} \exp\left(-\frac{t}{2} - \frac{d(x,y)^2}{2t}\right).$$

In the expression (1.2) for the heat kernel, the term $|x-y|^2/t$ reflects that time scales like the square of the distance (Brownian space-time scaling). There are Markov processes with other space-time scaling behavior. We describe two such examples: jump processes on $\mathbb{R}^n$ and diffusions on fractals.

Let (Y_t) be the symmetric α-stable process (where $\alpha \in (0,2)$) on $\mathbb{R}^n$; that is, (Y_t) is a Lévy process (stationary, independent increments) such that

$$\mathbb{E}[e^{i\xi\cdot(Y_{t+s}-Y_s)}] = \exp(-t|\xi|^\alpha), \quad \text{for all } t,s \geqslant 0 \text{ and } \xi \in \mathbb{R}^n.$$

Then the law of (Y_t) admits a density (heat kernel) p_t satisfying the following estimate: there exists $C \in [1,\infty)$ such that

$$C^{-1}\min\left(t^{-n/\alpha}, \frac{t}{|x-y|^{n+\alpha}}\right) \leqslant p_t(x,y) \leqslant C\min\left(t^{-n/\alpha}, \frac{t}{|x-y|^{n+\alpha}}\right), \quad \text{for all } t>0, x,y \in \mathbb{R}^n. \tag{1.3}$$

This formula suggest the space time scaling relation distance$^\alpha$ scales like time.

A rich family of heat kernels arise from diffusions on fractals. Barlow and Perkins constructed a diffusion process on the standard Sierpiński gasket [BP]. The law of this diffusion process at any time $t>0$ admits a density with respect to the d_f-Hausdorff measure, where $d_f = \log_2 3$ is the Hausdorff dimension of the Sierpiński gasket. Let $d_w = \log_2 5$. The transition density p_t admits the following estimates: there exists $C_1, C_2, C_3, C_4 \in (0,\infty)$ such that for any x, y in the Sierpiński gasket and any $t \in (0,1)$, we have

$$\frac{C_3}{t^{d_f/d_w}}\exp\left(-C_4\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \leqslant p_t(x,y) \leqslant \frac{C_1}{t^{d_f/d_w}}\exp\left(-C_2\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \tag{1.4}$$

Bounds of the form (1.4) are called *sub-Gaussian heat kernel estimates* is now known to hold on many fractals.

Here is the outline for this course. We will begin by covering fundamental aspects of the theory of Dirichlet forms, a powerful framework for constructing and analyzing symmetric Markov processes. We will then survey key results characterizing spaces that satisfy heat kernel estimates analogous to (1.2), (1.3), and (1.4). We will illustrate these results by applying them to a number of examples. One important class of examples is the boundary trace process. Consider the reflected Brownian motion on the $n+1$-dimensional upper half space $\mathbb{R}^n \times [0,\infty)$. The boundary trace process is obtained by removing the path of the reflection Brownian motion in the interior $\mathbb{R}^n \times (0,\infty)$ in a certain sense. A classical result of Spitzer states that the resulting process is a jump process on the boundary $\partial\mathbb{H}^{n+1} = \mathbb{R}^n$ and coincides with the symmetric 1-stable process (Cauchy process). By modifying the reflected Brownian motion to a different reflected diffusion process, Molchanov and Ostrowski [MO] discovered that any α-stable process with $\alpha \in (0,2)$ can be obtained as trace of a diffusion process on the upper half space. This was rediscovered in an analytic setting by Caffarelli and Silvestre [CS]. We will study the behavior of boundary trace process for more general diffusions and domains.

2 Dirichlet forms and symmetric Markov processes

We refer to the standard references [FOT, CF, BH91] for a comprehensive introduction to the theory of Dirichlet forms.

2.1 Semigroup and resolvent

Definition 2.1 (semigroup of operators)**.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We say that a family of linear operators $\{T_t : \mathcal{H} \to \mathcal{H} | t > 0\}$ is a *semigroup* if it satisfies

(1) Each T_t is a symmetric operator; that is, $\langle T_t(f), g \rangle = \langle f, T_t(g) \rangle$ for all $t > 0$ and $f, g \in \mathcal{H}$.

(2) $\{T_t : t > 0\}$ satisfy the semigroup property; $T_{t+s} = T_t T_s$ for all $t, s > 0$.

(3) Each T_t is a contraction operator, $\|T_t(u)\| \leqslant \|u\|$ for all $t > 0, u \in \mathcal{H}$.

(4) (strongly continuous) For all $u \in \mathcal{H}$, we have $\lim_{t \downarrow 0} \|T_t(u) - u\| = 0$.

It is more precise to call it a symmetric, strongly continuous, contraction semigroup. We will use the abbreviated term semigroup.

A Markov process $(Y_t)_{t \geqslant 0}$ on a space X defines an operator

$$P_t f(x) = \mathbb{E}_x[f(Y_t)] = \mathbb{E}[f(Y_t) | Y_0 = x],$$

for all $t > 0, x \in X$ and a suitable class of functions f. By the Markov property, for any $t, s > 0, x \in X$ and $f : X \to \mathbb{R}$ bounded, we have the semigroup property

$$P_{t+s} f(x) = \mathbb{E}_x[f(Y_{t+s})] = \mathbb{E}_x\left[\mathbb{E}_x[f(Y_{t+s}) | \mathcal{F}_t]\right] = \mathbb{E}_x\left[\mathbb{E}_{Y_t}[f(Y_s)]\right] = \mathbb{E}_x[(P_s f)(Y_t)] = P_t(P_s(f))(x).$$

This estimate implies that the expected distance traveled by the diffusion process (B_t) satsifes the bound $\mathbb{E}[|B_t - B_0|] \asymp t^{1/d_w}$ for all $t \in (0, 1)$. This should be compared with the expected distance traveled by the Brownian motion on $\mathbb{R}^n$ which is comparable to $\sqrt{t}$.

Example 2.2. (i) The Brownian (or heat) semigroup is given by $P_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is

$$P_t f(x) := \int_{\mathbb{R}^n} \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t}\right) f(y)\, dy, \quad \text{for all } t > 0, f \in L^2(\mathbb{R}^n).$$

This corresponds to the standard Brownian motion on $\mathbb{R}^n$.

(ii) Let γ_n denote the standard Gaussian measure on $\mathbb{R}^n$; $\gamma_n(dx) = (2\pi)^{-n/2} e^{-|x|^2/2}\, dx$. The Ornstein-Uhlenbeck process $(X_t)_{t \geqslant 0}$ on $\mathbb{R}^n$ is a Markov process that is associated with the stochastic differential equation

$$dX_t = -X_t\, dt + \sqrt{2} dB_t,$$

where (B_t) is the standard Brownian motion on $\mathbb{R}^n$. Alternately,

$$X_t = e^{-t}X_0 + e^{-t}B_{e^{2t}-1},$$

where (B_t) is again the standard Brownian motion. Then the Ornstein-Uhlenbeck semigroup is defined by $P_t : L^2(\mathbb{R}^n, \gamma_n) \to L^2(\mathbb{R}^n, \gamma_n)$ as

$$P_t f(x) := \int_{\mathbb{R}^n} f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right)\gamma_n(dy), \quad \text{for all } f \in L^2(\mathbb{R}^n, \gamma_n), t > 0.$$

(iii) Let $X = \{1, \dots, n\}$ and $c : X \times X \to [0, \infty)$ be such that $c(x, y) = c(y, x) \geqslant 0$ for all $x, y \in X$. Let $m : X \to [0, \infty)$ be

$$m(x) := \sum_{z \in X} c(x, y).$$

Let us assume that $m(x) > 0$ for all $x \in X$. This defines a *discrete time* Markov chain $(Z_n)_{n \in \mathbb{N} \cup \{0\}}$ with transition probabilities given by

$$P(x, y) := \frac{c(x, y)}{m(x)}, \quad \text{for all } x, y \in X.$$

This defines a linear operator $Q : L^2(E, m) \to L^2(E, m)$ given by

$$Qf(x) = \mathbb{E}[f(Z_1)|Z_0 = x] = \sum_{y \in X} P(x, y)f(y).$$

By the Markov property, we have $Q^k f(x) = \mathbb{E}[f(Z_k)|Z_0 = x]$ for all $k \in \mathbb{N} \cup \{0\}$ with $Q^0 = I$, where Q^k denotes the k-fold composition of Q. A standard construction of a continuous time process (Y_t) from the discrete time process $(Z_n)_{n \in \mathbb{N} \cup \{0\}}$ is obtained by waiting at every state $x \in E$ for an exponential time Exp(1) before jumping to a state y with probability $Q(x, y)$. If $(N(t))_{t \geqslant 0}$ denotes a Poisson process with rate 1 (independent of (Z_n)) that determines the waiting times, we have that the semigroup corresponding to process $Y_t := Z_{N(t)}$ is given by

$$P_t f(x) = \mathbb{E}[f(Y_t)|Y_0 = x] = \sum_{k=0}^{\infty} \mathbb{P}(N(t) = k)\mathbb{E}[f(Z_k)|Z_0 = x] = \sum_{k=0}^{\infty} e^{-t}\frac{t^k}{k!}Q^k f(x).$$

Then $(P_t)_{t \geqslant 0}$ is a semigroup on $L^2(X, m)$.

Exercise 2.3. Verify that the examples above are a semigroup (in the sense of Definition 2.1).
Hint: In order to show strong continuity, it might help to look at Lemma 2.37.

As we will see, it is often easier to construct or analyze the Laplace transform of a semigroup.

Definition 2.4 (resolvent)**.** A resolvent on $\mathcal{H}$ is a family of linear operators $\{G_\alpha : \mathcal{H} \to \mathcal{H} | \alpha > 0\}$ on $\mathcal{H}$ such that

(1) Each G_α is a symmetric operator; ; that is, $\langle G_\alpha(f), g\rangle = \langle f, G_\alpha(g)\rangle$ for all $t > 0$ and $f, g \in \mathcal{H}$.

(2) $\{G_\alpha : \alpha > 0\}$ satisfy the resolvent equation

$$G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0, \quad \text{for all } \alpha, \beta > 0. \tag{2.1}$$

(3) (contraction property) For any $u \in \mathcal{H}, \alpha > 0$, we have $\|\alpha G_\alpha u\| \leqslant \|u\|$.

(4) (strongly continuous) For any $u \in \mathcal{H}$, we have $\lim_{\alpha\to\infty} \|\alpha G_\alpha(u) - u\| = 0$.

The resolvent can be viewed as the Laplace transform of the semigroup $t \mapsto T_t$. To describe this we need to define integrals of Hilbert space valued function.

Definition 2.5. Let $I \subset \mathbb{R}$ be an interval and $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. We say that a function $f : I \to \mathcal{H}$ is *weakly measurable* if for any $v \in \mathcal{H}$, the functions $t \mapsto \langle v, f(t)\rangle$ and $t \mapsto \|f(t)\|$ are measurable functions on I. If a weakly measurable function $f : I \to \mathcal{H}$ satisfies $\int_I \|f(t)\|\, dt < \infty$, we say that f is *integrable*. If $f : I \to \mathcal{H}$ is weakly measurable and integrable, then by the Riesz-Fréchet representation theorem, there exists a unique $x \in \mathcal{H}$ such that

$$\langle v, x\rangle = \int_I \langle v, f(t)\rangle\, dt, \quad \text{for all } v \in \mathcal{H}. \tag{2.2}$$

We denote x as the integral $\int_I f(t)\, dt \in \mathcal{H}$.

The following basic properties of this integral are easily verified.

Exercise 2.6. Let $I \subset \mathbb{R}$ be an interval and $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space.

1. If $f : I \to \mathcal{H}$ is continuous, then f is weakly measurable.
2. If $f : I \to \mathcal{H}$ is weakly measurable and integrable, then we have the triangle inequality
$$\left\| \int_I f(t)\, dt \right\| \leqslant \int_I \|f(t)\|\, dt.$$
3. Let $\mathcal{H}_1$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}_1$ be a bounded linear map. If $f : I \to \mathcal{H}$ is continuous and integrable, then $T \circ f : I \to \mathcal{H}_1$ is weakly measurable and integrable. Furthermore, we have
$$T\left(\int_I f(t)\, dt\right) = \int_I (T \circ f)(t)\, dt.$$

Every semigroup defines a resolvent as its Laplace transform.

Exercise 2.7. Consider a strongly continuous contraction semigroup $\{T_t : t > 0\}$ on $\mathcal{H}$. For any $\alpha > 0, u \in \mathcal{H}$ consider the integral

$$G_\alpha(u) = \int_0^\infty e^{-\alpha t} T_t(u)\, dt.$$

Show that $\{G_\alpha : \mathcal{H} \to \mathcal{H} | \alpha > 0\}$ defines a resolvent (in the sense of Definition 2.4).
Hint: The standard approach to showing the resolvent identity is to use

$$e^{-\alpha t} - e^{-\beta t} = (\beta - \alpha) \int_0^t e^{-\alpha(t-s)} e^{-\beta s}\, ds$$

in the integral for $G_\alpha(u) - G_\beta(u)$ and interchange the order of integration and use semigroup property.

Here is another probabilistic approach to show the resolvent identity[1]: Assume $\alpha > \beta$ and consider two independent random variables ξ_β and $\xi_{\alpha-\beta}$ with exponential distributions with rate α. Then here is an outline of the probabilistic approach.

1. $\xi_\alpha := \xi_\beta \wedge \xi_{\alpha-\beta}$ is exponentially distributed with rate α.
2. $\mathbb{E}[T_{\xi_\alpha}(u)] = \alpha G_\alpha(u), \mathbb{E}[T_{\xi_\beta}(u)] = \beta G_\beta(u)$.
3. $\mathbb{P}(\xi_\alpha = \xi_\beta) = 1 - \mathbb{P}(\xi_\alpha < \xi_\beta) = \frac{\beta}{\alpha}$.
4. Conditioned on the event $\{\xi_\alpha < \xi_\beta\}$, the distribution on $\xi_\beta - \xi_\alpha$ is exponential with parameter β (memoryless property).
5. Write

$$T_{\xi_\beta} = \mathbb{1}_{\{\xi_\alpha = \xi_\beta\}} T_{\xi_\alpha} + \mathbb{1}_{\{\xi_\alpha < \xi_\beta\}} T_{\xi_\alpha} \circ T_{\xi_\beta - \xi_\alpha} = \mathbb{1}_{\{\xi_\alpha = \xi_\beta\}} T_{\xi_\alpha} + \mathbb{1}_{\{\xi_\alpha < \xi_\beta\}} T_{\xi_\beta - \xi_\alpha} \circ T_{\xi_\alpha},$$

   and take expectations on both sides to derive the resolvent identity.

Another way to rephrase the probabilistic approach is as follows. Let ξ_α and ξ_β be independent exponential random variables with rates α and β with $\alpha > \beta$ and let Z be an independent (of ξ_α, ξ_β) Bernoulli random variable with $\mathbb{P}(Z = 1) = 1 - \mathbb{P}(Z = 0) = \frac{\beta}{\alpha}$. Then verify that the random variable

$$\widetilde{\xi}_\beta := \mathbb{1}_{\{Z=1\}} \xi_\alpha + \mathbb{1}_{\{Z=0\}} (\xi_\alpha + \xi_\beta)$$

is also exponentially distributed with parameter β. Writing

$$T_{\widetilde{\xi}_\beta} = \mathbb{1}_{\{Z=1\}} T_{\xi_\alpha} + \mathbb{1}_{\{Z=0\}} T_{\xi_\alpha} \circ T_{\xi_\beta} = \mathbb{1}_{\{Z=1\}} T_{\xi_\alpha} + \mathbb{1}_{\{Z=0\}} T_{\xi_\beta} \circ T_{\xi_\alpha}$$

and taking expectations as above yields the resolvent identity.

[1] Thanks to Ryoichiro Noda for sharing this argument due to David Croydon.

2.2 Generators and background on self-adjoint operators

The generator provides an *infinitesimal* description of the semigroup. Let $A : \mathcal{D}(A) \to \mathcal{H}$ be a denseley defined operator on a Hilbert space $\mathcal{H}$; that is, $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$.

Definition 2.8 (Generator of a semigroup)**.** Let $\{T_t : \mathcal{H} \to \mathcal{H} | t > 0\}$ be a semigroup. Then the *generator* A of the semigroup $\{T_t : t > 0\}$ is the operator $A : \mathcal{D}(A) \to \mathcal{H}$ is defined as

$$A(u) := \lim_{t \downarrow 0} \frac{T_t(u) - u}{t}, \quad \text{for all } u \in \mathcal{D}(A),$$

where the domain $\mathcal{D}(A)$ is

$$\mathcal{D}(A) = \{u \in \mathcal{H} : \lim_{t \downarrow 0} \frac{T_t(u) - u}{t} \text{ exists}\}.$$

Often the generator is easier to compute than the semigroup as we illustrate.

Example 2.9. We will slightly generalize Example 2.2-(iii). Let $X = \{1, \ldots, n\}$ and $c : X \times X \to [0, \infty)$ be such that $c(x, y) = c(y, x) \geqslant 0$ for all $x, y \in X$. Let $m : X \to [0, \infty)$ be

$$m(x) := \sum_{y \in X} c(x, y).$$

Assume that $m(x) > 0$ for all $x \in X$. This defines a *discrete time* Markov chain $(Z_n)_{n \in \mathbb{N} \cup \{0\}}$ with transition probabilities given by

$$P(x, y) := \frac{c(x, y)}{m(x)}, \quad \text{for all } x, y \in X.$$

Let $\lambda : X \to (0, \infty)$ be a function determining a continuous time process as follows: at each state $x \in X$, the process waits (independently of other waiting times and transitions) an exponential time with parameter $\lambda(x)$[2] before jumping to a new state y with probability $P(x, y)$. Call this process $(Y_t^\lambda)_{t \geqslant 0}$ on X. If $\lambda \equiv 1$, the number of transitions (jumps) is a Poisson process which helped in the computation of semigroup. Since λ need not be a constant function, the computation of semigroup corresponding semigroup $P_t^\lambda : \mathbb{R}^X \to \mathbb{R}^X$ is not as explicit as Example 2.2-(iii). Nevertheless, we can compute the generator by observing that

$$P_t^\lambda f(x) = \mathbb{E}_x[f(Y_t^\lambda)] = e^{-\lambda(x)t} f(x) + \sum_{y \in X} (1 - e^{-\lambda(x)t}) P(x, y) f(y) + O(t^2),$$

for all $f \in \mathbb{R}^X, t \in (0, 1), x \in X$. This implies

$$L^\lambda f(x) = \lim_{t \downarrow 0} \frac{P_t^\lambda f(x) - f(x)}{t} = -\lambda(x) f(x) + \lambda(x) \sum_{y \in X} P(x, y) f(y) = -\lambda(x)(I - Q) f(x).$$

[2]exponential random variable with parameter λ has mean λ^{-1}, so larger λ means smaller wait times

Since P is m-symmetric, it is easy to see that L^λ is $\widetilde{m}$-symmetric with $\widetilde{m}(x) = \lambda(x)^{-1}m(x)$ as

$$\langle -L^\lambda f, g\rangle_{L^2(\widetilde{m})} = \langle (I-Q)f, g\rangle_{L^2(m)} = \langle f, (I-Q)g\rangle_{L^2(m)} = \langle f, -L^\lambda g\rangle_{L^2(\widetilde{m})}, \tag{2.3}$$

for all $f, g \in L^2(X, \widetilde{m})$. Using the symmetry of L^λ in $L^2(X, \widetilde{m})$, we can conclude that (P_t^λ) is also symmetric in $L^2(X, \widetilde{m})$ (we will justify this in Proposition 2.21).

If the discrete time random walk on X is irreducible, then using limiting behavior of discrete time Markov chains, one can interpret $\widetilde{m}$ (normalized to be a probability measure) as the asymptotic occupation time of a Markov chain; that is

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{1}_{\{Y_s=y\}}\, ds = \frac{\widetilde{m}(y)}{\sum_{z\in X}\widetilde{m}(z)}, \quad \mathbb{P}_x\text{-a.s. for all } x, y \in X.$$

Using the contraction property of the semigroup, we see that the generator is a non-positive definite operator.

Lemma 2.10. *$\mathcal{D}(A)$ defined above a subspace of $\mathcal{H}$, $A : \mathcal{D}(A) \to \mathcal{H}$ is a linear map, and A is non-positive definite; that is,*

$$\langle A(u), u\rangle \leqslant 0, \quad \textit{for all } u \in \mathcal{D}(A).$$

Proof. The first two claims are easy consequence of the definition of A. If $u \in \mathcal{D}(A)$, then

$$\langle A(u), u\rangle = \lim_{t\downarrow 0} \frac{1}{t}\langle T_t(u) - u, u\rangle.$$

By Schwarz's inequality, for any $u \in \mathcal{H}, t > 0$, we have $\langle T_t(u) - u, u\rangle = \langle T_t(u), u\rangle - \|u\|^2 \leqslant \|T_t(u)\|\,\|u\| - \|u\|^2 \leqslant 0$ (since T is a contraction). □

Lemma 2.11. *Let $\{G_\alpha : \alpha > 0\}$ be a resolvent on $\mathcal{H}$. Then for each $\alpha > 0$, G_α is injective and non-negative definite.*

Proof. Let $\beta > 0$. Since G_β is linear, it suffices to show that G_β has a trivial kernel. Let $u \in \mathcal{H}$ be such that $G_\beta(u) = 0$. By the resolvent equation, for any $\beta > 0$ we have

$$G_\alpha(u) = G_\beta(u) + (\beta - \alpha)G_\alpha(G_\beta(u)) = 0.$$

Hence by strong continuity $u = \lim_{\alpha\to\infty} \alpha G_\alpha(u) = 0$. Hence G_β is injective for each $\beta > 0$.

To show that G_α is non-negative definite, we need to verify that $\langle u, G_\alpha u\rangle \geqslant 0$ for all $u \in \mathcal{H}$. To this end, fix $u \in \mathcal{H}$ and define $f(\alpha) = \langle u, G_\alpha u\rangle$. By the contraction property of the semigroup

$$\|G_\alpha(u) - G_\beta(u)\| \leqslant |\alpha - \beta|\alpha^{-1}\beta^{-1}\,\|u\|\,.$$

Therefore, $\lim_{\beta\to\alpha} G_\beta(u) = G_\alpha(u)$. For $\alpha \neq \beta$, by the resolvent equation and symmetry of the resolvent operators, we have

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = -\langle u, G_\alpha(G_\beta(u))\rangle = -\langle G_\alpha(u), G_\beta(u)\rangle.$$

Letting $\beta \to \alpha$ and using $\lim_{\beta\to\alpha} G_\beta(u) = G_\alpha(u)$, we obtain that $\alpha \mapsto f(\alpha)$ is differentiable and

$$f'(\alpha) = -\|G_\alpha(u)\|^2 \leqslant 0.$$

By the strong continuity, we have $\lim_{\alpha\to\infty} \alpha f(\alpha) = \lim_{\alpha\to\infty}\langle u, \alpha G_\alpha(u)\rangle = \|u\|^2 \in [0,\infty)$. Hence $\lim_{\alpha\to\infty} f(\alpha) = 0$. Since f is non-increasing, we conclude $f(\alpha) = \langle u, G_\alpha u\rangle \geqslant 0$ for all $\alpha \in (0,\infty)$. □

By Exercise 2.7, every semigroup defines a resolvent as a Laplace transform. So one might wonder if we can compute the generator of semigroup directly from the resolvent. This motivates the following definition.

Definition 2.12 (Generator of a resolvent)**.** Let $\{G_\alpha : \alpha > 0\}$ be a resolvent on $\mathcal{H}$. The *generator* of $\{G_\alpha : \alpha > 0\}$ is defined as the linear operator $A : \mathcal{D}(A) \to \mathcal{H}$ defined by

$$\mathcal{D}(A) = G_\alpha(\mathcal{H}), \quad A(u) = \alpha u - G_\alpha^{-1}(u), \quad \text{for any } \alpha > 0, u \in \mathcal{D}(A),$$

where $G_\alpha^{-1} : G_\alpha(\mathcal{H}) \to \mathcal{H}$ is the inverse of G_α (recall from Lemma 2.11 that G_α is injective).

The fact that the above operator is well-defined is an easy exercise in the use of resolvent equation that we state below.

Exercise 2.13. Let $\{G_\alpha : \alpha > 0\}$ be a resolvent on $\mathcal{H}$.

(i) For any $\alpha, \beta > 0$, show that $G_\alpha(\mathcal{H}) = G_\beta(\mathcal{H})$.

(ii) For any $\alpha, \beta > 0$ and any $u \in G_\alpha(\mathcal{H}) = G_\beta(\mathcal{H})$, show that $\alpha u - G_\alpha^{-1}(u) = \beta u - G_\beta^{-1}(v)$.

Hint: Use the resolvent equation (2.1).

Let $\mathcal{H}$ be a Hilbert space. Let $T : \mathcal{D}(T) \to \mathcal{H}$ be a densely defined operator ($\overline{\mathcal{D}(T)} = \mathcal{H}$). Then define

$$\mathcal{D}(T^*) = \{x \in \mathcal{H} | y \mapsto \langle T(y), x\rangle \text{ is a bounded operator on } \mathcal{D}(T)\}.$$

If $x \in \mathcal{H}$, then by Hahn-Banach theorem, there is a unique extension of the map $y \mapsto \langle x, A(y)\rangle$ to $\mathcal{H}$. Hence by Riesz-Fréchet representation theorem, there exists (a unique) $T^*(z) \in \mathcal{H}$ such that

$$\langle T(y), x\rangle = \langle y, T^*(x)\rangle, \quad \text{for all } x \in \mathcal{D}(T^*).$$

It is easy to verify that $T^* : \mathcal{D}(T^*) \to \mathcal{H}$ is a linear map. For an operator $T : \mathcal{D}(T) \to \mathcal{H}$, let $\mathcal{G}(T) = \{(u, T(u)) : u \in \mathcal{D}(T)\} \subset \mathcal{H} \times \mathcal{H}$ denote the graph of T. We view $\mathcal{H} \times \mathcal{H}$ is a Hilbert space with inner product $\langle (x_1, y_1), (x_2, y_2)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$ for all $(x_1, y_1), (x_2, y_2) \in \mathcal{H} \times \mathcal{H}$.

Definition 2.14. (i) We say an operator $T : \mathcal{D}(T) \to \mathcal{H}$ is *closed* if $\mathcal{G}(T)$ is a closed set (in $\mathcal{H} \times \mathcal{H}$).

(ii) We say a densely defined operator $T : \mathcal{D}(T) \to \mathcal{H}$ is *symmetric* if for all $x, y \in \mathcal{D}(T)$, we have

$$\langle T(x), y\rangle = \langle x, T(y)\rangle.$$

Equivalently, T^* is an extension of T ($\mathcal{D}(T^*) \supset \mathcal{D}(T)$ and $T^*(x) = T(x)$ for all $x \in \mathcal{D}(T)$).

(iii) We say that an operator $T : \mathcal{D}(T) \to \mathcal{H}$ is *self-adjoint* if $T = T^*$.

For a subspace M of a Hilbert space $\mathcal{H}$, by $M^\perp$ we denote the *orthogonal complement* defined by

$$M^\perp = \{u \in \mathcal{H} : \langle u, m\rangle = 0 \quad \text{for all } m \in M\}.$$

The following lemma expresses the graph of the adjoint T^* in terms of the graph of T.

Lemma 2.15. *Let $T : \mathcal{D}(T) \to \mathcal{H}$ be a densely defined operator and let $\mathcal{G}(T) = \{(u, T(u)) : u \in \mathcal{H}\} \subset \mathcal{H} \times \mathcal{H}$ denote the graph of T. Let $V : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ be defined by $V(x, y) = (y, -x)$ for all $(x, y) \in \mathcal{H} \times \mathcal{H}$. Then $\mathcal{G}(T^*) = V(\mathcal{G}(T))^\perp$.*

Proof. $(x, y) \in \mathcal{G}(T^*)$ if and only if

$$\langle T(u), x\rangle = \langle u, y\rangle, \quad \text{for all } u \in \mathcal{D}(T).$$

This can be rewritten as

$$\langle V(u, T(u)), (x, y)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle (T(u), -u), (x, y)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle T(u), x\rangle - \langle u, y\rangle = 0, \quad \text{for all } u \in \mathcal{D}(T).$$

□

Clearly, every self-adjoint operator is symmetric and closed (by Lemma 2.15).

Corollary 2.16. *Let $T : \mathcal{D}(T) \to \mathcal{H}$ be an injective self-adjoint operator with range $\mathcal{R}(T)$. Then $T^{-1} : \mathcal{R}(T) \to \mathcal{H}$ is a densely defined, self-adjoint operator.*

Proof. Let us show that $\mathcal{R}(T)$ is dense. This is equivalent to showing that $\mathcal{R}(T)^\perp = \{0\}$. Let $y \in \mathcal{R}(T)^\perp$ as $\overline{\mathcal{R}(T)} = (\mathcal{R}(T)^\perp)^\perp$. Then $(y, 0) \in V(\mathcal{G}(T))^\perp = \mathcal{G}(T^*) = \mathcal{G}(T)$. Since T is injective, $y = 0$.

Let $V, S : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ be defined by $V(x, y) = (y, -x)$ and $S(x, y) = (y, x)$ for all $(x, y) \in \mathcal{H} \times \mathcal{H}$. Then by Lemma 2.15 and the observation that $V \circ S = -S \circ V$, we have

$$\mathcal{G}((T^{-1})^*) = V\left(\mathcal{G}(T^{-1})\right)^\perp = V(S(\mathcal{G}(T)))^\perp = S(V(\mathcal{G}(T)))^\perp.$$

Since $S : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ is a unitary (inner-product preserving) bijection with $S^2 = \mathrm{Id}$, we have $S(V(\mathcal{G}(T)))^\perp = S(V(\mathcal{G}(T))^\perp)$ and hence by Lemma 2.15, we have

$$\mathcal{G}((T^{-1})^*) = S(V(\mathcal{G}(T))^\perp) = S(\mathcal{G}(T^*)) = S(\mathcal{G}(T)) = \mathcal{G}(T^{-1}).$$

□

Exercise 2.17. Let (Ω, μ) be a measure space and let $\mathcal{H} = L^2(\Omega, \mu)$. Let $\lambda : \Omega \to \mathbb{R}$ be a measurable function (not necessarily bounded). Show that the multiplication operator $M_\lambda : \mathcal{D}(M_\lambda) \to \mathcal{H}$ with domain

$$\mathcal{D}(M_\lambda) := \{f \in L^2(\Omega, \mu) : \lambda f \in L^2(\Omega, \mu)\}$$

defined by $M_\lambda(f) = f(\cdot)\lambda(\cdot)$ is a densely defined, self-adjoint operator.

Theorem 2.18 (Spectral theorem)**.** *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a measure space (Ω, μ) and a unitary map $U : L^2(\Omega, \mu) \to \mathcal{H}$ and a measurable function $\lambda : \Omega \to \mathbb{R}$ such that*

$$U^{-1}AU = M_\lambda, \quad \mathcal{D}(A) = \{U(f) : f \in \mathcal{D}(M_\lambda)\},$$

where $M_\lambda : \mathcal{D}(M_\lambda) \to L^2(\Omega, \mu)$ is as defined in Exercise 2.17.

Furthermore, if A is non negative definite (respectively, non positive definite); that is, $\langle A(u), u\rangle \geqslant 0$ (respectively, $\langle A(u), u\rangle \leqslant 0$) for all $u \in \mathcal{D}(A)$, then $\lambda(x) \geqslant 0$ (respectively, $\lambda(x) \leqslant 0$) for μ-almost every $x \in \Omega$.

Definition 2.19 (Functional calculus)**.** Let A be a non-positive self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $U : L^2(\Omega, \mu) \to \mathcal{H}$ and $\lambda : \Omega \to \mathbb{R}$ be as given in Theorem 2.18. Then for any Borel function $f : (-\infty, 0] \to \mathbb{R}$, we define $f(A) : \mathcal{D}(f(A)) \to \mathcal{H}$ denote the operator defined by

$$U^{-1}f(A)U = M_{f(\lambda(\cdot))}, \quad \mathcal{D}(f(A)) = \{U(g) : g \in \mathcal{D}(M_{f(\lambda(\cdot))})\}$$

It is easy to verify that $f(A)$ is a self-adjoint operator on $\mathcal{H}$. Note that if f is bounded, then $f(A)$ is a bounded operator with $\mathcal{D}(f(A)) = \mathcal{H}$. If f is non-negative (respectively, non-positive), then $f(A)$ is a non-negative definite (respectively, non-positive definite) operator.

Proposition 2.20. *Let $\{T_t : t > 0\}$ be a a strongly continuous contraction semigroup $\{T_t : t > 0\}$ on $\mathcal{H}$ and let us denote the associated resolvent by $\{G_\alpha : \alpha > 0\}$ defined as*

$$G_\alpha(u) := \int_0^\infty e^{-\alpha t} T_t(u)\, dt, \quad \text{for all } u \in \mathcal{H}, \alpha > 0. \tag{2.4}$$

Let A_s amd A_r denote the generators of the semigroup $\{T_t : t > 0\}$ and the resolvent $\{G_\alpha : \alpha > 0\}$ respectively. Then $A_s = A_r$ and is a non-positive definite self-adjoint operator.

Proof. Let $u \in \mathcal{D}(A_r)$. Then $u = G_\alpha(v)$ for some $v \in \mathcal{H}$ and

$$\begin{aligned}\frac{1}{t}\left(e^{-\alpha t}T_t(u) - u\right) &= \frac{1}{t}\left(e^{-\alpha t}T_t\left(\int_0^\infty e^{-\alpha s}T_s(v)\, ds\right) - \int_0^\infty e^{-\alpha s}T_s(v)\, ds\right)\\ &= \frac{1}{t}\left(\left(\int_t^\infty e^{-\alpha s}T_s(v)\, ds\right) - \int_0^\infty e^{-\alpha s}T_s(v)\, ds\right)\end{aligned}$$

$$= -\frac{1}{t}\left(\int_0^t e^{-\alpha s}T_s(v)\,ds\right) \xrightarrow{t\to 0} -v.$$

This implies $u \in \mathcal{D}(A_s)$ since

$$A_s(u) = \lim_{t\downarrow 0}\frac{T_t(u)-u}{t} = \lim_{t\downarrow 0}\frac{e^{-\alpha t}T_t(u)-u}{t} + \lim_{t\downarrow 0}\frac{T_t(u)(1-e^{-\alpha t})}{t} \tag{2.5}$$

$$= -v + \alpha u = \alpha u - G_\alpha^{-1}(u) = A_r(u). \tag{2.6}$$

This show that A_r is an restriction of A_s.

It remains to show that $\mathcal{D}(A_s) \subset \mathcal{D}(A_r)$. To this end, let $u \in \mathcal{D}(A_s)$. Then by the calculation in (2.5)

$$v := -\lim_{t\downarrow 0}\frac{1}{t}\left(e^{-\alpha t}T_t(u)-u\right) = -A_s(u) + \alpha u.$$

Define $w := u - G_\alpha v$. We have

$$\begin{aligned}\lim_{t\downarrow 0}\frac{1}{t}\left(e^{-\alpha t}T_t(w)-w\right) &= \lim_{t\downarrow 0}\frac{1}{t}\left(e^{-\alpha t}T_t(u)-u\right) - \lim_{t\downarrow 0}\frac{1}{t}\left(e^{-\alpha t}T_t(G_\alpha(v)) - G_\alpha(v)\right)\\ &= -v + \lim_{t\downarrow 0}\frac{1}{t}\left(-\int_t^\infty e^{-\alpha s}T_s(v)\,ds + \int_0^\infty e^{-\alpha s}T_s(v)\,ds\right) = -v + v = 0.\end{aligned}$$

Therefore $w \in \mathcal{D}(A_s)$ and

$$0 = \lim_{t\downarrow 0}\frac{1}{t}\left(e^{-\alpha t}T_t(w)-w\right) = A_s(w) - \alpha w.$$

Therefore

$$0 = \alpha\langle w, w\rangle - \langle A_s(w), w\rangle \geqslant \alpha\langle w, w\rangle,$$

and hence $w = 0$, or equivalently, $u = G_\alpha(v) \in \mathcal{D}(A_r)$.

By Corollary 2.16 G_α^{-1} is a self-adjoint operator and hence $A_r = \alpha - G_\alpha^{-1}$ is self-adjoint. By Lemma 2.10, $A_r = A_s$ is a non-positive definite operator. □

We now state a converse to Proposition 2.20. The resolvent and semigroup corresponding to the generator can be defined using functional calculus (Definition 2.19).

Proposition 2.21. *Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{D}(A) \to \mathcal{H}$ be a non-positive definite, self-adjoint operator.*

(a) Then $\{T_t = \exp(tA) | t > 0\}$ and $\{G_\alpha = (\alpha - A)^{-1} | \alpha > 0\}$ are a semigroup and resolvent respectively. Furthermore, $\{G_\alpha : \alpha > 0\}$ is the resolvent corresponding to the semigroup $\{T_t = \exp(tA) | t > 0\}$.

(b) The generators of the semigroup $\{T_t > 0 | t > 0\}$ and the resolvent $\{G_\alpha | \alpha > 0\}$ in (a) coincide with A. Furthermore, there is a unique semigroup and a unique resolvent whose generators are A.

Proof. (a) Let $U : L^2(\Omega, \mu) \to \mathcal{H}$ be an unitary operator and $\lambda : \Omega \to (-\infty, 0]$ be as given in the spectral theorem (Theorem 2.18). Then we have

$$U^{-1}T_tU = M_{\exp(t\lambda(\cdot))}, \quad U^{-1}G_\alpha U = M_{(\alpha-\lambda(\cdot))^{-1}}, \quad \text{for all } t > 0, \alpha > 0.$$

The symmetry of T_t follows from symmetry of $M_{\exp(t\lambda)}$. For all $f, g \in \mathcal{H}, t > 0$ we have

$$\begin{aligned}\langle T_t(f), g\rangle &= \int_\Omega M_{\exp(t\lambda)}(U^{-1}(f))U^{-1}(g)\,d\mu = \int_\Omega \exp(t\lambda)U^{-1}(f)U^{-1}(g)\,d\mu \\ &= \int_\Omega U^{-1}(f)M_{\exp(t\lambda)}(U^{-1}(g))\,d\mu = \langle f, T_t(g)\rangle.\end{aligned}$$

The proof of the semigroup property is similar as for all $t, s > 0$

$$T_tT_s = UM_{\exp(t\lambda)}U^{-1}UM_{\exp(s\lambda)}U^{-1} = UM_{\exp(t\lambda)}M_{\exp(s\lambda)}U^{-1} = UM_{\exp((t+s)\lambda)}U^{-1} = T_{t+s}.$$

To show strong continuity, note that for any $f \in \mathcal{H}$ and using the fact that $\lambda \leqslant 0$ m-a.e., we have $|(\exp(t\lambda) - 1)U^{-1}(f)| \leqslant U^{-1}(f)$ m-a.e. for all $t > 0$. Hence by dominated convergence theorem, we have

$$\lim_{t\downarrow 0} \|T_t(f) - f\|^2 = \lim_{t\downarrow 0} \int_\Omega \left|(\exp(t\lambda) - 1)U^{-1}(f)\right|^2 d\mu = 0, \quad \text{for all } f \in \mathcal{H}.$$

The proof that $\{G_\alpha = UM_{(\alpha-\lambda(\cdot))^{-1}}U^{-1} | \alpha > 0\}$ defines a resolvent is similar and left as an exercise.

To verify that $\{G_\alpha : \alpha > 0\}$ is the resolvent corresponding to the semigroup $\{T_t = \exp(tA) | t > 0\}$, for any $f \in \mathcal{H}$, we have

$$\begin{aligned}\int_0^\infty e^{-\alpha t}T_t(f)\,dt &= \int_0^\infty U\left(e^{-\alpha t}M_{\exp(t\lambda)}(U^{-1}(f))\right)\,dt = U\left(\int_0^\infty e^{-\alpha t}M_{\exp(t\lambda)}(U^{-1}(f))\,dt\right) \\ &= U\left(\int_0^\infty \exp(-\alpha t + \lambda t)U^{-1}(f)\,dt\right) = U\left(M_{(\alpha-\lambda(\cdot))^{-1}}(U^{-1}(f))\right) = G_\alpha(f).\end{aligned}$$

(b) For any $f \in L^2(\Omega, \mu)$, we have the *pointwise* limit

$$\lim_{t\downarrow 0} \frac{M_{\exp(t\lambda(\cdot))}(f) - f}{t} = \lim_{t\downarrow 0} \frac{(\exp(t\lambda(\cdot) - 1)(f(\cdot)}{t} = M_\lambda(f),$$

and that since $\lambda \leqslant 0$ m-a.e., we have

$$\left|\frac{\exp(t\lambda(\cdot)) - 1}{t}\right| = \frac{1 - \exp(t\lambda(\cdot))}{t} \uparrow \lambda(\cdot), m\text{-a.e. as } t \downarrow 0.$$

So by the dominated convergence theorem,

$$\lim_{t\downarrow 0} \frac{M_{\exp(t\lambda(\cdot))}(f) - f}{t} = \lim_{t\downarrow 0} \frac{(\exp(t\lambda(\cdot) - 1)(f(\cdot)}{t} = M_\lambda(f),$$

in $L^2(\Omega,\mu)$ if and only if $f \in \mathcal{D}(M_\lambda) = \{f \in L^2(\Omega,\mu) : \lambda f \in L^2(\Omega,\mu)\}$. This implies that the generator of the semigroup $\{T_t = UM_{\exp(t\lambda(\cdot))}U^{-1} : t > 0\}$ is $A = UM_\lambda U^{-1}$. Since the generators of a semigroup and the corresponding resolvent coincide (by Proposition 2.20), we have that A is the generator of the resolvent $\{G_\alpha : \alpha > 0\}$.

By the injectivity of Laplace transform, it suffices to show that the resolvent is uniquely determined by the generator. Suppose $\{G_\alpha : \alpha > 0\}$ and $\{G'_\alpha : \alpha > 0\}$ are two resolvent with generator A, for any $f \in \mathcal{H}, \alpha > 0$, $w := G_\alpha(f) - G'_\alpha(f)$ satisfies

$$(\alpha - A)(w) = (\alpha - A)(G_\alpha(f) - G'_\alpha(f)) = f - f = 0.$$

Since $-A$ is non-negative definite, we have

$$0 = \langle(\alpha - A)(w), w\rangle \geqslant \alpha\langle w, w\rangle \geqslant 0,$$

and hence $w = 0$. Since $f \in \mathcal{H}$ is arbitrary, $G_\alpha = G'_\alpha$. □

Exercise 2.22. Show that $\{G_\alpha = UM_{(\alpha-\lambda(\cdot))^{-1}}U^{-1} | \alpha > 0\}$ defined in the proof above is a resolvent.

In the next exercise, we outline a direct proof of the uniqueness of the semigroup with a given generator (without relying on the injectivity of Laplace transform and the corresponding uniqueness result for the resolvent).

Exercise 2.23. Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{D}(A) \to \mathcal{H}$ be a non-positive definite, self-adjoint operator. Let $\{T_t : t > 0\}$ be a semigroup with generator A.

(i) Show that for any $x \in \mathcal{D}(A), t > 0$, we have $T_t(x) \in \mathcal{D}(A)$ and

$$\frac{d}{dt}(T_t(x)) = A(T_t(x)) = T_t(A(x)).$$

(ii) Using (i), show that if $\{T_t : t > 0\}$ and $\{\widetilde{T}_t : t > 0\}$ are two semigroups with generator A, then for any $x \in \mathcal{D}(A)$ the function $E_x : (0,\infty) \to [0,\infty)$ defined by

$$E_x(t) = \left\|T_t(x) - \widetilde{T}_t(x)\right\|^2, \quad \text{for all } t > 0$$

satisfies

$$\frac{d}{dt}E_x(t) = 2\langle T_t(x) - \widetilde{T}_t(x), A(T_t(x) - \widetilde{T}_t(x))\rangle \leqslant 0, \quad \text{for all } t > 0$$

and $\lim_{t\downarrow 0} E_x(t) = 0$.

(iii) Conclude from (ii) that $T_t(x) = \widetilde{T}_t(x)$ for all $x \in \mathcal{H}$ and $t > 0$.

The following exercise can be viewed as an inverse Laplace transform formula for the semigroup corresponding to a resolvent.

Exercise 2.24. Let $\{G_\alpha : \alpha > 0\}$ be a resolvent on $\mathcal{H}$. Show that for all $t > 0, f \in \mathcal{H}$

$$T_t(f) = \lim_{\alpha\to\infty} e^{-t\alpha} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} (\alpha G_\alpha)^n(f) \tag{2.7}$$

defines a semigroup and that $\{G_\alpha : \alpha > 0\}$ is the resolvent corresponding to the semigroup $\{T_t : t > 0\}$. Hint: Use the proof of Proposition 2.21 to express $G_\alpha = UM_{(\alpha-\lambda(\cdot))^{-1}}U^{-1}$ and show that $T_t = UM_{\exp(t\lambda(\cdot))}U^{-1}$.

2.3 Closed quadratic forms

Definition 2.25. Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. A *quadratic form* $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a dense subspace of $\mathcal{F}$ (called the domain of the quadratic form) of $\mathcal{H}$ such that it satsfies the following properties:

(i) (bi-linearity) For all $a_1, a_2 \in \mathbb{R}$ and $f_1, f_2, g \in \mathcal{F}$, we have $\mathcal{E}(a_1 f_1 + a_2 f_2, g) = a_1\mathcal{E}(f_1, g) + a_2\mathcal{E}(f_2, g)$.

(ii) (symmetry) $\mathcal{E}(f, g) = \mathcal{E}(g, f)$ for all $f, g \in \mathcal{H}$.

(iii) (non-negative definite) $\mathcal{E}(f, f) \geqslant 0$ for all $f \in \mathcal{F}$.

We say that a quadratic form is said to *closed* if $\mathcal{F}$ is a Hilbert space equipped with the inner product $\mathcal{E}_1 : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$

$$\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \langle f, g\rangle, \quad \text{for all } f, g \in \mathcal{F}.$$

Familiar properties of inner product such as Schwarz inequality and triangle inequality hold for (non-negative definite) quadratic forms with identical proofs.

Exercise 2.26. Let $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be a quadratic form on a Hilbert space $\mathcal{H}$. Show that for all $f, g \in \mathcal{F}$, we have the Schwarz inequality

$$|\mathcal{E}(u, v)| \leqslant \mathcal{E}(u, u)^{1/2}\mathcal{E}(v, v)^{1/2}$$

and the triangle inequality

$$\mathcal{E}(f + g, f + g)^{1/2} \leqslant \mathcal{E}(f, f)^{1/2} + \mathcal{E}(g, g)^{1/2}.$$

There is a one-to-one correspondence between closed quadratic forms and non-positive definite self-adjoint operators A.

Theorem 2.27. *Let A be a non-positive definite, self-adjoint operator on $\mathcal{H}$. Then $(\mathcal{E}, \mathcal{F})$ is a closed quadratic form, where*

$$\mathcal{E}(f, g) = \langle\sqrt{-A}(f), \sqrt{-A}(g)\rangle, \quad f, g \in \mathcal{F} := \mathcal{D}(\sqrt{-A}). \tag{2.8}$$

Conversely, any closed quadratic form on $\mathcal{H}$ arises from a non-positive definite, self-adjoint operator on $\mathcal{H}$ as given in (2.8).

Proof. Since $(\mathcal{F}, \mathcal{E}_1)$ is an inner product space, we need to verify completeness. Note that $\sqrt{-A}$ is a non-negative definite, self-adjoint operator. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{F}, \mathcal{E}_1)$. This implies that $(f_n, \sqrt{-A}(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H} \times \mathcal{H}$ that belongs to the graph $\mathcal{G}(\sqrt{-A})$. Since the graph of every self-adjoint operator is closed (by Lemma 2.15), we conclude that there exists $f \in \mathcal{F}$ such that $\lim_{n\to\infty} \mathcal{E}_1(f - f_n, f - f_n) = 0$. Hence $(\mathcal{F}, \mathcal{E}_1)$ is Hilbert space.

Define $\mathcal{E}_\alpha(f, g) := \mathcal{E}(f, g) + \alpha\langle f, g\rangle$ for all $f, g \in \mathcal{F}$. Let $\{G_\alpha = (\alpha - A)^{-1} : \alpha > 0\}$ denote the resolvent generated by A. Using the spectral theorem, it is easy to verify that

$$G_\alpha(\mathcal{H}) \subset \mathcal{F}, \quad \mathcal{E}_\alpha(G_\alpha(f), g) = \langle f, g\rangle, \quad \text{for all } f \in \mathcal{H}, g \in \mathcal{F}. \tag{2.9}$$

Conversely, let $(\mathcal{E}, \mathcal{F})$ be a closed quadratic form. Since $(\mathcal{F}, \mathcal{E}_\alpha)$ is a Hilbert space for each $\alpha > 0$ and for any $u \in \mathcal{H}$, the function $v \mapsto \langle u, v\rangle$ where $v \in \mathcal{F}$ is a bounded linear functional in the Hilbert space $(\mathcal{F}, \mathcal{E}_\alpha)$. Hence by Riesz-Fréchet representation theorem, there exists (a unique) $G_\alpha(u)$ in $\mathcal{F}$ such that

$$\mathcal{E}_\alpha(G_\alpha(u), v) = \langle u, v\rangle, \quad \text{for all } u \in \mathcal{H}, v \in \mathcal{F} \text{ and } \alpha > 0. \tag{2.10}$$

We claim that $\{G_\alpha : \alpha > 0\}$ is a resolvent. For the symmetry of G_α note that for each $u, v \in \mathcal{H}$, we have

$$\langle G_\alpha(u), v\rangle \overset{(2.10)}{=} \mathcal{E}_\alpha(G_\alpha(u), G_\alpha(v)) \overset{(2.10)}{=} \langle u, G_\alpha(v)\rangle.$$

To obtain the resolvent equation, note that for any $u \in \mathcal{H}, v \in \mathcal{F}$, we have

$$\begin{aligned}
\mathcal{E}_\alpha(G_\beta(u) - (\alpha - \beta)G_\alpha G_\beta(u), v) &= \mathcal{E}_\alpha(G_\beta(u), v) - (\alpha - \beta)\mathcal{E}_\alpha(G_\alpha(G_\beta(u)), v) \\
&\overset{(2.10)}{=} \mathcal{E}_\alpha(G_\beta(u), v) - (\alpha - \beta)\langle G_\beta(u), v\rangle = \mathcal{E}_\beta(G_\beta(u), v) \\
&\overset{(2.10)}{=} \langle u, v\rangle \overset{(2.10)}{=} \mathcal{E}_\alpha(G_\alpha(u), v).
\end{aligned}$$

Since $(\mathcal{F}, \mathcal{E}_\alpha)$ is a Hilbert space, we obtain the resolvent equation

$$G_\alpha(u) = G_\beta(u) - (\alpha - \beta)G_\alpha G_\beta(u), \quad \text{for all } u \in \mathcal{H}.$$

For each $\alpha > 0$, αG_α is a contraction since

$$\|(\alpha G_\alpha)(u)\| \, \|G_\alpha(u)\| = \alpha\langle G_\alpha(u), G_\alpha(u)\rangle \leqslant \mathcal{E}_\alpha(G_\alpha(u), G_\alpha(u)) = \langle u, G_\alpha(u)\rangle \leqslant \|u\| \, \|G_\alpha(u)\|,$$

for all $u \in \mathcal{H}$. For strong continuity, we use the contraction property to obtain

$$\begin{aligned}
\alpha \|\alpha G_\alpha(u) - u\|^2 &\leqslant \mathcal{E}_\alpha(\alpha G_\alpha(u) - u, \alpha G_\alpha(u) - u\rangle \overset{(2.10)}{=} \alpha^2\langle G_\alpha(u), u\rangle + \mathcal{E}(u, u) - \alpha\langle u, u\rangle \\
&\leqslant \alpha \|\alpha G_\alpha(u)\| \, \|u\| - \alpha \|u\|^2 + \mathcal{E}(u, u) \leqslant \mathcal{E}(u, u), \quad \text{for all } u \in \mathcal{F}.
\end{aligned}$$

Hence

$$\lim_{\alpha\to\infty} \|\alpha G_\alpha(u) - u\| = 0, \quad \text{for all } u \in \mathcal{F}.$$

Since $\mathcal{F}$ is dense in $\mathcal{H}$, for any $u \in \mathcal{H}$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Now since αG_α is a contraction, we have

$$\|\alpha G_\alpha(u) - u\| \leqslant \|(\alpha G_\alpha)(u - u_n)\| + \|\alpha G_\alpha(u_n) - u_n\| + \|u_n - u\| \leqslant 2\,\|u - u_n\| + \|\alpha G_\alpha(u_n) - u_n\|\,.$$

Let $\alpha \to \infty$ and then $n \to \infty$ to obtain

$$\limsup_{\alpha\to\infty} \|\alpha G_\alpha(u) - u\| \leqslant 2\,\|u - u_n\| \xrightarrow{n\to\infty} 0.$$

This concludes the proof of strong continuity and hence $\{G_\alpha : \alpha > 0\}$ is a resolvent.

Let A be the generator of the resolvent $(\mathcal{E}', \mathcal{F}')$ closed quadratic form corresponding to the non-positive self-adjoint operator A as defined in (2.8). By (2.9), we have $G_\alpha(\mathcal{H}) \subset \mathcal{F}'$ and hence

$$\mathcal{E}'(G_\alpha(u), G_\alpha(v)) \overset{(2.9)}{=} \langle G_\alpha(u), v\rangle \overset{(2.10)}{=} \mathcal{E}_\alpha(G_\alpha(u), G_\alpha(v)), \quad \text{for all } u, v \in \mathcal{H}.$$

Hence $\mathcal{E}'$ and $\mathcal{E}$ coincide on $G_\alpha(\mathcal{H}) \times G_\alpha(\mathcal{H})$. By (2.10), $G_\alpha(\mathcal{H})$ is dense in the Hilbert space $(\mathcal{F}, \mathcal{E}_\alpha)$. Similarly, by (2.9), $G_\alpha(\mathcal{H})$ is dense in the Hilbert space $(\mathcal{F}', \mathcal{E}'_\alpha)$. Since both $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}', \mathcal{F}')$ are closed quadratic forms that coincide on a dense set, they are equal. Hence the correspondence given in (2.8) is a bijection between non-positive definite, self-adjoint operators and closed quadratic forms. □

The quadratic form corresponding to the generator of a semigroup and resolvent can be described directly as outlined below.

Exercise 2.28. Let A be a non-positive definite, self-adjoint operator on $\mathcal{H}$ that is the generator of a semigroup $\{P_t = \exp(tA) : t > 0\}$ and resolvent $\{G_\alpha = (\alpha - A)^{-1} : \alpha > 0\}$. Let $(\mathcal{E}, \mathcal{F})$ denote the closed quadratic form corresponding to A as given by Theorem 2.27. Using the spectral theorem, show the following

(a) For any $f \in \mathcal{H}$, the function

$$t \mapsto \frac{1}{t}\langle (I - P_t)f, f\rangle$$

is non-increasing and non-negative function on $(0, \infty)$. Furthermore the quadratic form $(\mathcal{E}, \mathcal{F})$ is given by

$$\mathcal{F} = \left\{ f \in \mathcal{H} \,|\, \lim_{t\downarrow 0} \frac{1}{t}\langle (I - P_t)f, f\rangle < \infty \right\},$$

and

$$\mathcal{E}(f, f) = \lim_{t\downarrow 0} \frac{1}{t}\langle (I - P_t)f, f\rangle, \quad \text{for all } f \in \mathcal{F}.$$

(b) For any $f \in \mathcal{H}$, the function

$$\alpha \mapsto \alpha\langle (I - \alpha G_\alpha)f, f\rangle$$

is non-decreasing and non-negative function on $(0, \infty)$. Furthermore the quadratic form $(\mathcal{E}, \mathcal{F})$ is given by

$$\mathcal{F} = \left\{ f \in \mathcal{H} | \lim_{\alpha \uparrow \infty} \alpha \langle (I - \alpha G_\alpha) f, f \rangle < \infty \right\},$$

and

$$\mathcal{E}(f, f) = \lim_{\alpha \uparrow \infty} \alpha \langle (I - \alpha G_\alpha) f, f \rangle, \quad \text{for all } f \in \mathcal{F}.$$

To summarize, there is a one-to-one correspondence between

(a) Closed non-negative definite quadratic forms (Definition 2.25)

(b) Non-positive definite self-adjoint operators (Definition 2.14)

(c) Semigroups (Definition 2.1)

(d) Resolvents (Definition 2.4).

The following exercise outlines an argument for the computation of Dirichlet form for the Brownian motion. It assumes familiarity with Fourier transform of tempered distributions.

Exercise 2.29. Let m denote the Lebesgue measure and let $P_t : L^2(\mathbb{R}^n, m) \to L^2(\mathbb{R}^n, m)$ denote the Brownian semigroup

$$P_t f(x) := \int_{\mathbb{R}^n} \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t} \right) f(y)\, dy, \quad \text{for all } t > 0, f \in L^2(\mathbb{R}^n).$$

Show using Exercise 2.28-(a) that the corresponding Dirichlet form is given by $\mathcal{F} = \{f \in L^2(\mathbb{R}^n) : \frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}^n), \text{for all } i = 1, \ldots, n\}$, $\mathcal{E}(f, f) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla f|^2 \, dm$ for all $f \in \mathcal{F}$, where $\nabla f = \left(\frac{\partial f}{\partial x_i} \right)_{i=1}^n$ is the distributional gradient of f.
Hint: The following facts about Fourier transform and distributional derivatives are relevant. Fourier Transform of a function of a function $f \in L^2(\mathbb{R}^n)$ is given by

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^n\mathbf{x}, \quad \text{for all } \mathbf{k} \in \mathbb{R}^n$$

Inverse Fourier Transform of $\hat{f} \in L^2(\mathbb{R}^n)$ is

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^n\mathbf{k}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

The Parseval's Formula states

$$\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 d^n\mathbf{x} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{f}(\mathbf{k})|^2 d^n\mathbf{k}, \quad \text{for all } f \in L^2(\mathbb{R}^n.)$$

The Gaussian probability density function g with zero mean and covariance matrix tI (where I is the identity matrix)

$$g(\mathbf{x}) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{2t}}$$

has Fourier transform given by

$$\hat{g}(\mathbf{k}) = e^{-\frac{t|\mathbf{k}|^2}{2}}.$$

Fourier trasform of convolution is the product of Fourier transform $\widehat{f_1 * f_2} = \hat{f}_1 \hat{f}_2$ for all $f_1, f_2 \in L^2(\mathbb{R}^n)$. Fourier transform of the distributional derivative of $f \in L^2(\mathbb{R}^n)$ is

$$\widehat{\frac{\partial f}{\partial x_i}}(\mathbf{k}) := -k_i \hat{f}(\mathbf{k}), \quad \text{for all } \mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{R}^n \text{ and } i = 1, \ldots, n.$$

2.4 Beurling-Deny criterion and Markov operators

We introduce the definition of Dirichlet form, a notion due to Beurling and Deny [BD].

Definition 2.30. Let $(X, \mathcal{M}, m)$ be a σ-finite measure space. A Dirichlet form on $L^2(X, m)$ is a *quadratic form* (in the sense of Definition 2.25; that is, bi-linear, symmetric, non-negative definite, closed, densely defined) $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ on $L^2(X, m)$ such that it satisfies the *Markov property*: for all $u \in \mathcal{F}$, we have $\tilde{u} := (0 \vee u) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\tilde{u}, \tilde{u}) \leqslant \mathcal{E}(u, u)$.

At this point the terminology Markov property might seem strange. As we will later justify (see Theorem 2.36), the Markov property of quadratic form is equivalent to the Markovian property of the corresponding semigroup (or resolvent).

Definition 2.31. We say that a bounded linear map $T : L^2(X, m) \to L^2(X, m)$ is Markovian if for any $f \in L^2(X, m)$ such that $0 \leqslant f \leqslant 1$ m-almost everywhere, we have $0 \leqslant T(f) \leqslant 1$ m-a.e.

Exercise 2.32. Let $T : L^2(X, m) \to L^2(X, m)$ be a bounded linear operator. Then show that the following are equivalent.

(a) For any $f \in L^2(X, m)$ such that $0 \leqslant f \leqslant 1$ m-almost everywhere, we have $0 \leqslant T(f) \leqslant 1$ m-a.e.

(b) For any $f \in L^2(X, m)$ such that $f \leqslant 1$ m-almost everywhere, we have $T(f) \leqslant 1$ m-a.e.

The following exercise gives a description of all Dirichlet forms on a finite set. This special case of the *Beurling-Deny decomposition* of regular Dirichlet forms admits an elementary proof as outlined below. We use $\mathbb{1}_A$ to denote the indicator of a set A.

Exercise 2.33. Let $X = \{1, \ldots, m\}$, where $n \in \mathbb{N}$ and let m be a measure on X with $m(\{i\}) > 0$ for all $i \in X$. Let $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be a Dirichlet form on $L^2(X, m)$.

(i) Prove that $\mathcal{F} = L^2(X, m)$.

(ii) Show that for any $i, j \in X$ with $i \neq j$, we have

$$\mathcal{E}(\mathbb{1}_{\{i\}}, \mathbb{1}_{\{j\}}) \leqslant 0.$$

Hint: Consider $f = \mathbb{1}_{\{i\}} - \epsilon \mathbb{1}_{\{j\}}$ for $\epsilon \geqslant 0$.

(iii) Show that for any $i \in X$, we have $\mathcal{E}(\mathbb{1}_{\{i\}}, \mathbb{1}_X) \geqslant 0$. **Hint**: Consider $f = \mathbb{1}_X + \epsilon \mathbb{1}_{\{i\}}$ for $\epsilon \geqslant 0$.

(iv) Using (ii) and (iii), show that there exists $c_{ij} \geqslant 0$, for all $1 \leqslant i < j \leqslant n$ and $k(i) \geqslant 0$ for all $i \in X$ such that

$$\mathcal{E}(f, f) = \sum_{1 \leqslant i < j \leqslant n} c_{ij}(f(i) - f(j))^2 + \sum_{i=1}^{n} k(i) f(i)^2, \quad \text{for all } f \in \mathcal{F} = \mathbb{R}^X.$$

The next exercise illustrates the existence of closed, non-negative definite, symmetric quadratic forms which do not satisfy the Markovian property.

Exercise 2.34. Let $X = \{1, 2\}$ be equipped with the counting measure m and let $(\mathcal{E}, \mathcal{F})$ define a symmetric, bi-linear form on $L^2(X, m)$ given by $\mathcal{F} = L^2(X, m)$,

$$\mathcal{E}(f, f) = (f(1) - f(2))^2 + f(1)^2 + \lambda \sum_{i=1}^{2} f(i)^2,$$

where $\lambda \in \mathbb{R}$ is a parameter, $f \in \mathcal{F}$. For $f, g \in \mathcal{F}$, we define $\mathcal{E}(f, g)$ by polarization.

(a) Show that $(\mathcal{E}, \mathcal{F})$ is a non-negative definite, closed, quadratic form on $L^2(X, m)$ if and only if $\lambda \geqslant -\frac{1}{2}(3 - \sqrt{5})$.

(b) Show that $(\mathcal{E}, \mathcal{F})$ is a Markovian, non-negative definite, closed, quadratic form on $L^2(X, m)$ if and only if $\lambda \geqslant 0$.

(c) Conclude that there exist (contraction, strongly continuous) semigroups that do not satisfy the Markovian property[3].

Lemma 2.35. *Let $P : L^2(X, m) \to L^2(X, m)$ be a bounded, linear, m-symmetric, Markovian operator and let $F : \mathbb{R} \to \mathbb{R}$ be a 1-Lipschitz function (that is, $|F(a) - F(b)| \leqslant |a - b|$ for all $a, b \in \mathbb{R}$) such that $F(0) = 0$.*

(i) Then for any $n \in \mathbb{N}$, for all pairwise disjoint sets $A_1, \ldots, A_n$ such that $m(A_i) < \infty$ for all $i = 1, \ldots, n$, and for all $a_1, \ldots, a_n \in \mathbb{R}$, writing $f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$, we have

$$\langle (I - P) f, f \rangle_{L^2(X,m)} = \sum_{i=1}^{n} \mu_i a_i^2 + \frac{1}{2} \sum_{1 \leqslant i < j \leqslant n} \alpha_{i,j}(a_i - a_j)^2, \tag{2.11}$$

[3]Thanks to Renan Gross for asking a question that inspired this exercise and for helpful discussions.

where

$$\alpha_{i,j} := \langle \mathbb{1}_{A_i}, P\mathbb{1}_{A_j}\rangle_{L^2(X,m)} \geqslant 0, \quad \mu_i = m(A_i) - \sum_{k=1}^{n} \alpha_{i,k} \geqslant 0, \quad \textit{for all } 1 \leqslant i, j \leqslant n. \tag{2.12}$$

(ii) For any $g \in L^2(X,m)$, then $\widetilde{g} := F(g)$ satisfies

$$\langle (I-P)\widetilde{g}, \widetilde{g}\rangle_{L^2(X,m)} \leqslant \langle (I-P)g, g\rangle_{L^2(X,m)}. \tag{2.13}$$

Proof. (i) Note that by the m-symmetry of P, we have $\alpha_{i,j} = \langle \mathbb{1}_{A_i}, P\mathbb{1}_{A_j}\rangle_{L^2(X,m)} = \langle P\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\rangle_{L^2(X,m)} = \alpha_{j,i}$ for all $1 \leqslant i, j \leqslant n$. By linearity of P and the symmetry $\alpha_{i,j} = \alpha_{j,i}$, we have

$$\begin{aligned}
\langle (I-P)f, f\rangle_{L^2(X,m)} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \langle (I-P)\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\rangle_{L^2(X,m)} \\
&= \sum_{i=1}^{n} a_i^2 m(A_i) - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \alpha_{i,j} \\
&= \sum_{i=1}^{n} \mu_i a_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_i^2 \alpha_{i,j} - a_i a_j \alpha_{i,j}\right) \\
&= \sum_{i=1}^{n} \mu_i a_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_i^2 \alpha_{i,j} + a_j^2 \alpha_{i,j} - 2a_i a_j \alpha_{i,j}\right) \\
&= \sum_{i=1}^{n} \mu_i a_i^2 + \frac{1}{2}\sum_{1\leqslant i<j\leqslant n} \alpha_{i,j}(a_i - a_j)^2.
\end{aligned}$$

Since $0 \leqslant P\mathbb{1}_{A_j} \leqslant 1$ m-a.e. by the Markovian property of P, we have $\alpha_{i,j} = \langle \mathbb{1}_{A_i}, P\mathbb{1}_{A_j}\rangle_{L^2(X,m)} \geqslant 0$. Since $A_1, \ldots, A_n$ are pairwise disjoint, by the Markovian property of P, we have

$$\sum_{k=1}^{n} P\left(\mathbb{1}_{A_k}\right) = P\left(\sum_{k=1}^{n} \mathbb{1}_{A_k}\right) \leqslant 1, \quad m\text{-a.e.}$$

Hence

$$\sum_{k=1}^{n} \alpha_{i,k} = \langle \mathbb{1}_{A_i}, \sum_{k=1}^{n} P\left(\mathbb{1}_{A_k}\right)\rangle \leqslant \int_X \mathbb{1}_{A_i}\, dm = m(A_i), \quad \text{for all } 1 \leqslant i \leqslant n,$$

or equivalently $\mu_i \geqslant 0$ for all $1 \leqslant i \leqslant n$.

(ii) If g is a simple function as given in (i), then the desired estimate (2.13) follows from (2.11) and (2.12). In general, for any $g \in L^2(X,m)$, there exists a sequence of simple function $(g_n)_{n\in\mathbb{N}}$ such that $g_n \xrightarrow{n\to\infty} g$ pointwise, $|g_n| \uparrow |g|$ pointwise, and $g_n \xrightarrow{n\to\infty} g$ in $L^2(X,m)$. Since $|F(a)| \leqslant |a|$ for all $a \in \mathbb{R}$, by the dominated convergence

theorem $\widetilde{g_n} := F(g_n) \xrightarrow{n\to\infty} \widetilde{g} := F(g)$ in $L^2(X,m)$. Hence by the continuity of P, $P(\widetilde{g_n}) \xrightarrow{n\to\infty} P\widetilde{g}$ in $L^2(X,m)$ hence by using (2.13) for simple functions (since $\widetilde{g_n}$ are also simple functions) we obtain

$$\begin{aligned}\langle (I-P)\widetilde{g}, \widetilde{g}\rangle_{L^2(X,m)} &= \lim_{n\to\infty} \langle (I-P)\widetilde{g_n}, \widetilde{g_n}\rangle_{L^2(X,m)} \\ &\overset{(2.11),(2.12)}{\leqslant} \lim_{n\to\infty} \langle (I-P)g_n, g_n\rangle_{L^2(X,m)} = \lim_{n\to\infty} \langle (I-P)g, g\rangle_{L^2(X,m)}.\end{aligned}$$

□

Let $(X, \mathcal{B}, m)$ be a σ-finite measure space.

Theorem 2.36. *Let $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be a closed quadratic form on $L^2(X,m)$. Let $\{T_t : t > 0\}$, $\{G_\alpha : \alpha > 0\}$ and $A : \mathcal{D}(A) \to L^2(X,m)$ denote the associated semigroup, resolvent and generator associated with $\mathcal{E}$ respectively. Then the following are equivalent:*

(a) T_t is a Markovian operator for each $t > 0$.

(b) αG_α is a Markovian operator for each $t > 0$.

(c) For all $f \in \mathcal{D}(A)$, we have $\langle A(f), (f-1)^+\rangle \leqslant 0$.

(d) For all $u \in \mathcal{F}$, we have $\widetilde{u} := (0 \vee u) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\widetilde{u}, \widetilde{u}) \leqslant \mathcal{E}(u,u)$.

Proof. The implications (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (a) follow easily from (2.4) and (2.7).

Next, let us show that (d) $\Longrightarrow$ (b). Let $u \in \mathcal{F}$ such that $0 \leqslant u \leqslant 1$ m-a.e. and $\alpha > 0$. We consider the function $\Psi_u : \mathcal{F} \to \mathbb{R}$ defined by

$$\Psi(w) := \mathcal{E}(w,w) + \alpha\langle w - \alpha^{-1}u, w - \alpha^{-1}u\rangle.$$

For all $w \in \mathcal{F}$, we have

$$\begin{aligned}\Psi(w) - \Psi(G_\alpha(u)) &= \mathcal{E}_\alpha(w,w) - 2\langle w,u\rangle + \alpha^{-1}\langle u,u\rangle \\ &\quad - \mathcal{E}_\alpha(G_\alpha(u), G_\alpha(u)) + 2\langle G_\alpha(u), u\rangle - \alpha^{-1}\langle u,u\rangle \\ &= \mathcal{E}_\alpha(w,w) - 2\langle w,u\rangle + \mathcal{E}_\alpha(G_\alpha(u), G_\alpha(u)) \\ &= \mathcal{E}_\alpha(w,w) - 2\mathcal{E}_\alpha(G_\alpha(u), w) + \mathcal{E}_\alpha(G_\alpha(u), G_\alpha(u)) \\ &= \mathcal{E}_\alpha(G_\alpha(u) - w, G_\alpha(u) - w),\end{aligned}$$

and hence

$$\Psi(w) \leqslant \Psi(G_\alpha(u)) \quad \text{if and only if} \quad w = G_\alpha(u). \tag{2.14}$$

Now, define $v = \alpha^{-1}\left((0 \vee (\alpha G_\alpha(u))) \wedge 1\right) = (0 \vee G_\alpha(u)) \wedge \alpha^{-1}$. So by (d), we have $v \in \mathcal{F}$ and

$$\mathcal{E}(v,v) \leqslant \alpha^{-2}\mathcal{E}(\alpha G_\alpha(u), \alpha G_\alpha(u)) = \mathcal{E}(G_\alpha(u), G_\alpha(u)). \tag{2.15}$$

Since $u(x) \in [0,1]$ for m-a.e., we have for m-a.e. $x \in X$,

$$\left|v(x) - \alpha^{-1}u(x)\right| = \left|\left((0 \vee G_\alpha(u)(x)) \wedge \alpha^{-1}\right) - \alpha^{-1}u(x)\right| \leqslant \left|G_\alpha(u)(x) - \alpha^{-1}u(x)\right|. \tag{2.16}$$

By (2.15), (2.16), we have $\Psi(v) \leqslant \Psi(G_\alpha(u))$ and hence by (2.16), we conclude that $v = (0 \vee G_\alpha(u)) \wedge \alpha^{-1} = G_\alpha(u)$. Thus αG_α is a Markovian operator.

The implications (a) $\Longrightarrow$ (d) and (b) $\Longrightarrow$ (d) follows from Lemma 2.35 (by setting $F(t) = (0 \vee t) \wedge 1$ in Lemma 2.35-(ii)) and Exercise 2.28.

Next, let us show (b) $\Longrightarrow$ (c). By using Lemma 2.35 (by setting $F(t) = t \wedge 1$) and Exercise 2.28(b), we obtain that for all $f \in \mathcal{F}$, $f \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(f \wedge 1, f \wedge 1) \leqslant \mathcal{E}(f,f)$. In particular for $f \in \mathcal{D}(A)$, we have $f \wedge 1 = f - (f-1)_+$ and hence $\mathcal{E}(f - (f-1)_+, f - (f-1)_+) \leqslant \mathcal{E}(f,f)$ or equivalently,

$$2\langle A(f), (f-1)_+\rangle = -2\mathcal{E}(f, (f-1)_+) \leqslant -\mathcal{E}((f-1)_+, (f-1)_+) \leqslant 0.$$

It remains to show (c) $\Longrightarrow$ (b). By Exercise 2.32, it suffices to show that if $f \in L^2(X,m)$ satisfies $f \leqslant 1$ m-a.e., then $\alpha G_\alpha(f) \leqslant 1$ for all $\alpha > 0$. Fix $\alpha > 0$ and f as above and set $g := \alpha G_\alpha(f)$ so that $g \in \mathcal{D}(A)$ and

$$\alpha g - A(g) = \alpha f. \tag{2.17}$$

By (2.17) and (c) $\langle A(g), (g-1)^+\rangle \leqslant 0$, we obtain $\langle g, (g-1)_+\rangle \leqslant \langle f, (g-1)_+\rangle$, or equivalently,

$$\int_{\{g \geqslant 1\}} (g-f)(g-1)_+\rangle \leqslant 0. = \int_X (g-f)(g-1)_+\rangle \geqslant 0.$$

Since $f \leqslant 1$ m-a.e., the above inequality implies that $g = \alpha G_\alpha(f) \leqslant 1$ m-a.e. □

In order to show the existence of heat kernel, it is useful to consider the Markovian semigroup (T_t) on $L^2(X,m)$ also as an operator on $L^1(X,m)$. By Markov property, we have $|T_t(f)| \leqslant T_t(|f|)$ m-a.e. for any $f \in L^1(X,m) \cap L^2(X,m)$ (since T_t is positivity preserving). Let A_n be a sequence of increasing measurable sets with $\cup_n A_n = X$ and $m(A_n) < \infty$ for all n. By using Markov property and symmetry of T_t, for all $f \in L^1(X,m) \cap L^2(X,m)$, we have

$$\|T_t(f)\|_1 = \lim_{n\to\infty} \int \mathbb{1}_{A_n} |T_t(f)|\, dm \leqslant \lim_{n\to\infty} \int \mathbb{1}_{A_n} T_t(|f|)\, dm = \lim_{n\to\infty} \int T_t(\mathbb{1}_{A_n})|f|\, dm \leqslant \int |f|\, dm.$$

Hence by the density of $L^1(X,m) \cap L^2(X,m)$ in $L^1(X,m)$, T_t uniquely extends as a contraction on L^1; that is

$$\|P_t(f)\|_1 \leqslant \|f\|_1, \quad \text{for all } f \in L^1(X,m). \tag{2.18}$$

By the Riesz-Thorin interpolation theorem, T_t extends as a contraction to $L^p(X,m)$ for all $p \in [1,2]$. By considering the (Banach space) adjoint of $T_t : L^p(X,m) \to L^p(X,m)$ for $p \in [1,2)$, we obtain a contraction operator $T_t : L^q(X,m) \to L^q(X,m)$ for all $q \in (2,\infty]$. Since we only need the case $p = 1$ or $p = 2$ we don't provide further details on the Riesz-Thorin interpolation theorem or the duality argument.

The following lemma provides a convenient sufficient condition for the strong continuity property of a semigroup.

Lemma 2.37. *Let $\mathcal{H} = L^2(X, m)$ be a Hilbert space over $\mathbb{R}$, where m is a measure on X. Consider a family of linear operators $\{T_t : L^2(X, m) \to L^2(X, m) | t > 0\}$ such that each T_t is a Markovian, contraction operator. Furthermore if $\mathcal{L} \subset L^1(X, m) \cap L^\infty(X, m)$ is a dense subspace of $L^2(X, m)$ such that for any $f \in \mathcal{L}$, we have*

$$\lim_{t \downarrow 0} T_t(f)(x) = f(x) \quad \text{for } m\text{-a.e. } x \in X. \tag{2.19}$$

Then for any $u \in L^2(X, m)$, we have $\lim_{t\downarrow 0} \|T_t(u) - u\|_2 = 0$.

Proof. Since T_t is a contraction, we have

$$\|T_t(f) - f\|_2^2 = \|T_t(f)\|_2^2 + \|f\|_2^2 - 2\langle f, T_t(f)\rangle \leqslant 2\|f\|_2^2 - 2\langle f, T_t(f)\rangle, \quad \text{for all } f \in L^2(X, m). \tag{2.20}$$

If $f \in \mathcal{L}$, by (2.19) and the dominated convergence theorem (dominating function is $\|f\|_\infty f \in L^1(X, m)$ by the Markovian property of T_t; by Exercise 2.32 we have $\|T_t(f)\|_\infty \leqslant \|f\|_\infty$) we have

$$\lim_{t \downarrow 0} \langle f, T_t(f)\rangle = \|f\|_2^2. \tag{2.21}$$

Hence by (2.20) and (2.21) we have $\lim_{t\downarrow 0} \|T_t(f) - f\|_2 = 0$ for all $f \in \mathcal{L}$. By the density of $\mathcal{L}$ in $L^2(X, m)$ and the contraction property of T_t, we have $\lim_{t\downarrow 0} \|T_t(f) - f\|_2 = 0$ for all $f \in L^2(X, m)$ □

2.5 Regular Dirichlet forms and Fukushima's theorem

The theory of Dirichlet forms is useful to *construct* and *analyze* Markov processes. We have already seen the construction of a Markovian semigroup from the Dirichlet form. There are two issues with defining a Markov process on X from a Markovian semigroup on $L^2(X, m)$. The first issue is that it is possible for $T_t\mathbb{1}(x) < 1$, which implies that the semigroup at time t correponding to the process started at x has a probability strictly less than one to belong in the state space X (sub-probability measure). This issue is handled by adding an absorbing state (called the cemetery state Δ) and thus defining a transition probability measure on $X_\Delta := X \cup \{\Delta\}$. The second issue is that functions in $L^2(X, m)$ are not pointwise defined. So the resulting process is well-defined up to modification on 'small' sets.

Definition 2.38. Let $(X, \mathcal{M}, m)$ be a σ-finite metric measure space, and let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$. We say the Dirichlet form $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ is *regular*, if it satisfies the following properties:

(a) X is a locally compact separable metrizable topological space X, with $\mathcal{M}$ the associated Borel σ-field, a Radon measure m on X with full support (a Borel measure m on X which is finite on any compact subset of X and strictly positive on any non-empty open subset of X).

(b) The vector space $\mathcal{F} \cap C_c(X)$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(C_c(X), \|\cdot\|_\infty)$.

A fundamental theorem of Fukushima the assumption of the regularity of the Dirichlet form allows us to construct a m-symmetric Markov process on X whose semigroup coincides with the semigroup of the Dirichlet form. Assume for the moment that the semigroup corresponding to a Dirichlet form is defined over a space of pointwise well-defined functions. Then the Markov property along with Riesz–Markov–Kakutani representation theorem would then imply the existence of a sub-probability measure on X which can be made into a transition probability on $X \cup \Delta$, where Δ is an absorbing cemetery state. Fukushima observed that the assumption of regularity ensures that every function in the domain of the form can be modified to continuous outside a small set (quasi-continuous). This allows us to overcome the difficulty of functions in L^2 being not pointwise well-defined.

Theorem 2.39 (Fukushima's theorem)**.** *[FOT, Theorem 7.2.1] Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$. Let $\mathcal{B}$ denote the Borel σ-field on X. Let $X \cup \{\Delta\}$ denote the one-point compactification of X and let $\mathcal{B}_\Delta = \mathcal{B} \cup \{B \cup \Delta | B \in \mathcal{B}\}$. For each $x \in X \cup \{\Delta\}$, there is a $X_\Delta := X \cup \{\Delta\}$-valued stochastic process $(\Omega, (\mathcal{F}_t)_{t \geqslant 0}, \{Y_t : t \in [0, \infty]\}, \{\mathbb{P}_x\}_{x \in X \cup \{\Delta\}})$ that satisfies the following properties:*

1. *$(\mathcal{F}_t)_{t \geqslant 0}$ is a right continuous filtration on Ω and (Y_t) is $(\mathcal{F}_t)_{t \geqslant 0}$-adapted; that is $Y_t : (\Omega, \mathcal{F}_t) \to (X \cup \Delta, \mathcal{B}_\Delta)$ is measurable for each $t \geqslant 0$.*

2. *For each $E \in \mathcal{B}_\Delta$ and $t > 0$, the function $x \mapsto \mathbb{P}_x(Y_t \in E)$ is measurable on $(X \cup \Delta, \mathcal{B}_\Delta)$.*

3. *(Markov property) For each $x \in X, t, s \geqslant 0$ and $E \in \mathcal{B}$ we have*
$$\mathbb{P}_x(Y_{t+s} \in E | \mathcal{F}_t) = \mathbb{P}_{Y_t}(Y_s \in E).$$

4. *(cemetery is absorbing) $Y_\infty(\omega) = \Delta$ for all $\omega \in \Omega$, $\mathbb{P}_\Delta(Y_t = \Delta) = 1$ for all $t \geqslant 0$. More generally, $\mathbb{P}_x(Y_t(\omega) = \Delta) = 1$ for all $t \geqslant \zeta(\omega)$, where $\zeta(\omega)$ denotes the* lifetime of the process
$$\zeta(\omega) := \inf\{t \geqslant 0 : Y_t(\omega) = \Delta\}.$$

5. *(normal process) For each $x \in X$, we have $\mathbb{P}_x(Y_0 = x) = 1$.*

6. *(càdlàg paths) $t \mapsto Y_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$.*

7. *(time shift operators) For each $t \geqslant 0$, there exists a* time shift operator *$\theta_t : \Omega \to \Omega$ such that $Y_s \circ \theta_t = Y_{t+s}$.*

8. *(strong Markov property) For any probability measure μ on $(X \cup \{\Delta\}, \mathcal{B}_\Delta)$ and for any $(\mathcal{F}_t)$-stopping time T, and for all $s \geqslant 0$, we have*
$$\mathbb{P}_\mu\left(Y_{T+s} \in E | \mathcal{F}_T\right) = \mathbb{P}_{Y_T}(Y_s \in E), \quad \text{for all } E \in \mathcal{B}_\Delta.$$

9. *The process is* quasi-left continuous *on $(0, \infty)$: for any sequence of $(\mathcal{F}_t)_{t \geqslant 0}$ stopping times $T_n \uparrow T$ and any probability measure μ on $(X \cup \{\Delta\}, \mathcal{B}_\Delta)$, we have*
$$\mathbb{P}_\mu\left(\lim_{n \to \infty} Y_{T_n} = Y_T, T < \infty\right) = \mathbb{P}_\mu(T < \infty).$$

The semigroup (P_t^Y) corresponding to the process (Y_t) defined by $P_t^Y f(x) = \mathbb{E}_x[f(Y_t)\mathbb{1}_{\{t<\zeta\}}]$ coincides with the semigroup corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ (the term $\mathbb{1}_{\{t<\zeta\}}$ is usually dropped with the convention that every function f on X is extended to $X \cup \{\Delta\}$ by setting $f(\Delta) = 0$).

The Markov process in Fukushima's theorem is not quite unique because the semigroup associated to a Dirichlet form is not well-defined pointwise. However, it is essentially unique outside a very small set as shown in [FOT, Theorem 4.2.8]. We describe the uniqueness below.

Let $X \cup \{\Delta\}$-valued stochastic process $(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, \{Y_t : t \in [0, \infty]\}, \{\mathbb{P}_x\}_{x\in X\cup\{\Delta\}})$ be a stochastic process as above. We say that subset $\mathcal{N}$ of X is *properly exceptional* for the Markov process $\{Y_t : t \in [0, \infty]\}$ if $\mathcal{N}$ is a Borel set with $m(\mathcal{N}) = 0$ and

$$\mathbb{P}_x\left(\{\omega \in \Omega | Y_t(\omega) \in X_\Delta \backslash \mathcal{N}, Y_{t-}(\omega) \in X_\Delta \backslash \mathcal{N}\}, \text{for all } t \geqslant 0\right) = 1, \quad \text{for all } x \in X \backslash \mathcal{N}.$$

The process constructed in Fukushima's theorem is unique in the following sense. Suppose $\{Y_t : t \geqslant 0\}$ and $\{\widetilde{Y}_t : t \geqslant 0\}$ be two Markov processes that satisfy the conclusion of Theorem 2.39, then there is a common properly exceptional set $\mathcal{N}$ for both $\{Y_t : t \geqslant 0\}$ and $\{\widetilde{Y}_t : t \geqslant 0\}$ such that for all $x \in X\backslash\mathcal{N}$, for all $t \geqslant 0$ and $E \in \mathcal{B}$, we have

$$\mathbb{P}_x(Y_t \in E) = \widetilde{\mathbb{P}}_x(\widetilde{Y}_t \in E).$$

Let us recall the definition of quasi-continuous functions. First, we define the 1-*capacity* $\mathrm{Cap}_1(A)$ of $A \subset X$ with respect to $(X, m, \mathcal{E}, \mathcal{F})$ by

$$\mathrm{Cap}_1(A) := \inf\big\{\mathcal{E}_1(f, f) \;\big|\; f \in \mathcal{F},\ f \geqslant 1 \ m\text{-a.e. on a neighborhood of } A\big\}, \tag{2.22}$$

where $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(X,m)}$ as defined before. A subset $\mathcal{N}$ of X is said to be $\mathcal{E}$-*polar* if $\mathrm{Cap}_1(\mathcal{N}) = 0$. For $A \subset X$ and a statement $\mathcal{S}(x)$ on $x \in A$, we say that $\mathcal{S}$ holds $\mathcal{E}$-*quasi-everywhere on* A ($\mathcal{E}$-*q.e. on* A for short), or $\mathcal{S}(x)$ holds for $\mathcal{E}$-*quasi-every* $x \in A$ ($\mathcal{E}$-*q.e.* $x \in A$ for short), if $\mathcal{S}(x)$ holds for any $x \in A\backslash\mathcal{N}$ for some $\mathcal{E}$-polar $\mathcal{N} \subset X$.

Polar sets and properly exceptional are defined using the Dirichlet form $(\mathcal{E}, \mathcal{F})$ and the corresponding Markov process $(Y_t)_{t\geqslant 0}$ respectively. These two notions are closely related as follows: any properly exceptional set $\mathcal{N} \subset X$ for $(Y_t)_{t\geqslant 0}$ is $\mathcal{E}$-polar, c any $\mathcal{E}$-polar subset of X is included in some properly exceptional set $\mathcal{N}$ for $(Y_t)_{t\geqslant 0}$.

Definition 2.40 (Quasi-continuous function)**.** Let $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(X, m)$. Let $f \in L^2(X, m)$. A *quasi-continuous version* of f is a (pointwise defined) function $\widetilde{f} : X \to \mathbb{R}$, $\widetilde{f} = f$ m-a.e. and for $\epsilon > 0$ there exists an open set G of X such that $\mathrm{Cap}_1(G) < \epsilon$ and $\widetilde{f}\Big|_{X\backslash G}$.

A crucial ingredient in the proof of Theorem 2.39 is the existence of quasicontinuous modification of functions in $\mathcal{F}$ [FOT, Theorem 2.1.3].

Theorem 2.41. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$. Then every function $f \in \mathcal{F}$ admits a quasi-continuous modification $\widetilde{f} : X \to \mathbb{R}$ such that $f = \widetilde{f}$ m-a.e. Furthermore any two quasi-continuous modifications are equal quasi-everywhere.*

We will use the notation $\widetilde{f}$ to denote the quasi-continuous modification of a function $f \in \mathcal{F}$.

Remark 2.42. The assumption of local compactness in the definition of regular Dirichlet forms does not allow for some infinite dimensional examples. There is a fruitful generalization of regular Dirichlet form that allows for such examples called *quasi-regular Dirichlet forms*. We refer to [AM, CF] for more on this theory. The theory of quasi-regular Dirichlet forms allows to handle many infinite-dimensional spaces of interest; see [DeS, HKK, Suz, Stu25] for some recent examples.

We discuss conditions on a regular Dirichlet form that ensure continuity of the sample paths of the associated Markov process. For a function $f \in L^2(X, m)$, we denote the support of the measure $f \cdot m$ as $\mathrm{supp}_m(f)$.

Definition 2.43. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$. We say that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *strongly local* $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with $\mathrm{supp}_m[f]$, $\mathrm{supp}_m[g]$ compact and $\mathrm{supp}_m[f - a\mathbb{1}_X] \cap \mathrm{supp}_m[g] = \varnothing$ for some $a \in \mathbb{R}$.
We say that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *local* $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with $\mathrm{supp}_m[f]$, $\mathrm{supp}_m[g]$ compact and $\mathrm{supp}_m[f] \cap \mathrm{supp}_m[g] = \varnothing$ for some $a \in \mathbb{R}$.

By [FOT, Theorem 4.5.3], for a process (Y_t) associated with a strongly local Dirichlet form, we have for q.e. $x \in X$,

$$\mathbb{P}_x\big([0, \infty) \ni t \mapsto Y_t \in X_\Delta \text{ is continuous}\big) = 1.$$

For a local Dirichlet form, the corresponding process is allowed to have a discontinuity at its lifetime (that is, a jump to cemetery state). By [FOT, Theorem 4.5.3], for a process (Y_t) associated with a local Dirichlet form, we have for q.e. $x \in X$,

$$\mathbb{P}_x\big([0, \zeta) \ni t \mapsto Y_t \in X \text{ is continuous}\big) = 1.$$

Every regular Dirichlet form can be canonically decomposed into three parts: strongly-local part (diffusion), jumping part (jumping between two points in X) and killing part (junp from a point in X to the cemetery state Δ). This is called the Beurling-Deny decomposition. We state this first before discussing the probabilistic meaning.

Theorem 2.44 (Beurling-Deny decomposition)**.** *For a regular Dirichlet fpr $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ there exists a unique triple $(\mathcal{E}^{(c)}, J, \kappa)$ of a strongly local non-negative definite symmetric bilinear form $\mathcal{E}^{(c)} \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$, a symmetric Radon measure J on $X \times X \setminus \mathrm{diag}_X$ and a Radon measure κ on X, such that $J((X \times \mathcal{N}_1) \cap X^2_{\mathrm{od}}) = 0 = \kappa(\mathcal{N}_1)$ for any $\mathcal{E}$-polar $\mathcal{N}_1 \in \mathcal{B}(X)$ and*

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \frac{1}{2} \int_{X \times X \setminus \mathrm{diag}_X} (\widetilde{u}(x) - \widetilde{u}(y))(\widetilde{v}(x) - \widetilde{v}(y))\, J(dx\, dy) + \int_X \widetilde{u}(x)\widetilde{v}(x)\, \kappa(dx)$$

for any $u, v \in \mathcal{F}$, where $\widetilde{u}, \widetilde{v}$ denote $\mathcal{E}$-quasi-continuous versions of u, v respectively and $\mathrm{diag}_X \subset X \times X$ denotes the diagonal subset. We call $\mathcal{E}^{(c)}, J, \kappa$ the strongly local part*, the* jumping measure *and the* killing measure*, respectively, of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$.*

We remark that the condition $J((X \times \mathcal{N}_1) \cap X^2_{\mathrm{od}}) = 0 = \kappa(\mathcal{N}_1)$ for any $\mathcal{E}$-polar $\mathcal{N}_1 \in \mathcal{B}(X)$ ensures that the above integrals are well-defined as quasi-continuous modifications are well-defined up to quasi-everywhere equivalence. It might be possible for κ to be singular to m and J be singular to $m \times m$ and hence use of quasi-continuous versions is needed in general.

For a probabilistic description of jumping and killing measures, let us make a simplifying assumption that $\mathbb{1}_A \in \mathcal{F}$ and is quasi-continuous for all Borel sets $A \in \mathcal{B}(X)$ (for example, a finite set as the state space). In this case, for any pair of disjoint Borel sets $A, B \in \mathcal{B}(X)$ we have

$$J(A\times B) = -\mathcal{E}(\mathbb{1}_A, \mathbb{1}_B) = -\lim_{t\downarrow 0} t^{-1}\langle (I-T_t)(\mathbb{1}_A), \mathbb{1}_B\rangle = \lim_{t\downarrow 0} t^{-1}\langle T_t(\mathbb{1}_A), \mathbb{1}_B\rangle = \lim_{t\downarrow 0} t^{-1}\langle \mathbb{1}_A, T_t(\mathbb{1}_B)\rangle$$

which has the probabilistic interpretation as jump intensity of the associated process (Y_t) as

$$J(A \times B) = \lim_{t\downarrow 0} t^{-1} \int_A \mathbb{P}_x(Y_t \in B)\, m(dx).$$

Similarly for any Borel set $A \in \mathcal{B}(X)$, we have

$$\kappa(A) = \mathcal{E}(\mathbb{1}_A, \mathbb{1}_X) = \mathcal{E}(\mathbb{1}_A, \mathbb{1}_X) = \lim_{t\downarrow 0} t^{-1}\langle \mathbb{1}_A, (I - T_t)\mathbb{1}_X\rangle,$$

which has the probabilistic interpretation as the killing (jump intensity to the cemetery state Δ) of the associated process (Y_t) as

$$\kappa(A) = \lim_{t\downarrow 0} t^{-1} \int_A \mathbb{P}_x(Y_t = \Delta)\, m(dx).$$

The strongly local part of the Beurling-Deny decomposition is somewhat mysterious and is not as explicit as the jumping and killing parts.

Example 2.45. Let m denote the Lebesgue measure on $\mathbb{R}^2$ and let $A \subset \mathbb{R}^2$ denote the straight line from $(0,0)$ to $(0,1)$. Let $(\mathcal{E}, \mathcal{F})$ denote the Dirichlet form corresponding to the two-dimensional Brownian motion given by $\mathcal{F} = W^{1,2}(\mathbb{R}^2)$, and $\mathcal{E}(f,f) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla f|^2\, dm$ for all $f \in \mathcal{F}$ (see Exercise 2.29). Then $\mathrm{Cap}_1(A) > 0 = m(A)$. This shows that there are non-$\mathcal{E}$-polar sets with measure zero (see Exercise 3.10 for another example). On the other hand by the inequality, $\mathrm{Cap}_1(A) \geqslant m(A)$, every set of 1-capacity zero has measure zero.

2.6 Irreducibility, recurrence and transience

Familiar probabilistic notions such as irreducibility, recurrence and transience can be defined at the level of Dirichlet forms.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ and let $(P_t)_{t>0}$ denote the corresponding semigroup.

Definition 2.46. We say that a measurable set $A \subset X$ is *$\mathcal{E}$-invariant*, if it satisfies

$$P_t(\mathbb{1}_A f) = \mathbb{1}_A P_t(f), \quad m\text{-a.e. on } X \text{ for any } f \in L^2(X, m) \text{ and any } t \in (0, \infty).$$

Equivalently, $\mathbb{1}_A u \in \mathcal{F}$ for any $u \in \mathcal{F}$ and

$$\mathcal{E}(u,u) = \mathcal{E}(\mathbb{1}_A u, \mathbb{1}_A u) + \mathcal{E}(\mathbb{1}_{A^c} u, \mathbb{1}_{A^c} u).$$

We say that a Dirichlet form $(\mathcal{E},\mathcal{F})$ is *irreducible* if $m(A)m(X\backslash A) = 0$ for any $\mathcal{E}$-invariant set A.

In order to define recurrence and transience, we recall the definition of extended Dirichlet space.

Definition 2.47 (Extended Dirichlet space)**.** We define the *extended Dirichlet space* $\mathcal{F}_e$ of a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$ as the space of m-equivalence classes of functions $f\colon X \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n = f$ m-a.e. on X for some $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ with $\lim_{k\wedge l\to\infty} \mathcal{E}(f_k - f_l, f_k - f_l) = 0$. Then the limit $\mathcal{E}(f,f) := \lim_{n\to\infty} \mathcal{E}(f_n, f_n) \in \mathbb{R}$ exists and is independent of a choice (why?) of such $\{f_n\}_{n\in\mathbb{N}}$ for each $f \in \mathcal{F}_e$, so that $\mathcal{E}$ is canonically extended to $\mathcal{F}_e \times \mathcal{F}_e$.

We say that a Markovian semigroup $(P_t)_{t\geqslant 0}$ is *transient* if for any $f \in L^1(X,m) \cap L^\infty(X,m)$ with $f \geqslant 0$-m-a.e., we have

$$Gf := \lim_{T\to\infty} \int_0^T P_t(f)\, dm < \infty, \quad m\text{-a.e.}$$

We say that a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$ is *transient* if there exists a $f \in L^1(X,m) \cap L^\infty(X,m)$ such that f is strictly positive m-a.e., and satisfying

$$\int_X |u| f\, dm \leqslant \sqrt{\mathcal{E}(u,u)}, \quad \text{for all } u \in \mathcal{F}.$$

The following theorem gives equivalent definitions of transience [FOT, Theorems 1.5.3 and 1.6.2].

Theorem 2.48. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(X,m)$ and let $(P_t)_{t>0}$ denote the corresponding semigroup. The following are equivalent:*

1. *The semigroup $(P_t)_{t>0}$ is transient.*
2. *The Dirichlet form $(\mathcal{E},\mathcal{F})$ is transient.*
3. *$(\mathcal{F}_e,\mathcal{E})$ is a Hilbert space.*

We say that a m-symmetric Markov semigroup $(P_t)_{t\geqslant 0}$ is *recurrent*, if for all non-negative $f \in L^1(X,m)$, we have

$$Gf := \lim_{T\to\infty} \int_0^T P_t f\, dt \in \{0,\infty\} \quad m\text{-a.e. on } X.$$

We say that a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$ is *recurrent* if $\mathbb{1}_X \in \mathcal{F}_e$ and $\mathcal{E}(\mathbb{1}_X, \mathbb{1}_X) = 0$. Analogous to Theorem 2.48, we have the following result [FOT, Theorem 1.6.3].

Theorem 2.49. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ and let $(P_t)_{t>0}$ denote the corresponding semigroup. The following are equivalent:*

1. *The semigroup $(P_t)_{t>0}$ is recurrent.*
2. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is recurrent.*

The following exercise concerns the Dirichlet form for n-dimensional Bessel process.

Exercise 2.50. Let $n \in (0, \infty)$. Consider the measure $m(dx) = x^{n-1}\, dx$ on $X = (0, \infty)$. Let $\mathcal{F}$ denote the set of all functions $f : X \to \mathbb{R}$ such that f is absolutely continuous on X and satisfies

$$\int_{(0,\infty)} \left(|f'(x)|^2 + |f(x)|^2\right) m(dx) = \int_{(0,\infty)} \left(|f'(x)|^2 + |f(x)|^2\right) x^{n-1}\, dx < \infty.$$

Then $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(X, m)$, where

$$\mathcal{E}(f, f) = \int_X f'(x) g'(x)\, m(dx), \quad \text{for all } f, g \in \mathcal{F}.$$

1. Show that $(\mathcal{E}, \mathcal{F})$ is irreducible.
2. Show that $(\mathcal{E}, \mathcal{F})$ is recurrent if $n \in (0, 2]$ and is transient if $n \in (2, \infty)$.
   Hint: For the case $n \in (2, \infty)$, the following inequality is useful to obtain

   $$|f(x) - f(y)|^2 \leqslant (n-2)^{-1}(x^{2-n} - y^{2-n})\mathcal{E}(f, f), \quad \text{for all } f \in \mathcal{F}, x, y \in I \text{ with } x < y.$$

Let (Y_t) denote the Markov process associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$. For a Borel set $B \subset X$, we denote by T_B, the hitting time of set B defined as

$$T_B := \inf\{t > 0 | Y_t \in B\}.$$

If $(\mathcal{E}, \mathcal{F})$ is irreducible, then for all non $\mathcal{E}$-polar Borel set $B \subset X$, and for q.e. $x \in X$, we have [CF, Theorem 3.5.6-(i)]

$$\mathbb{P}_x(T_B < \infty) > 0.$$

If $(\mathcal{E}, \mathcal{F})$ is transient, then the path wanders to infinity whenever the lifetime is infinite; that is, (see [CF, Theorem 3.5.2])

$$\mathbb{P}_x(\zeta = \infty, \lim_{t\to\infty} X_t = \Delta) = \mathbb{P}_x(\zeta = \infty), \quad \text{for q.e. } x \in X.$$

On the other hand, by [CF, Theorem 3.5.6-(ii)] if $(\mathcal{E}, \mathcal{F})$ is recurrent and irreducible, then the process visits any non-$\mathcal{E}$-polar Borel set $B \subset X$ infinitely often, that is,

$$\mathbb{P}_x\left(T_B \circ \theta_n < \infty \text{ for all } n \in \mathbb{N}\right) = 1, \quad \text{for q.e. } x \in X.$$

3 Heat kernel

Definition 3.1. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$ with associated semigroup $(P_t)_{t>0}$. A family $\{p_t\}_{t>0}$ of $[0, \infty]$-valued Borel measurable functions on $X \times X$ is called the *heat kernel*, if p_t is an integral kernel of the operator P_t for any $t \in (0, \infty)$, that is, for any $t \in (0, \infty)$ and any $f \in L^2(X, m)$,

$$P_t f(x) = \int_X p_t(x, y) f(y)\, dm(y) \qquad \text{for } m\text{-a.e. } x \in X.$$

3.1 Existence of heat kernel via ultracontractivity

In general, a heat kernel need not exist (consider the semigroup consisting of identity maps). From a probabilistic perspective, if the heat kernel exists, $p_t(x, y)\, m(dy)$ can be viewed as the law of the corresponding Markov process started at x at time t. So the existence of heat kernel can be viewed as the absolute continuity (with respect to the reference measure m) of the law of a Markov process.

The following very general result is useful to show the existence of a heat kernel (see [DS, Theorem 6, VI.8.6]).

Proposition 3.2. *Suppose $(X, \mathcal{M}, m)$ be a σ-finite, separable measure space and let $T : L^1(X, m) \to L^\infty(X, m)$ be a bounded linear operator. Then there exists a jointly measurable function $K : X \times X \to \mathbb{R}$ such that $K \in L^\infty(X \times X, m \times m)$ with $\|K\|_\infty = \|T\|_{1\to\infty}$ and*

$$Tf(x) = \int_X K(x, y) f(y)\, m(dy), \qquad m\text{-a.e. for each } f \in L^1(X, m).$$

A linear operator that is bounded from L^1 to L^∞ as above is said to be *ultracontractive*. Ultracontractivity of semigroup offers a way to show the existence of heat kernel.

Proposition 3.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$ and let $\{P_t : t > 0\}$ denote the associated semigroup. Assume that there exists $C_1 \geqslant 0, C_2 > 0$ and $n \geqslant 1$ such that we have the* Nash inequality

$$\|f\|_2^{1+n/2} \leqslant \|f\|_1 \left(C_1 \|f\|_2^2 + C_2 \mathcal{E}(f, f)\right)^{n/4}, \quad \text{for all } f \in L^1(X, m) \cap \mathcal{F}. \tag{3.1}$$

Then the semigroup $\{P_t : t > 0\}$ admits a heat kernel $\{p_t(\cdot, \cdot)\}$ such that

$$\operatorname*{ess\,sup}_{x,y \in X} p_t(x, y) \leqslant \max\left(2C_1, \frac{nC_2}{t}\right)^{n/2}.$$

Proof. First we show the estimate

$$\|P_t(f)\|_2 \leqslant \max\left(2C_1, \frac{nC_2}{2t}\right)^{n/4} \|f\|_1, \quad \text{for all } t > 0 \text{ and } f \in L^1(X, m). \tag{3.2}$$

By the density of $L^1 \cap L^2$ in L^1 and linearity of P_t, it suffices to consider $f \in L^1(X,m) \cap L^2(X,m)$ with $\|f\|_1 = 1$. By the spectral theorem, we note that $P_t(f) \in \mathcal{D}(A)$ for all $t > 0$, where A is the generator and

$$\frac{d}{dt}\|P_t(f)\|^2 = 2\langle \frac{d}{dt}P_t(f), P_t(f)\rangle = 2\langle AP_t(f), P_t(f)\rangle = -2\mathcal{E}(P_t(f), P_t(f)).$$

Setting $\Psi(t) := \|P_t(f)\|_2^2$ by the above equality and (3.1), we obtain

$$\begin{aligned}\Psi(t)^{1+2/n} &= \|P_t(f)\|_2^{2+4/n} \overset{(3.1)}{\leqslant} \|P_t(f)\|_1^{4/n}\left(C_1\|P_t(f)\|_2^2 + C_2\|\mathcal{E}(P_t(f), P_t(f))\|\right)\\ &\overset{(2.18)}{\leqslant} C_1\Psi(t) - 2^{-1}C_2\Psi'(t).\end{aligned}$$

If $\Psi(t) \geqslant (2C_1)^{n/2}$, then $\Psi(t)^{1+2/n} - C_1\Psi(t) \geqslant \Psi(t)(\Psi(t)^{2/n} - C_1) \geqslant 2^{-1}\Psi(t)^{1+2/n}$. This leads to the differential inequality,

$$\Psi(t)^{1+2/n} \leqslant -C_2\Psi'(t)$$

for all $t > 0$ such that $\Psi(t) \geqslant (2C_1)^{n/2}$ (recall that Ψ is non-increasing). This can be written as

$$\frac{d}{dt}\left(\Psi(t)^{-2/n}\right) = -\frac{2}{n}\Psi(t)^{-(1+2/n)}\Psi'(t) \geqslant \frac{2}{nC_2}$$

which implies (3.2).

Now by duality of L_p spaces and symmetry of P_t, we have $\|P_t\|_{1\to 2} = \|P_t\|_{2\to\infty}$. To see this, note that

$$\begin{aligned}\|P_t\|_{2\to\infty} &= \sup_{f\in L^2, \|f\|_2=1}\|P_t(f)\|_\infty = \sup_{f\in L^1\cap L^\infty, \|f\|_2=1}\|P_t(f)\|_\infty\\ &= \sup_{\substack{f\in L^1\cap L^\infty, \|f\|_2=1,\\ g\in L^1, \|g\|_1=1}}\int_X gP_f(f)\,dm = \sup_{\substack{f\in L^1\cap L^\infty, \|f\|_2=1,\\ g\in L^1\cap L^\infty, \|g\|_1=1}}\langle P_t(f), g\rangle\\ &= \sup_{\substack{f\in L^1\cap L^\infty, \|f\|_2=1,\\ g\in L^1\cap L^\infty, \|g\|_1=1}}\langle f, P_t(g)\rangle = \sup_{g\in L^1\cap L^\infty, \|g\|_1=1}\|P_t(g)\|_2 = \|P_t\|_{1\to 2}.\end{aligned}$$

Now by the semigroup property and submultiplicativity of operator norms, we have

$$\|P_t\|_{1\to\infty} \leqslant \left\|P_{t/2}\right\|_{1\to 2}\left\|P_{t/2}\right\|_{2\to\infty} = \left\|P_{t/2}\right\|_{1\to 2}^2 \overset{(3.2)}{\leqslant} \max\left(2C_1, \frac{nC_2}{t}\right)^{n/2}.$$

The existence of heat kernel and the upper bound follows from Proposition 3.2 □

We remark that conversely the existence of such heat kernel bounds imply the Nash inequality assumed [Cou, CKS].

Example 3.4. J. Nash obtained the following inequality on $\mathbb{R}^n$: there exists $C > 0$ such that

$$\|f\|_2^{1+n/2} \leqslant C \|f\|_1 \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx \right)^{n/4}, \quad \text{for all } f \in L^1(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n). \tag{3.3}$$

Therefore Proposition 3.3, gives the upper bound $p_t(x,y) \lesssim t^{-n/2}$ for all $x, y \in \mathbb{R}^n$ for the heat kernel for Brownian motion on $\mathbb{R}^n$. The advantage of this method is that it is robust to perturbations. Consider a measurable symmetric positive definite matrix valued function $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that there exists $\Lambda \in [1, \infty)$

$$\Lambda^{-1} \|\xi\|^2 \leqslant \xi^t \mathcal{A}(x) \xi \leqslant \Lambda \|\xi\|^2, \quad \text{for all } x, \xi \in \mathbb{R}^n.$$

Then the Dirichlet form $\mathcal{E}(f,f) = \int_{\mathbb{R}^n} (\nabla f(x))^t \mathcal{A}(x) \nabla(f) \, dx$ on $W^{1,2}(\mathbb{R}^n)$ corresponding to uniformly elliptic operator $\operatorname{div}(\mathcal{A}(x)\nabla(\cdot))$ also satisfies the above Nash inequality (since $\mathcal{E}(f,f) \asymp \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx$) and hence the same upper bound. From a probabilistic viewpoint the diffusion generated by a uniformly elliptic operator can be viewed as a perturbation of the Brownian motion that allows for different speeds of diffusion at different points in the space and also allows for anisotropic (tendency for the diffusion process to prefer certain direction over others) behavior.

Let us quickly sketch the proof of Nash for the case when $f \in C^\infty(\mathbb{R}^n)$ is smooth and well-behaved at infinity [Nas]. The Fourier transform $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx$ satisfies the Parseval's identity $(2\pi)^{-n/2} \left\| \widehat{f} \right\|_2 = \|f\|_2$ and the elementary bound $\left\| \widehat{f} \right\|_\infty \leqslant \|f\|_1$. Again by Parseval's identity and the Fourier transform of derivative formula, we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} |\xi|^2 \left| \widehat{f}(\xi) \right|^2 d\xi.$$

Denoting ω_n as the volume of unit ball in $\mathbb{R}^n$, we have

$$\begin{aligned}
\|f\|_2^2 &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left| \widehat{f}(\xi) \right|^2 d\xi \\
&\leqslant (2\pi)^{-n} \int_{\{|\xi|<R\}} \left\| \widehat{f} \right\|_\infty^2 d\xi + (2\pi)^{-n} R^{-2} \int_{\{|\xi|\geqslant R\}} |\xi|^2 \left| \widehat{f}(\xi) \right|^2 d\xi \\
&\leqslant (2\pi)^{-n} \|f\|_1^2 \, \omega_n R^n + R^{-2} \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx.
\end{aligned}$$

Minimizing the above expression as a function of R, we obtain the Nash inequality when f is smooth and well-behaved at infinity. The general case follows from the same argument by viewing f as a tempered distribution.

Remark 3.5. The Nash inequality (3.3) for the case $n \geqslant 3$ is a simple consequence of Hölder inequality

$$\|f\|_2 \leqslant \|f\|_{2n/(n-2)}^{n/(n+2)} \|f\|_1^{2/(n+2)}.$$

and the Sobolev inequality

$$\|f\|_{2n/(n-2)} \lesssim \|\nabla f\|_2, \quad \text{for all } f \in W^{1,2}(\mathbb{R}^n).$$

Remarkably, one can obtain Sobolev inequality from the Nash inequality using an elementary 'truncation argument' [BCLS].

We describe an example due to C. Berg (1976) who showed that it is possible for the heat kernel to exist for certain times but *fail* to exist for other times [Berg].

Example 3.6. Consider the one-dimensional torus $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ (more popularly known as the circle) equipped with the normalized Haar measure m (normalized to be a probability measure). The image of Brownian motion on $\mathbb{R}$ under the quotient is defined to be the Brownian motion on $\mathbb{R}$. It is easy to see that the heat kernel $p_t(x, y) := g_t(x - y)$ for all $x, y \in \mathbb{T}$ is given by

$$g_t(x) := \sqrt{\frac{2\pi}{t}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(x + 2\pi k)^2}{2t}\right), \quad \text{for all } x \in \mathbb{T}, t > 0.$$

Let (B_t) denote the Brownian motion on $t > 0$ defined as the projection of 1-dimensional Brownian motion under the quotient map $\mathbb{R} \to \mathbb{R}/(2\pi\mathbb{Z})$. Now for a sequence $(a_1, a_2, \ldots,)$ of positive real numbers consider the process $Y_t := (B^1_{ta_1}, B^2_{ta_2}, \ldots)$ on the infinite dimensional torus $\mathbb{T}^\infty = \prod_{k=1}^\infty \mathbb{T}$, where $B^1, B^2, \ldots$ are independent Brownian motions on $\mathbb{T}$. The n-th component is faster than the usual Brownian motion by a factor a_n (assuming $a_n > 1$). Let m_∞ denote the Haar (probability) measure on $\mathbb{T}^\infty$ (product of infinitely many copies of m). Let μ_t denote the law of Y_t stated at 0; that is,

$$\mu_t(dx) := \prod_{n=1}^\infty g_{a_n t}(x) \cdot m(dx).$$

By Kakutani's dichotomy theorem ([Kak]), μ_t is mutually absolutely continuous with respect to m_∞ if and only if

$$\prod_{n\in\mathbb{N}} \int_{\mathbb{T}} \sqrt{g_{a_n t}}\, dm > 0,$$

and $\mu_t \perp m_\infty$ if and only if

$$\prod_{n\in\mathbb{N}} \int_{\mathbb{T}} \sqrt{g_{a_n t}}\, dm = 0.$$

After some estimates, this leads to the condition that μ_t is absolutely continuous with respect to m_∞ if and only if

$$\sum_{n=1}^\infty e^{-ta_n} < \infty.$$

For example if $a_n := \frac{1}{\alpha}\log(n + 1)$ for all $n \in \mathbb{N}$ where $\alpha > 0$, then $\mu_t \perp m_\infty$ if and only if $t \leqslant \alpha$ and is absolutely continuous with respect to m_∞ for all $t > \alpha$.

The dichotomy between singularity and absolute continuity is conjectured to hold even when the Brownian motion in individual components are dependent; that is, when the diagonal covariance matrix of the underlying Gaussian is replaced with arbitrary covariance. We refer to [Sal10, Problem 4] for details and a precise statement.

3.2 Brownian motion on the Sierpiński gasket

Let $V_0 = \{q_1, q_2, q_3\}$, where

$$q_1 := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad q_2 := (0,0), \quad q_3 := (1,0).$$

Let $S := \{1,2,3\}$ and set $f_j : \mathbb{R}^2 \to \mathbb{R}^2$ as $f_j(x) := \frac{1}{2}(x + q_j)$ for all $j \in S$. Define inductively for $m \in \mathbb{N}$

$$V_m = \bigcup_{j \in S} f_j(V_{m-1}).$$

By an induction argument, $V_m \subset V_{m+1}$ for all $m \in \mathbb{N} \cup \{0\}$. Set $V_* = \bigcup_{m=0}^{\infty} V_m$ and let $K = \overline{V_*}$ denote the closure of V_* in $\mathbb{R}^2$. Since $V_* = \bigcup_{j \in S} f_j(V^*)$, we have

$$K = \bigcup_{j \in S} f_j(K).$$

We set $F_j = f_j\big|_K$ for all $j \in S$. We define words of lengths with alphabet S as $W_m = S^m = w_1 \dots w_m : w_1, \dots, w_m \in S$ for all $m \in \mathbb{N} \cup \{0\}$ (with $W_0 = \{\varnothing\}$ consisting of the empty word). For $m \in \mathbb{N} \cup \{0\}$, $w = w_1 \dots w_m \in W_m$ and $F_w := F_{w_1} \circ \dots \circ F_{w_m}$ (with $F_\varnothing = I$) and $x_0 := q_1$.

The approach behind constructing a Dirichlet form on K is to construct a limit of a sequence of Dirichlet forms on V_m as $m \to \infty$. Equivalently, from a probabilistic perspective, the diffusion on Sierpiński gasket is constructed as a limit of random walks on a sequence of graph approximations. For $x, y \in V_m$, we say that $x \overset{m}{\sim} y$ if and only if $x \neq y$ and $x, y \in F_w(V_0)$ for some $w \in W_m$.

We define quadratic forms $E^{(m)}, \mathcal{E}^{(m)} : V_m \times V_m \to \mathbb{R}$ for all $u, v \in \mathbb{R}^{V_m}$

$$E^{(m)}(u,v) = \frac{1}{2} \sum_{\substack{x,y \in V_m, \\ x \overset{m}{\sim} y}} (u(x) - u(y)(v(x) - v(y))), \quad \mathcal{E}^{(m)}(u,v) = \left(\frac{5}{3}\right)^m E^{(m)}(u,v). \quad (3.4)$$

The reason for the re-scaling factor $\left(\frac{5}{3}\right)^m$ is due to the following fact. For any $m \in \mathbb{N} \cup \{0\}, u \in \mathbb{R}^{V_m}$ there exists a unique extension $H_{m.m+1}(u) \in \mathbb{R}^{V_{m+1}}$ of u such that

$$E^{(m+1)}(H_{m.m+1}(u), H_{m.m+1}(u)) = \min_{\substack{v \in \mathbb{R}^{V_{m+1}}, \\ v|_{V_m} = u}} E^{(m+1)}(v,v) = \frac{3}{5} E^{(m)}(u,u). \quad (3.5)$$

The analysis can be reduced to the case $m = 0$. Suppose $(u(q_1), u(q_2), u(q_3)) = (a,b,c) \in \mathbb{R}^3$. For $j, k \in S$ with $j \neq k$, we set $q_{kj} = F_k(q_j)$, so that $F_k(K) \cap F_j(K) = \{q_{jk}\}$. Let $v \in \mathbb{R}^{V_1}$ be an extension of u with $(x,y,z) := (v(q_{23}), v(q_{31}), v(q_{12}))$, then

$$E^{(1)}(v,v) = (y-z)^2 + (a-y)^2 + (a-z)^2 + (b-x)^2 + (x-z)^2 + (z-b)^2 + (x-c)^2 + (c-y)^2 + (y-x)^2.$$

We need to minimize this expression as a function of x, y, z. This leads to

$$4x = b + z + y + c, \quad 4y = c + x + z + a, \quad 4z = a + y + x + b,$$

or equivalently,

$$x = \frac{a+2b+2c}{5}, \quad y = \frac{2a+b+2c}{5}, \quad z = \frac{2a+2b+c}{5}.$$

For v chosen as above, it is easy to compute that $E^{(1)}(v,v) = \frac{3}{5}E^{(0)}(u,u)$. This completes the proof of (3.5).

By (3.5), for any $u \in \mathbb{R}^{V_*}$, the sequence $\left(\mathcal{E}^{(m)}(u\big|_{V_m}, u\big|_{V_m})\right)_{m\in\mathbb{N}\cup\{0\}}$ is non-decreasing and hence $\lim_{m\to\infty}\mathcal{E}^{(m)}(u\big|_{V_m}, u\big|_{V_m}) \in [0,\infty]$ exists. Define

$$\mathcal{F}_* := \{u \in \mathbb{R}^{V_*} | \lim_{m\to\infty}\mathcal{E}^{(m)}(u\big|_{V_m}, u\big|_{V_m}) < \infty\}, \quad \mathcal{E}^{(*)}(u,u) = \lim_{m\to\infty}\mathcal{E}^{(m)}(u\big|_{V_m}, u\big|_{V_m}),$$

for all $u \in \mathcal{F}_*$. It is easy to see that $\mathcal{F}_*$ is a subspace of $\mathbb{R}^{V_*}$ since $\mathcal{E}^{(*)}(au,au) = a^2\mathcal{E}^{(*)}(u,u)$ for all $a \in \mathbb{R}, u \in \mathbb{R}^{V_*}$, and $\mathcal{E}^{(*)}(u+v)^{1/2} \leqslant \mathcal{E}^{(*)}(u,u)^{1/2} + \mathcal{E}^{(*)}(v,v)^{1/2}$ for all $u,v \in \mathcal{F}_*$ (see Exercise 2.26). By bi-linearity we have that $\lim_{m\to\infty}\mathcal{E}^{(m)}(u\big|_{V_m}, u\big|_{V_m}) \in \mathbb{R}$ exists for all $u,v \in \mathcal{F}_*$.

The following self-similarity follows easily from the definition of the energies.

Exercise 3.7. For any $u \in \mathbb{R}^{V_*}$, we have

$$\mathcal{E}^{(m+1)}(u\big|_{V_{m+1}}, u\big|_{V_{m+1}}) = \frac{5}{3}\sum_{j\in S}\mathcal{E}^{(m)}(u\circ F_j\big|_{V_m}, u\circ F_j\big|_{V_m}), \tag{3.6}$$

and hence

$$\mathcal{F}_* = \{u \in \mathbb{R}^{V_*} : u\circ F_j \in \mathcal{F}_* \text{ for all } j \in S\}, \quad \mathcal{E}^{(*)}(u,u) = \frac{5}{3}\sum_{j\in S}\mathcal{E}^{(*)}(u\circ F_j, u\circ F_j). \tag{3.7}$$

More generally, for any $m \in \mathbb{N}$, we have

$$\mathcal{F}_* = \{u \in \mathbb{R}^{V_*} : u\circ F_\tau \in \mathcal{F}_* \text{ for all } \tau \in W_m\}, \quad \mathcal{E}^{(*)}(u,u) = \left(\frac{5}{3}\right)^m \sum_{\tau\in W_m}\mathcal{E}^{(*)}(u\circ F_\tau, u\circ F_\tau). \tag{3.8}$$

Proposition 3.8. *For all $x,y \in V_*$ and $u \in \mathcal{F}_*$, we have*

$$|u(x)-u(y)|^2 \leqslant 400|x-y|^\alpha\mathcal{E}^{(*)}(u,u) \tag{3.9}$$

where $|\cdot|$ denotes the Euclidean distance in $\mathbb{R}^2$ and $\alpha = \log_2(5/3)$. In particular, every $u \in \mathcal{F}_*$ is uniformly continuous in V_* and hence admits a unique continuous extension to its closure $K = \overline{V_*}$.

Proof. Note that the graph corresponding to V_1 with edges given by $\overset{1}{\sim}$ has diameter 2. Hence for any $y,z \in V_1$ there exists $q \in V_1$ such that one of hte following hold: $y = z = q$ or $y \overset{1}{\sim} q = z$ or $y \overset{1}{\sim} q \overset{1}{\sim} z$. In all these cases, for any $v \in \mathbb{R}^{V_1}$ we have

$$|v(y)-v(z)| \leqslant |v(y)-v(q)| + |v(q)-v(z)| \leqslant \sqrt{2}\left(|v(y)-v(q)|^2 + |v(q)-v(z)|^2\right)^{1/2}$$

$$\leqslant \sqrt{2}E^{(1)}(v,v) \overset{(3.4)}{=} \sqrt{6/5}\mathcal{E}^{(1)}(v,v). \tag{3.10}$$

Now let $u \in \mathcal{F}_*, x \in V_*$ be arbitrary. Then there exists $m \in \mathbb{N}, w = w_1 \dots w_m \in W_m, j \in S$ such that $x = F_w(q_j)$. Set for $1 \leqslant k \leqslant m$, $x_k = F_{w_1 \dots w_{k-1}}(q_j)$ (with $F_\varnothing = I$ for $j = 0$). Then for all $k = 1, \dots, m$ we have

$$\begin{aligned}
|u(x_{k-1}) - u(x_k))| &= \left| u \circ F_{w_1 \dots w_{k-1}}(F^{-1}_{w_1 \dots w_{k-1}}(x_{k-1})) - u \circ F_{w_1 \dots w_{k-1}}(F^{-1}_{w_1 \dots w_{k-1}}(x_k)) \right| \\
&\overset{(3.10)}{\leqslant} \sqrt{6/5}\mathcal{E}^{(1)}(u \circ F_{w_1 \dots w_{k-1}}\big|_{V_1}, u \circ F_{w_1 \dots w_{k-1}}\big|_{V_1})^{1/2} \leqslant \sqrt{6/5}\mathcal{E}^{(*)}(u \circ F_w, u \circ F_w) \\
&\leqslant \sqrt{\frac{6}{5}} \left(\frac{3}{5}\right)^{(k-1)/2} \left(\left(\frac{5}{3}\right)^{(k-1)} \sum_{\tau \in W_{k-1}} \mathcal{E}^{(*)}(u \circ F_\tau, u \circ F_\tau) \right)^{1/2} \\
&\overset{(3.8)}{=} \sqrt{\frac{6}{5}} \left(\frac{3}{5}\right)^{(k-1)/2} \mathcal{E}^{(*)}(u,u)^{1/2}.
\end{aligned} \tag{3.11}$$

Hence we obtain

$$\begin{aligned}
|u(q_j) - u(x)| &\leqslant \sum_{k=1}^{m} |u(x_{k-1}) - u(x_k))| \overset{(3.11)}{\leqslant} \sum_{k=1}^{m} \sqrt{\frac{6}{5}} \left(\frac{3}{5}\right)^{(k-1)/2} \mathcal{E}^{(*)}(u,u)^{1/2} \\
&\leqslant \sum_{k=1}^{\infty} \sqrt{\frac{6}{5}} \left(\frac{3}{5}\right)^{(k-1)/2} \mathcal{E}^{(*)}(u,u)^{1/2} \leqslant 5\mathcal{E}^{(*)}(u,u)^{1/2}.
\end{aligned} \tag{3.12}$$

Thus for any $x, y \in V_*$ and any $u \in \mathcal{F}_*$, we have

$$|u(x) - u(y)| \leqslant |u(x) - u(q_1)| + |u(y) - u(q_1)| \overset{(3.12)}{\leqslant} 10\mathcal{E}^{(*)}(u,u)^{1/2}. \tag{3.13}$$

Since for any $w \in W_m, m \in \mathbb{N} \cup \{0\}$, $F_w(K)$ is a subset of an equilateral triangle with side length 2^{-m}, for any $w, v \in W_m$ with $F_w(K) \cap F_v(K) \neq \varnothing$, we have

$$|x - y| \leqslant 2^{-m} + 2^{-m} = 2^{-m+1}, \quad \text{for all } x \in F_v(K), y \in F_w(K). \tag{3.14}$$

Let $x, y \in K$ be arbitrary with $x \neq y$. Let $n := n_{x,y} = \max\{m \in \mathbb{N} \cup \{0\} :$ there exist $v, w \in W_m$ with $x \in F_v(V_*), y \in F_w(V_*), F_v(K) \cap F_w(K) \neq \varnothing\}$. Let $q \in F_v(K) \cap F_w(K) \subset V_*$, where $v, w \in W_n$ with $x \in F_v(V_*), y \in F_w(V_*)$ as above. Then

$$\begin{aligned}
|u(x) - u(y)| &\leqslant |u(x) - u(q)| + |u(y) - u(q)| = \left| u \circ F_v(F_v^{-1}(x)) - u \circ F_w(F_w^{-1}(y)) \right| \\
&\overset{(3.13)}{\leqslant} 10 \left(\mathcal{E}^{(*)}(u \circ F_v, u \circ F_v)^{1/2} + \mathcal{E}^{(*)}(u \circ F_w, u \circ F_w)^{1/2} \right) \\
&\leqslant 10\sqrt{2} \left(\mathcal{E}^{(*)}(u \circ F_v, u \circ F_v) + \mathcal{E}^{(*)}(u \circ F_w, u \circ F_w) \right)^{1/2} \\
&\overset{v \neq w}{\leqslant} 10\sqrt{2} \left(\frac{3}{5}\right)^{n/2} \left(\left(\frac{5}{3}\right)^{n} \sum_{\tau \in W_n} \mathcal{E}^{(*)}(u \circ F_\tau\big|_{V_*}, u \circ F_\tau\big|_{V_*}) \right)^{1/2} \\
&\overset{(3.8)}{=} 10\sqrt{2} \left(\frac{3}{5}\right)^{n/2} \mathcal{E}^{(*)}(u,u)^{1/2}.
\end{aligned} \tag{3.15}$$

Let $i, j \in S$ be such that $x \in F_{vi}(K), y \in F_{wj}(K)$, where $vi, wj \in W_{n+1}$ denotes concatenation of words. By the maximality on n, $F_{vi}(K) \cap F_{wj}(K) = \varnothing$ and hence $|x-y| \geqslant \frac{\sqrt{3}}{2} 2^{-n-1}$ (draw a picture to see the possibilities). Hence from (3.15), we estimate

$$|u(x)-u(y)|^2 \leqslant (200) \cdot 2^{-\alpha n} \mathcal{E}^{(*)}(u,u) \leqslant 200 \left(\frac{4}{\sqrt{3}}\right)^{\alpha} |x-y|^{\alpha} \mathcal{E}^{(*)}(u,u) \leqslant 400 |x-y|^{\alpha} \mathcal{E}^{(*)}(u,u).$$

□

Let $\mathcal{F} \subset C(K)$ be defined as

$$\mathcal{F} := \{u \in C(K) : \lim_{m \to \infty} \mathcal{E}^{(m)}(u\big|_{V_m}, u\big|_{V_m}) < \infty\} = \{u \in C(K) : u\big|_{V_*} \in \mathcal{F}_*\}.$$

The standard Dirichlet form on Sierpińksi gasket $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is defined as

$$\mathcal{E}(u,v) := \lim_{m \to \infty} \mathcal{E}^{(m)}(u\big|_{V_m}, v\big|_{V_m}), \quad \text{for all } u, v \in \mathcal{F}.$$

The following properties of $(\mathcal{E}, \mathcal{F})$ follow easily from the construction. From (3.7), we have the self-similarity property

$$\mathcal{F} = \{u \in C(K) : u \circ F_j \in \mathcal{F} \text{ for all } j \in S\}, \quad \mathcal{E}(u,u) = \frac{5}{3} \sum_{j \in S} \mathcal{E}(u \circ F_j, u \circ F_j). \tag{3.16}$$

By Proposition 3.8 and the density of V_* in K, we have

$$|u(x)-u(y)|^2 \leqslant 400 |x-y|^{\alpha} \mathcal{E}(u,u), \quad \text{for all } u \in \mathcal{F}, \tag{3.17}$$

where $\alpha = \log_2(5/3)$. In particular, by (3.17), we have

$$\mathbb{R}\mathbb{1}_K = \{a\mathbb{1}_K | a \in \mathbb{R}\} = \{u \in \mathcal{F} | \mathcal{E}(u,u) = 0\}.$$

For any 1-Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ and for any $u \in \mathcal{F}$, we have $F \circ u \in \mathcal{F}$ and $\mathcal{E}(F \circ u, F \circ u) \leqslant \mathcal{E}(u,u)$. In particular, for any $u \in \mathcal{F}$, we have $\widetilde{u} := u_+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\widetilde{u}, \widetilde{u}) \leqslant \mathcal{E}(u,u)$. The subspace $\mathcal{F}$ is closed under pointwise multiplication (that is, an algebra). More precisely, we have for all $u, v \in \mathcal{F}$, we have $uv \in \mathcal{F}$ and (denoting by $\|\cdot\|_\infty$, the sup norm)

$$\mathcal{E}(uv, uv) \leqslant 2 \|u\|_\infty^2 \mathcal{E}(v,v) + 2 \|v\|_\infty^2 \mathcal{E}(u,u). \tag{3.18}$$

The estimate (3.18) follows from showing an analogous estimate for $\mathcal{E}^{(m)}$ for all $m \in \mathbb{N}$ using the elementary inequality

$$((uv)(x) - (uv)(y))^2 \leqslant 2u(x)^2 (v(x)-v(y))^2 + 2v(y)^2 (u(x)-u(y))^2, \quad \text{for all } x, y \in K.$$

Next, we show that $\mathcal{F}$ is dense subspace of the Banach space $(C(K), \|\cdot\|_\infty)$. Since $\mathcal{F}$ is an algebra, by the Stone-Weierstrass theorem (cf. [**?**, Corollary 4.50]), it suffices to show

that $\mathcal{F}$ separates points and that for any $x \in K$, there exists $v \in \mathcal{F}$ such that $v(x) \neq 0$. There it suffices to show the following: for any non-empty finite subset $V \subset K$, we have

$$\{u\big|_V : u \in \mathcal{F}\} = \mathbb{R}^V.$$

Since the inclusion $\{u\big|_V : u \in \mathcal{F}\} \subset \mathbb{R}^V$ is trivial, it suffices to show that $\{u\big|_V : u \in \mathcal{F}\} \supset \mathbb{R}^V$. To this end, it suffices to consider V with $\#V \geqslant 2$ as $\mathbb{R}\mathbb{1}_K \subset \mathcal{F}$. Let $g \in \mathbb{R}^V$ be arbitrary. Choose $m \in \mathbb{N}$ such that $2^{1-m} < \min_{\substack{x,y\in V,\\ x\neq y}} |x-y|$. Then for any $x, y \in V$ with $x \neq y$ and for any $v, w \in W_m$ with $x \in F_v(K), y \in F_w(K)$, we have

$$F_v(K) \cap F_w(K) = \varnothing. \tag{3.19}$$

For each $x \in V$, pick $w_x \in W_m$ such that $x \in F_{w_x}(K)$. Define $h \in \mathbb{R}^{V_m}$ such that for all $x \in V$ and for all $z \in V_m \cap F_{w_x}(K)$, we have $h(z) = g(x)$, and h is arbitrarily defined at other vertices (such a function h exists due to (3.19)). There exists $u \in \mathcal{F}$ such that $u\big|_{V_n} = H_{n-1,n} \circ \cdot \circ H_{m.m+1}h$. It is easy to see that $u\big|_{F_{w_x}(K)} \equiv g(x)$ for all $x \in V$ and hence $u\big|_V = g$. This concludes the proof that in the Banach space $(C(K), \|\cdot\|_\infty)$, we have

$$\overline{\mathcal{F}}^{\|\cdot\|_\infty} = C(K). \tag{3.20}$$

We summarize the construction of Dirichlet form for Brownian motion on the Sierpiński gasket below.

Proposition 3.9. *Let m be a Radon probability measure on the Sierpiński gasket K with full support. Then $(\mathcal{E}, \mathcal{F})$ is a strongly local, regular, Dirichlet form on $L^2(K, m)$.*

Proof. Since $C(K)$ is dense in $L^2(K, m)$, by (3.20), we have that $\mathcal{F}$ is a dense subspace of $L^2(K, m)$. The bi-linearity, symmetry, non-negative definiteness and Markov property of $(\mathcal{E}, \mathcal{F})$ follow from the corresponding properties for $\mathcal{E}^{(m)}$. Next, let us verify that $(\mathcal{E}, \mathcal{F})$ is closed. To this end, let $(f_n)_{n\in\mathbb{N}}$ be an $\mathcal{E}_1$-Cauchy sequence. By (3.9), for all $x, y \in K$ and for all $g \in \mathcal{F}$, we have (denoting every $g \in \mathcal{F}$ by its continuous version)

$$|g(x)|^2 \leqslant 2(|g(y)|^2 + 400\mathcal{E}(g, g))$$

By averaging over y with respect to m, we have

$$|g(x)|^2 \leqslant 2(\int_K |g(y)|^2 \, dm + 400\mathcal{E}(g, g)) \leqslant 800\mathcal{E}_1(g, g), \quad \text{for all } x \in K, g \in \mathcal{F}. \tag{3.21}$$

By (3.21) for each $x \in K$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and hence $\lim_{n\to\infty} f_n(x) = f(x)$ exists. Let $\epsilon > 0$ and $n \in \mathbb{N}$ be arbitrary. Then exists $N \in \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ with $k \wedge l \geqslant N$, we have

$$\mathcal{E}^{(n)}\left((f_k - f_l)\big|_{V_n}, (f_k - f_l)\big|_{V_n}\right) \leqslant \mathcal{E}(f_k - f_l, f_k - f_l) \leqslant \mathcal{E}_1(f_k - f_l, f_k - f_l) \leqslant \epsilon, \quad \text{for all } n \in \mathbb{N}.$$

By letting $l \to \infty$ and using the pointwise convergence of f_l to f, we obtain

$$\sup_{n\in\mathbb{N}} \mathcal{E}^{(n)}\left((f_k - f)\big|_{V_n}, (f_k - f)\big|_{V_n}\right) \leqslant \epsilon.$$

Hence $f \in \mathcal{F}$ and $\lim_{k\to\infty} \mathcal{E}(f_k - f, f_k - f) = 0$.

It remains to show that f_k converges to f in $L^2(K, m)$. By (3.21) and $\sup_{n\in\mathbb{N}} \mathcal{E}_1(f_n, f_n) < \infty$, we have

$$\sup_{n\in\mathbb{N}} \sup_{x\in K} |f_n(x)| < \infty.$$

Combining this with (3.9) and $\sup_{n\in\mathbb{N}} \mathcal{E}_1(f_n, f_n) < \infty$, we have that the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly boundeded and equicontinuous. Hence by the Arzela-Ascoli theorem, $(f_n)_{n\to\infty}$ converges to f in the sup norm and hence $f_n \xrightarrow[L^2(K,m)]{k\to\infty} f$. Therefore $(\mathcal{E}_1, \mathcal{F})$ is a Hilbert space. The regularity of $(\mathcal{E}, \mathcal{F})$ follows from (3.20) and $\mathcal{F} \subset C(K)$.

Let us verify the strong locality property. Let $f, g \in \mathcal{F}$ and $a \in \mathbb{R}$ be such that $\mathrm{supp}_m(f - a\mathbb{1}_K) \cap \mathrm{supp}_m(g) = \varnothing$. Then there exists $n \in \mathbb{N}$ such that

$$\mathrm{dist}(\mathrm{supp}_m(f - a\mathbb{1}_K), \mathrm{supp}_m(g)) = \inf_{\substack{x\in\mathrm{supp}_m(f-a\mathbb{1}_K),\\ y\in\mathrm{supp}_m(g)}} |x - y| > 2^{-n}.$$

Therefore for all $w \in W_n$, we have either $F_w(K) \cap \mathrm{supp}_m(f - a\mathbb{1}_K) = \varnothing$ or $F_w(K) \cap \mathrm{supp}_m(g) \neq \varnothing$ (or possibly both). Thus by the self-similarity property (3.16) we have

$$\mathcal{E}(f, g) = \sum_{w\in W_n} \left(\frac{5}{3}\right)^n \mathcal{E}(f \circ F_w, g \circ F_w) = 0.$$

□

Exercise 3.10. Let $(\mathcal{E}, \mathcal{F})$ denote the Dirichlet form on $L^2(K, m)$ in Proposition 3.9. Show that the only $\mathcal{E}$-polar subset of K is the empty set. **Hint**: Use (3.17).

Our next goal is to obtain the existence of heat kenrel for Brownian motion on the Sierpinski gasket. To this end, we need the following general notion of energy measures. The energy measure of a function f can be viewed as the generalization of the measure $A \mapsto \int_A |\nabla f(x)|^2 \, dx$.

Definition 3.11 (Energy measure; [FOT, (3.2.13), (3.2.14) and (3.2.15)])**.** Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(X, m)$. The *$\mathcal{E}$-energy measure* $\Gamma(f, f)$ of $f \in \mathcal{F}$ is defined, first for $f \in \mathcal{F} \cap L^\infty(X, m)$ as the unique ($[0, \infty]$-valued) Radon measure on X such that

$$\int_X g \, d\Gamma(f, f) = \mathcal{E}(f, fg) - \frac{1}{2}\mathcal{E}(f^2, g) \qquad \text{for all } g \in \mathcal{F} \cap C_{\mathrm{c}}(X), \tag{3.22}$$

next by $\Gamma(f, f)(A) := \lim_{n\to\infty} \Gamma\big((-n) \vee (f \wedge n), (-n) \vee (f \wedge n)\big)(A)$ for each $A \in \mathcal{B}(X)$ for $f \in \mathcal{F}$. For $f, g \in \mathcal{F}$, we define the (signed) energy measure $\Gamma(f, g)$ by polarization as

$$\Gamma(f, g)(A) = \frac{1}{4}\left(\Gamma(f + g, f + g)(A) - \Gamma(f - g, f - g)(A)\right).$$

The energy measure of the whole space is the Dirichlet energy, that is, $\mathcal{E}(f,f) = \Gamma(f,f)(X)$ for all $f \in \mathcal{F}$. By [FOT, (3.2.13) and (3.2.14)], we have the triangle inequality for energy measure

$$\left|\sqrt{\Gamma(f,f)(B)} - \sqrt{\Gamma(g,g)(B)}\right|^2 \leqslant \Gamma(f-g,f-g)(B) \leqslant \mathcal{E}(f-g,f-g), \tag{3.23}$$

for all Borel sets B and for all $f,g \in \mathcal{F}$.

The self-similarity property of energy (3.16) along with the definition of energy measure leads to the self-similarity of the corresponding energy measures.

Exercise 3.12. Show that for the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,m)$ in Proposition 3.9, the corresponding energy measure satisfies

$$\Gamma(f,f) = \left(\frac{5}{3}\right)^m \sum_{w\in W_m} (F_w)_*(\Gamma(f\circ F_w, f\circ F_w)), \quad \text{for all } m\in\mathbb{N}, \text{ and } f\in\mathcal{F}. \tag{3.24}$$

On the Sierpiński gasket, let $B(x,r)$ denote the open ball centered at $x \in K$ with radius $r > 0$ with respect to the Euclidean metric. We show the following *Poincaré inequality* on the Sierpiński gasket.

Proposition 3.13. *Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form on $L^2(K,m)$ in Proposition 3.9. Set $d_w = \log 5/\log 2$. There exists $C>0, A\geqslant 1$ such that for all $x\in K, 0<r\leqslant 1$, $f\in\mathcal{F}$, we have*

$$\int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^2 m(dy) \leqslant C r^{d_w} \int_{B(x,Ar)} d\Gamma(f,f), \tag{3.25}$$

where $f_{B(x,r)} = \frac{1}{m(B(x,r))}\int_{B(x,r)} f(z)\, m(dz)$.

Proof. Since $m(F_w(K)) = 3^{-m}$ and $\operatorname{diam}(F_w(K)) = 2^{-m}$ for all $w\in W_m, m\in\mathbb{N}$, there exists $C_1>0$ such that

$$C_1^{-1} r^{d_f} \leqslant m(B(x,r)) \leqslant C_1 r^{d_f}, \quad \text{for all } x\in K, 0<r\leqslant 1. \tag{3.26}$$

Therefore for all $x\in K, 0<r\leqslant 1$, $f\in\mathcal{F}$,

$$\int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^2 m(dy) \leqslant C_1 r^{d_f} \sup_{y,z\in B(x,r)} |f(y)-f(z)|^2. \tag{3.27}$$

Fix any $y,z\in B(x,r)$ and $f\in\mathcal{F}$. Let $n\in\mathbb{N}\cup\{0\}$ be largest integer such that there exist $v,w\in W_n$ such that $y\in F_v(K), z\in F_w(K)$ and $F_v(K)\cap F_w(K)\neq\varnothing$. Let $q\in F_v(K)\cap F_w(K)$, $v\neq w$. Similar to the proof of Proposition 3.8 (see (3.11)), we estimate

$$\begin{aligned}
|f(y)-f(z)|^2 &\leqslant 2\left(|f(y)-f(q)|^2 + |f(z)-f(q)|^2\right)\\
&\overset{(3.9)}{\leqslant} 800\left(\mathcal{E}(f\circ F_v, f\circ F_v) + \mathcal{E}(f\circ F_w, f\circ F_w)\right)\\
&\overset{(3.24)}{\leqslant} 800\left(\frac{3}{5}\right)^n \Gamma(f,f)\left(F_v(K)\cup F_w(K)\right)
\end{aligned} \tag{3.28}$$

As explained in the proof of Proposition 3.8, $\frac{\sqrt{3}}{2}2^{-n-1} \leqslant |y-z| < 2r$ and hence by (3.27) and (3.28), we obtain the desired Poincaré inequality (3.25) with $A = 1 + \frac{8}{\sqrt{3}}$ as

$$F_v(K) \cup F_w(K) \subset B(x, r + 2^{-n}) \subset B(x, (1 + 8/\sqrt{3})r).$$

□

For a function $f \in L^1(K, m)$ and $r > 0$, by $f_r : K \to \mathbb{R}$ we denote the function

$$f_r(x) := \fint_{B(x,r)} f\, dm = \frac{1}{m(B(x,r))} \int_{B(x,r)} f\, dm, \quad \text{for all } x \in K.$$

The following estimate is called the pseudo-Poincaré inequality. The difference from Poincaré inequality is that the integrals involved are global.

Lemma 3.14. *There exists $C > 0$ such that for all $f \in \mathcal{F}, r > 0$, we have*

$$\int_K |f(x) - f_r(x)|^2\, m(dx) \leqslant C r^{d_w} \mathcal{E}(f, f),$$

where $d_w := \log_2 5$.

Proof. It suffices to assume $0 < r < 1$ as the case $r \geqslant 1$ follows from Proposition 3.13. By Jensen's inequality and the volume estimate (3.26), we have

$$\begin{aligned}\int_K |f(x) - f_r(x)|^2\, m(dx) &\lesssim \int_K \fint_{B(x,r)} |f(x) - f(y)|^2\, m(dy)\, m(dx) \\ &\lesssim r^{-d_f} \int_K \int_K |f(x) - f(y)|^2 \mathbb{1}_{\{d(x,y) \leqslant r\}}\, m(dy)\, m(dx) \qquad (3.29)\end{aligned}$$

Let N denote a r-net (a maximal r-separated subset; any two distinct points in N are at least distance r apart and any set that strictly contains N is not r-separated). The maximality of N implies that

$$\sum_{n \in N} \mathbb{1}_{B(n,r)} \geqslant \mathbb{1}_K, \quad \sum_{n \in N} \mathbb{1}_{B(n,2r)}(x) \mathbb{1}_{B(n,2r)}(y) \geqslant \mathbb{1}_{\{d(x,y) \leqslant r\}}, \quad \text{for all } x, y \in K.$$

The balls $B(n, 2Ar)$, where A is the constant in (3.25) do not overlap too much in the sense that $\sum_{n \in N} \mathbb{1}_{B(n,2Ar)} \lesssim \mathbb{1}_K$ (due to the volume estimate (3.26); see also Exercise 3.15). Hence by (3.29), for all $f \in \mathcal{F}$, we have

$$\begin{aligned}\int_K |f(x) - f_r(x)|^2\, m(dx) &\lesssim \int_K \fint_{B(x,r)} |f(x) - f(y)|^2\, m(dy)\, m(dx) \\ &\lesssim r^{-d_f} \int_K \int_K |f(x) - f(y)|^2 \mathbb{1}_{\{d(x,y) \leqslant r\}}\, m(dy)\, m(dx) \\ &\lesssim r^{-d_f} \sum_{n \in N} \int_{B(n,2r)} \int_{B(n,2r)} |f(x) - f(y)|^2\, m(dy)\, m(dx)\end{aligned}$$

$$\lesssim \sum_{n\in N}\int_{B(n,2r)} \left|f(x)-f_{B(n,2r)}\right|^2 m(dx)$$

$$\overset{(3.25)}{\lesssim} r^{d_w}\sum_{n\in N}\int_K \mathbb{1}_{B(n,2Ar)}\, d\Gamma(f,f) \lesssim r^{d_w}\mathcal{E}(f,f). \tag{3.30}$$

□

The following exercise outlines a proof of the bounded overlap estimate used above.

Exercise 3.15. Let $0<r<R$ and let N be a an r-separated subset of $\mathbb{R}^n$ (that is; $d(x,y)\geqslant r$ for any pair of distinct points $x,y\in N$). Show that there exists C (independent of r,R) such that for all $x\in\mathbb{R}^n$, we have

$$\sum_{y\in N}\mathbb{1}_{B(y,R)}(x)\leqslant C\left(\frac{R}{r}\right)^n.$$

Hint: Consider the set of all $y\in N$ such that $d(x,y)\leqslant R$ and observe that

$$B(x,R+r)\supset\bigcup_{y\in N, d(x,y)\leqslant R} B(y,r)$$

and use the additivity of Lebesgue measure.

We obtain a Nash inequality for the Brownian motion on the Sierpiński gasket.

Proposition 3.16. *There exists $C_1,C_2\in(0,\infty)$ such that*

$$\|f\|_2^{1+n/2}\leqslant\|f\|_1\left(C_1\|f\|_2^2+C_2\|\mathcal{E}(f,f)\|\right)^{n/4},\quad \textit{for all } f\in L^1(K,m)\cap\mathcal{F},$$

where $n=2d_f/d_w=2\log 5/\log 3$.

Proof. Let $0\leqslant r\leqslant 1$ and $f\in L^1(K,m)\cap\mathcal{F}$. Then by Lemma 3.14,

$$\|f\|_2\leqslant\|f-f_r\|_2+\|f_r\|_2\lesssim r^{d_w/2}\mathcal{E}(f,f)^{1/2}+\|f_r\|_2. \tag{3.31}$$

By Cauchy-Schwarz inequality, for all $f\in L^1(X,m), r\in(0,1]$, we have

$$\|f_r\|_2^2\leqslant\|f_r\|_\infty\|f_r\|_1\lesssim r^{-d_f}\|f\|_1\|f_r\|_1.$$

By Jensen's inequality, we have

$$\|f_r\|_1\leqslant\int f_r(x)\,m(dx)\lesssim r^{-d_f}\int_K\int_K f(y)\mathbb{1}_{\{d(x,y)<r\}}\,m(dy)\,m(dx)\asymp\int_K f(y)\,m(dy)\leqslant\|f\|_1.$$

Combining the above tow estimates, we obtain

$$\|f_r\|_2\lesssim r^{-d_f/2}\|f\|_1,\quad \text{for all } f\in L^1(K,m) \text{ and } r\in(0,1].$$

Thus by (3.31), we obtain

$$\|f\|_2\leqslant\|f-f_r\|_2+\|f_r\|_2\lesssim r^{d_w/2}\mathcal{E}(f,f)^{1/2}+r^{-d_f/2}\|f\|_1,\quad \text{for all } f\in\mathcal{F}\cap L^1(K,m) \text{ and } r\in(0,1].$$

Optimizing over $r\in(0,1]$ yields the desired estimate. □

Hence the using the Nash inequality above, we obtain the existence of heat kernel and an upper bound using Proposition 3.3.

Proposition 3.17. *Let m denote the self-similar measure on the Sierpiński gasket K such that $m(F_w(K)) = 3^{-k}$ for all $k \in \mathbb{N}$ and $w \in W_k$. Then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, m)$ admits a heat kernel. There exists $C > 0$ such that*

$$p_t(x, y) \leqslant \frac{C}{t^{d_f/d_w}}, \quad \textit{for all } t \in (0, 1], x, y \in K, \textit{ where } d_f = \log_2(3), d_w = \log_2(5).$$

As mentioned earlier in (1.4) sharp two-sided bounds on the heat kernel was obtained by Barlow and Perkins and is referred to as sub-Gaussian heat kernel bounds [BP].

The argument for obtaining heat kernel bound using Nash inequality is versatile and also applies to jump processes such as the symmetric α-stable process as outlined in the exercise below.

Exercise 3.18. Let $\alpha \in (0, 2)$ and consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^n, m)$, where m is the Lebesgue measure and

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^n, m) | \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} \, dx \, dy < \infty \right\}, \quad \mathcal{E}(f, f) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} \, dx \, dy,$$

for all $f \in \mathcal{F}$. Then show that the following psedo-Poincare inequality holds: there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^n} |f(x) - f_r(x)|^2 \, m(dx) \leqslant C_1 r^\alpha \mathcal{E}(f, f), \quad \text{for all } f \in \mathcal{F},$$

where $f_r(x) := \fint_{B(x,r)} f \, dm$. Using this and adapting the proof of Nash inequality in Proposition 3.16, show that there exist a heat kernel $p_t(\cdot, \cdot)$ for all $t > 0$ and $C_2 > 0$ such that

$$p_t(x, y) \leqslant \frac{C_2}{t^{n/\alpha}}, \quad \text{for all } t > 0, x, y \in X.$$

Next, we give an explanation for the exponential term in the expression of the heat kernel of the Sierpiński gasket

$$p_t(x, y) \asymp \frac{C_1}{t^{d_f/d_w}} \exp\left(-C_2 \left(\frac{d(x, y)^{d_w}}{t} \right)^{1/(d_w - 1)} \right), \quad \text{for all } x, y \in K, t \in (0, 1).$$

Usually in order to obtain heat kernel lower bounds, first one obtains the following near diagonal bound: there exists $c > 0$ such that

$$p_t(x, y) \gtrsim \frac{1}{t^{d_f/d_w}} \mathbb{1}_{\{d(x,y) \leqslant c t^{1/d_w}\}}, \quad \text{for all } x, y \in K, t \in (0, 1).$$

In order to obtain the full heat kernel lower bound, we choose $n \in \mathbb{N}$ a sequence of points $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) < \frac{c(t/n)^{1/d_w}}{2}$. Set $B_i = B(x_i, \frac{c(t/n)^{1/d_w}}{4})$, so that by the near diagonal lower bound, for any $y_i \in B_i, y_{i+1} \in B_{i+1}$ we have

$$p_{t/n}(y_i, y_{i+1}) \gtrsim (t/n)^{-d_f/d_w}.$$

Such points $x_0, x_1, \ldots, x_n$ exist provided $d(x,y) \lesssim n\frac{c(t/n)^{1/d_w}}{2}$ or equivalently,

$$n \gtrsim \left(\frac{d(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}.$$

By the Chapman-Kolmogorov equation, we have the bound

$$p_t(x,y) \gtrsim \int_{B_2} \cdots \int_{B_{n-1}} p_{t/n}(x,y_1)p_{t/n}(y_1,y_2)\ldots p_{t/n}(y_{n-1},y)\, m(dy_{n-1}) \ \ldots \ m(dy_1),$$

which implies

$$p_t(x,y) \gtrsim (t/n)^{-d_f/d_w} \exp(-c_0 n)$$

for some $c_0 > 0$. By choosing n as small as possible (that is $n \asymp \left(\frac{d(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}$), we obtain the desired lower bound. The proof of heat kernel upper also uses a similar chaining argument.

The exact value of d_w obtained for Sierpiński gasket is not the typical for diffusion processes defined on fractals. For many 'infinitely ramified' fractals such as the Sierpiński carpet, the value of d_w is not known. In these cases, the existence of such an exponent is shown using estimates on resistances of approximating graphs. We refer the reader to [Bar98, Kig01] for more on diffusion on fractals.

3.3 General characterization of heat kernel bounds and Harnack inequalities

While the Nash inequality gives sharp on-diagonal upper bounds (bounds $p_t(x,x)$) in many examples such as Brownian motion in $\mathbb{R}^n$ or the Sierpiński gasket and α-stable process, it is not suited to handle situations where diffusion has very different heat kernel at different points in the space. In particular, Proposition 3.2 may not apply as a semigroup (P_t) might admit a heat kernel without being ultracontractive (that is $\|P_t\|_{1\to\infty} = \infty$ for all $t > 0$) as illustrated below.

Example 3.19. Consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^n, m_\alpha)$, where $m_\alpha(dx) = (1+|x|^2)^{\alpha/2}$, $\alpha \in (-n, \infty)$, $n \in \mathbb{N}$ with $n \geqslant 2$ with Dirichlet form given by

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, m_\alpha(dx),$$

for all $f \in \mathcal{F}$, where ∇f is the distributional gradient of f. Note that this includes Brownian motion as a special case $\alpha = 0$. The generator (integration by parts) can be viewed as Brownian motion with a radial drift as it equals

$$L_\alpha f = (1+|x|^{\alpha/2})^{-2}\mathrm{div}\left((1+|x|^{\alpha/2})^2 f\right) = \Delta f + \frac{\alpha x \cdot \nabla f}{(1+|x|^2)}.$$

This is called the weighted Laplacian as this operator is self-adjoint on $L^2(\mathbb{R}^n, m_\alpha)$. Note that integration by parts implies

$$\int_{\mathbb{R}^n} L_\alpha(f) g \, dm_\alpha = \int_{\mathbb{R}^n} f L_\alpha(g) \, dm_\alpha = -\int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, dm_\alpha, \quad \text{for all } f, g \in C_c^\infty(\mathbb{R}^n).$$

In this case the measure m_α satisfies the estimate

$$m_\alpha(B(x,r)) \asymp r^n (1 + |x| + r)^\alpha.$$

Grigor'yan and Saloff-Coste [GS] obtain the following Gaussian heat kernel bounds: there exist a heat kernel p_t such that

$$p_t(x,y) \asymp \frac{C}{m_\alpha(B(x,\sqrt{t}))} \exp\left(-c\frac{d(x,y)^2}{t}\right), \quad \text{for all } t > 0 \text{ and } x, y \in \mathbb{R}^n,$$

which can be written as

$$p_t(x,y) \asymp \frac{\exp\left(-cd(x,y)^2/t\right)}{t^{n/2}(1 + |x| + \sqrt{t})^\alpha}, \text{for all } t > 0 \text{ and } x, y \in \mathbb{R}^n. \tag{3.32}$$

If $\alpha \in (-n, 0)$, then by letting $x = y \to \infty$ in the above estimate, we obtain

$$\|P_t\|_{1\to\infty} = \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n} p_t(x,y) = \infty.$$

This example shows that ultracontractivity is sufficient but not necessary for the existence of heat kernel.

An important message from the above example is that the heat kernel can have different behavior at different parts of the space (unlike the earlier examples). To handle such behaviors at different parts of the space, instead of relying on *global* functional inequalities such as the Nash inequality, we needs to develop methods that takes into account different local behaviors. This is accomplished by using *local* versions of functional inequalities: that is energies and norms are considered on balls rather on the whole space. In particular, the key tool to obtain (3.32) in [GS] is a Poincaré inequality on balls. We already encounter such a Poincaré inequality on the Sierpiński gasket (see Proposition 3.13).

Next, we will survey the relationships between heat kernel estimates and Harnack inequalities. An important influence in the study of heat kernel bounds was works of de Giorgi, Nash and Moser [DeG, Nas, Mos61, Mos64] on Harnack inequality and Hölder continuity estimates for solutions to elliptic and parabolic equations corresponding to uniformly elliptic operators.

The classical *elliptic Harnack inequality* (1887) states that there exists $C > 1$ such that for any $x \in \mathbb{R}^n, r > 0$, and any non-negative harmonic function $h : B(x, 2r) \to [0, \infty)$ (that is, $\Delta h \equiv 0$ on $B(x, 2r)$), we have

$$\sup_{B(x,r)} h \leqslant C \inf_{B(x,r)} h.$$

Here the constant C is independent of x, r and h. The usefulness of Harnack inequality is due to its robustness; that is, it is stable under perturbations. The first evidence of such a stability is due to Moser (1964) who showed that the elliptic Harnack inequality holds if we replace Laplace operator with an uniformly elliptic operator (in divergence form) $\mathcal{A}$

$$\mathcal{A}f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial f}{\partial x_j}\Big),$$

where $(a_{ij}(x), x \in \mathbb{R}^d)$ is bounded, measurable and uniformly elliptic. This allowed Moser to give a new proof of Hölder regularity of the associated solutions a result earlier obtained by de Giorgi [DeG]. Using very different methods S. Y. Cheng and S. T. Yau [CY] obtained elliptic Harnack inequality obtained ellitpic Harnack inequality for Riemannian manifolds with non-negative Ricci curvature.

Since the kind of Hölder regularity obtained by Moser will be used later in our analysis of boundary trace process, we recall Moser's oscillation lemma. The idea of Moser is to apply Harnack inequality to $h(\cdot) - \inf_{B(x,2r)} h$ and $\sup_{B(x,2r)} h - h(\cdot)$ to obtain

$$\sup_{B(x,r)} h - \inf_{B(x,2r)} h \leqslant C(\inf_{B(x,r)} h - \inf_{B(x,2r)} h)$$

and

$$\sup_{B(x,2r)} h - \inf_{B(x,r)} h \leqslant C(\sup_{B(x,2r)} h - \sup_{B(x,r)} h).$$

Adding the above inequalities, and setting $\operatorname{osc}_A h = \sup_A h - \inf_A h$, we obtain

$$\operatorname*{osc}_{B(x,r)} h \leqslant \frac{C-1}{C+1} \operatorname*{osc}_{B(x,2r)} h.$$

The above estimate is called Moser's oscillation lemma. By iterating the above estimate, we obtain

$$|h(y_1) - h(y_2)| \lesssim \left(\frac{d(y_1, y_2)}{r}\right)^{\alpha} \|h\|_{\infty}, \quad \text{for all } y_1, y_2 \in B(x, r),$$

where $\alpha = \log_2\left(\frac{C-1}{C+1}\right)$. If h is harmonic and bounded in $\mathbb{R}^n$ by letting $r \to \infty$ in the above estimate, we obtain the Liouville property that every bounded harmonic function is constant.

The parabolic version for the Harnack inequality for the heat equation is (independently) due to Hadamard and Pini (1954). The *parabolic Harnack inequality* for non-negative solutions of heat equation states that there exist $C > 1$ such that for any $x \in \mathbb{R}^n, r > 0$ and a non-negative solution $u : (0, r^2) \times B(x, r) \to [0, \infty)$, we have

$$\sup_{Q_-} u \leqslant C \inf_{Q_+} u,$$

where $Q_- = (r^2/4.r^2/2) \times B(x.r/2)$ and $Q_+ := (3r^2/4.r^2) \times B(x.r/2)$. Here the constant C is independent of x, r, u. Note that the parabolic Harnack inequality implies the elliptic

Harnack inequality as $u(t,x) = h(x)$ satisfies the heat equation for any harmonic function h. Moser obtained the parabolic Harnack inequality for any uniformly elliptic operator (in divergence form) as mentioned above for the elliptic case [Mos64]. Moser adapted the above argument for Hölder continuity to obtain parabolic Hölder regularity that providing a new proof of a result of Nash [Nas]. P. Li and S. T. Yau obtained parabolic Harnack inequality for Riemannian manifolds with non-negative Ricci curvature [LY].

Since $u = P_t f$ is a solution to the heat equation, where (P_t) is the Brownian semigroup (sped up by a factor of 2) we can think of parabolic Harnack inequality is a *local* version of ultracontractivity. To see this, let $f \geqslant 0$, $f \in L^1 \cap L^2$ with $\|f\|_1 = 1$. Then for any $x \in \mathbb{R}^n, r > 0$,

$$\sup_{y \in B(x,r/2)} P_{3r^2/8} f(y) \leqslant C \inf_{(t,y) \in Q_+} P_t(f)(y) \leqslant C \frac{1}{m(B(x,r/2))} \int_{B(x,r/2)} P_{7r^2/8}(f)\, dm \leqslant \frac{C}{m(B(x,r/2))},$$

where the last inequality follows from the fact the semigroup is contraction in L^1. This yields *local* ultracontractivity bound

$$\left\| P_{3r^2/8} \right\|_{L^1(\mathbb{R}^n) \to L^\infty(B(x,r/2))} \leqslant \frac{C}{m(B(x,r/2))},$$

which implies

$$\operatorname*{ess\,sup}_{y \in B(x,r/2), z \in \mathbb{R}^n} p_{3r^2/8}(y,z) \leqslant \frac{C}{m(B(x,r/2))}.$$

Such local ultracontractivity bounds are enough to obtain the existence of heat kernel and provide sharp bounds in many cases such as Example 3.19. This suggests that the parabolic Harnack inequality can be viewed as a local version of $L^{-\infty}$ to L^∞ ultracontractivity estimate.

A *metric measure Dirichlet space*, or an *MMD space* $(X, d, m, \mathcal{E}, \mathcal{F})$ is a complete, locally compact metric space (X, d), equipped with a Radon measure m of full support and a strongly local, regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$.

We recall the definitions of harmonic function and caloric (solutions to 'heat' equation) for a MMD space.

Definition 3.20. A function $h \in \mathcal{F}$ is said to be $\mathcal{E}$*-harmonic* on an open subset U of X, if

$$\mathcal{E}(h, f) = 0 \qquad \text{for all } f \in \mathcal{F} \cap C_c(X) \text{ with } \operatorname{supp}_m[f] \subset U. \tag{3.33}$$

Let I be an open interval in $\mathbb{R}$. We say that a function $u : I \to L^2(X, m)$ is *weakly differentiable* at $t_0 \in I$ if for any $f \in L^2(X, m)$ the function $t \mapsto \langle u(t), f \rangle$ is differentiable at t_0, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(X, m)$. If u is weakly differentiable at t_0, then by the uniform boundedness principle, there exists a (unique) function $w \in L^2(X, m)$ such that

$$\lim_{t \to t_0} \left\langle \frac{u(t) - u(t_0)}{t - t_0}, f \right\rangle = \langle w, f \rangle, \qquad \text{for all } f \in L^2(X, m).$$

We say that the function w is the *weak derivative* of the function u at t_0 and write $w = u'(t_0)$.

Let I be an open interval in $\mathbb{R}$ and let Ω be an open subset of X. A function $u : I \to \mathcal{F}$ is said to be *caloric* in $I \times \Omega$ if u is weakly differentiable in the space $L^2(\Omega)$ at any $t \in I$, and for any $f \in \mathcal{F} \cap C_c(\Omega)$, and for any $t \in I$,

$$\langle u', f \rangle + \mathcal{E}(u, f) = 0. \tag{3.34}$$

In order to describe the space-time scaling, the following notion is used. Let $\Psi : [0, \infty) \to [0, \infty)$ be a homeomorphism, such that for all $0 < r \leqslant R$,

$$C^{-1} \left(\frac{R}{r}\right)^{\beta_1} \leqslant \frac{\Psi(R)}{\Psi(r)} \leqslant C \left(\frac{R}{r}\right)^{\beta_2}, \tag{3.35}$$

for some constants $1 < \beta_1 < \beta_2$ and $C > 1$. Such a function Ψ is said to be a **scale function**. For example, the scale function in case of the Brownian motion on $\mathbb{R}^n$ is $\Psi(r) = r^2$ reflected by the space-time scaling in the parabolic Harnack inequality mentioned above. We allow for quite general space-time scaling in the following definition.

Definition 3.21 (Harnack inequalities)**.** We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *elliptic Harnack inequality* (abbreviated as EHI), if there exist $C > 1$ and $\delta \in (0, 1)$ such that for all $x \in X$, $r > 0$ and for any $h \in \mathcal{F}$ that is non-negative on $B(x, r)$ and $\mathcal{E}$-harmonic on $B(x, r)$, we have

$$\operatorname*{ess\,sup}_{B(x,\delta r)} h \leqslant C \operatorname*{ess\,inf}_{B(x,\delta r)} h. \tag{EHI}$$

We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *parabolic Harnack inequality* with space-time scaling Ψ (abbreviated as PHI(Ψ)), if there exist $0 < C_1 < C_2 < C_3 < C_4 < \infty$, $C_5 > 1$ and $\delta \in (0, 1)$ such that for all $x \in X$, $r > 0$ and for any non-negative bounded caloric function u on the space-time cylinder $Q = (a, a + C_4\Psi(r)) \times B(x, r)$, we have

$$\operatorname*{ess\,sup}_{Q_-} u \leqslant C_5 \operatorname*{ess\,inf}_{Q_+} u, \tag{PHI(Ψ)}$$

where $Q_- = (a + C_1\Psi(r), a + C_2\Psi(r)) \times B(x, \delta r)$ and $Q_+ = (a + C_3\Psi(r), a + C_4\Psi(r)) \times B(x, \delta r)$.

The measures we consider will often satisfy the following volume doubling and reverse volume doubling properties.

Definition 3.22 (Volume doubling and Reverse volume doubling properties)**.** Let (X, d) be a metric space and let μ be a Borel measure on X.

(a) We say that μ satisfies the *volume doubling property* VD, or μ is a doubling measure on (X, d), or (X, d, μ) is VD, if there exists $C_D \in (1, \infty)$ such that

$$0 < \mu(B(x, 2r)) \leqslant C_D \mu(B(x, r)) < \infty \quad \text{for all } x \in X,\ r \in (0, \infty). \tag{VD}$$

(b) We say that μ satisfies the *reverse volume doubling property* RVD, or (X, d, μ) is RVD, if there exist $C_1, C_2 \in (1, \infty), \alpha \in (0, \infty)$ such that

$$\mu(B(x, R)) \geqslant C_1^{-1} \left(\frac{R}{r}\right)^{\alpha} \mu(B(x, r)) \tag{RVD}$$

for all $x \in X$, $0 < r \leqslant R < \operatorname{diam}_d(X)/C_2$.

Exercise 3.23. (i) Show that if a metric space (X, d) is connected and if m is a measure that satisfies the volume doubling property, then it also satisfies the reverse volume doubling property.

(ii) Consider the measure m_α on $\mathbb{R}^n$ in Example 3.19. Then show that m_α satisfies the volume doubling property if and only if $\alpha > -n$.

Associated with a scale function Ψ satisfying (3.35), we define

$$\Phi(s) = \sup_{r>0} \left(\frac{s}{r} - \frac{1}{\Psi(r)}\right). \tag{3.36}$$

For example, if $\Psi(r) \asymp r^\beta$ for some $\beta > 1$, then $\Phi(r) \asymp r^{\beta/(\beta-1)}$. We define sub-Gaussian heat kernel estimates with space-time scaling Ψ below.

Definition 3.24 (HKE(Ψ))**.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space, and let $\{P_t\}_{t>0}$ denote its associated Markov semigroup. A family $\{p_t\}_{t>0}$ of non-negative Borel measurable functions on $X \times X$ is called the *heat kernel* of $(X, d, m, \mathcal{E}, \mathcal{F})$, if p_t is the integral kernel of the operator P_t for any $t > 0$, that is, for any $t > 0$ and for any $f \in L^2(X, m)$,

$$P_t f(x) = \int_X p_t(x, y) f(y)\, dm(y) \qquad \text{for } m\text{-almost all } x \in X.$$

We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the sub-Gaussian **heat kernel estimates** HKE(Ψ), if there exist $C_1, c_1, c_2, c_3, \delta \in (0, \infty)$ and a heat kernel $\{p_t\}_{t>0}$ such that for any $t > 0$,

$$p_t(x, y) \leqslant \frac{C_1}{m\big(B(x, \Psi^{-1}(t))\big)} \exp\left(-c_1 t \Phi\left(c_2 \frac{d(x, y)}{t}\right)\right) \qquad \text{for } m\text{-a.e. } x, y \in X, \tag{3.37}$$

$$p_t(x, y) \geqslant \frac{c_3}{m\big(B(x, \Psi^{-1}(t))\big)} \qquad \text{for } m\text{-a.e. } x, y \in X \text{ with } d(x, y) \leqslant \delta \Psi^{-1}(t), \tag{3.38}$$

where Φ is as defined in (3.36). If the underlying metric is close to a geodesic metric (see Definition 3.25), then by the same argument sketched for Sierpiński gasket to obtain full lower bound from near diagonal lower bound, we obtain a lower bound that matches the upper bound in (3.37). We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *full* sub-Gaussian **heat kernel estimates** $\mathrm{HKE_f}(\Psi)$, if there exist $C_1, c_1, c_2, c_3, C_4, C_5 > 0$ and a heat kernel $\{p_t\}_{t>0}$ such that for any $t > 0$, we have (3.37) along with the lower bound

$$p_t(x, y) \geqslant \frac{c_3}{m\big(B(x, \Psi^{-1}(t))\big)} \exp\left(-C_4 t \Phi\left(C_5 \frac{d(x, y)}{t}\right)\right) \qquad \text{for } m\text{-a.e. } x, y \in X. \tag{3.39}$$

We clarify what we mean by 'close to a geodesic metric' mentioned above. The argument presented for heat kernel lower bounds for Sierpinski gasket relies on the following property of the underlying metric.

Definition 3.25. Let (X, d) be a metric space. We say that a sequence $\{x_i\}_{i=0}^N$ of points in X is an *ε-chain* between points $x, y \in X$ if

$$x_0 = x, \quad x_N = y, \quad \text{and} \quad d(x_i, x_{i+1}) < \varepsilon \quad \text{for all } i = 0, 1, \dots, N-1.$$

For any $\varepsilon > 0$ and $x, y \in X$, define

$$d_\varepsilon(x, y) = \inf_{\{x_i\} \text{ is } \varepsilon\text{-chain}} \sum_{i=0}^{N-1} d(x_i, x_{i+1}),$$

where the infimum is taken over all ε-chains $\{x_i\}_{i=0}^N$ between x, y with arbitrary N. We say that (X, d) satisfies the *chain condition* if there exists $K \in [1, \infty)$ such that

$$d_\epsilon(x, y) \leqslant K d(x, y) \quad \text{for all } \epsilon > 0, x, y \in X.$$

The following theorem provides a large class of examples of spaces satisfying sub-Gaussian heat kernel bounds.

The sub-Gaussian heat kernel estimates are known to imply Poincaré inequality, capacity estimates, cutoff energy inequality and exit time bounds.

Definition 3.26. Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space, $\Psi : [0, \infty) \to [0, \infty)$ be a scale function, and let $\Gamma(\cdot, \cdot)$ denote the corresponding energy measure.

(a) We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the **Poincaré inequality** $\mathrm{PI}(\Psi)$, if there exist constants $C_P, A_P \geqslant 1$ such that for all $(x, r) \in X \times (0, \infty)$ and all $f \in \mathcal{F}$,

$$\int_{B(x,r)} (f - f_{B(x,r)})^2 \, dm \leqslant C_P \Psi(r) \int_{B(x, A_P r)} d\Gamma(f, f), \qquad \mathrm{PI}(\Psi)$$

where $f_{B(x,r)} := m(B(x, r))^{-1} \int_{B(x,r)} f \, dm$.

(b) For open subsets U, V of X with $\overline{U} \subset V$, we say that a function $\phi \in \mathcal{F}$ is a *cutoff function* for $U \subset V$ if $0 \leqslant \phi \leqslant 1$, $\phi = 1$ on a neighbourhood of $\overline{U}$ and $\mathrm{supp}_m[\phi] \subset V$. Then we say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the **cutoff energy inequality** $\mathrm{CS}(\Psi)$, if there exist $C_S > 0, A_1, A_2, C_1 > 1$ such that the following holds: for all $x \in X$ and $0 < R < \mathrm{diam}(X, d)/A_2$, there exists a cutoff function $\phi \in \mathcal{F}$ for $B(x, R) \subset B(x, A_1 R)$ such that for all $f \in \mathcal{F}$,

$$\int_{B(x, A_1 R)} \widetilde{f}^2 \, d\Gamma(\phi, \phi) \leqslant C_1 \int_{B(x, A_1 R)} d\Gamma(f, f) + \frac{C_1}{\Psi(R)} \int_{B(x, A_1 R)} f^2 \, dm; \qquad \mathrm{CSS}(\Psi)$$

where $\widetilde{f}$ is a quasi-continuous version of $f \in \mathcal{F}$.

(c) For subsets $A, B \subset X$, we define

$$\mathcal{F}(A,B) := \{f \in \mathcal{F} : f \equiv 1 \text{ on a neighborhood of } A \text{ and } f \equiv 0 \text{ on a neighborhood of } B\},$$

and the capacity $\mathrm{Cap}(A,B)$ as

$$\mathrm{Cap}(A,B) = \inf\{\mathcal{E}(f,f) : f \in \mathcal{F}(A,B)\}.$$

We say that an MMD space $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies the *capacity bound* $\mathrm{cap}(\Psi)$ if there exist $C_1, A_1, A_2 > 1$ such that for all $R \in (0, \mathrm{diam}(X,d)/A_2)$, $x \in X$, we have

$$C_1^{-1}\frac{m(B(x,R))}{\Psi(R)} \leqslant \mathrm{Cap}(B(x,R), B(x,A_1R)^c) \leqslant C_1\frac{m(B(x,R))}{\Psi(R)}. \qquad \mathrm{cap}(\Psi)$$

The upper and lower bounds on capacity above will be denoted by $\mathrm{cap}(\Psi)_{\leqslant}$ and $\mathrm{cap}(\Psi)_{\geqslant}$ respectively.

(d) We say that the an MMD space $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies the *exit time bound* $\mathrm{E}(\Psi)$, if there exist $C, A \in (1,\infty), \delta \in (0,1)$ such that for all $x \in X, 0 < r < \mathrm{diam}(X,d)/A$ the corresponding Markov process satisfies

$$C^{-1}\Psi(r) \leqslant \operatorname*{ess\,inf}_{y \in B(x,\delta r)} \mathbb{E}_y[\tau_{B(x,r)}], \quad \operatorname*{ess\,sup}_{y \in B(x,r)} \mathbb{E}_y[\tau_{B(x,r)}] \leqslant C\Psi(r). \qquad \mathrm{E}(\Psi)$$

The cutoff energy inequality CS(Ψ) mentioned above was first introduced by Barlow and Bass [BB04]. The version above is from [Mur24] which combines simplifications introduced in [AB, BM18].

The following theorem is due to a number of authors [Gri, Sal92, Stu96, BB04, BBK, GHL15].

Theorem 3.27. *Let $(X,d,m,\mathcal{E},\mathcal{F})$ be an MMD space and let $\Psi : [0,\infty) \to [0,\infty)$ be a scale function. Then the following are equivalent:*

(a) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies PHI(Ψ).

(b) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies VD, EHI *and* cap(Ψ).

(c) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies VD, EHI *and* E(Ψ).

(d) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies VD *and* HKE(Ψ).

(e) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies VD, PI(Ψ) *and* CS(Ψ).

Moreover, if $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies any one of above equivalent conditions, then (X,d) is connected and (X,d,m) is RVD.

The equivalence between (a) and (e), (d) and (e) are of significance as it implies the stability of parabolic Harnack inequality and sub-Gaussian heat kernel estimates. For Brownian motion on Riemannian manifolds the equivalence between (a) and (e) for the case $\Psi(r) = r^2$ is independently due to Grigor'yan and Saloff-Coste [Gri, Sal92]. This was later generalized to MMD spaces by Sturm [Stu96]. In these results it turns out that the condition CS(Ψ) is unnecessary for the case $\Psi(r) = r^2$. This condition was introduced by Barlow and Bass [BB04] to obtain the stability of parabaolic Harnack inequality on graphs with arbitrary Ψ (see also [BBK] for Dirichlet spaces). We will present a number of applications of Theorem 3.27. A major conjecture is to show that each of the properties in Theorem 3.27 is equivalent to the conjunction of the properties VD, PI(Ψ) and cap(Ψ).

The relationship between full sub-Gaussian heat kenrel bounds $\mathrm{HKE_f}(\Psi)$ and the version HKE(Ψ) with near diagonal lower bounds is given by the following result obtained in [Mur20].

Theorem 3.28. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $\Psi : [0, \infty) \to [0, \infty)$ be a scale function. Then the following are equivalent:*

1. *$(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies $\mathrm{HKE_f}(\Psi)$.*
2. *$(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies HKE(Ψ), the volume doubling property and the underlying metric satisfies the chain condition.*

More precisely, it was shown in [Mur20] that $\mathrm{HKE_f}(\Psi)$ implies the chain condition for the underlying metric. The other implications were well-known.

Remark 3.29. Functional inequalities such as Poicaré inequality and cutoff energy inequalities are stable under perturbations due to Le Jan's domination principle, [LJ, Proposition 1.5.5]. If two regular Dirichlet forms $(\mathcal{E}, \mathcal{F})$ and $(\widetilde{\mathcal{E}}, \mathcal{F})$ be two regular Dirchlet forms on $L^2(X, m)$ such that there exists $C > 0$ satisfying

$$\mathcal{E}(f, f) \leqslant C\widetilde{\mathcal{E}}(f, f)$$

Then their corresponding energy measures also satisfy the same estimate: that is,

$$\Gamma(f, f)(A) \leqslant C\widetilde{\Gamma}(f, f)(A), \quad \text{for all Borel sets } A \text{ and } f \in \mathcal{F}.$$

The following proof due is to Mosco uses chain rule for energy measures [Mosco]. It states that for any $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$ with $\Phi(0, \dots, 0) = 0$, $f_1, \dots, f_m, g \in \mathcal{F} \cap L^\infty$, we have $\Phi(f_1, \dots, f_m) \in \mathcal{F}$ and

$$\Gamma(\Phi(f_1, \dots, f_m), g) = \sum_{i=1}^{m} \partial_i \Phi(f_1, \dots, f_m) \cdot \Gamma(f_i, g).$$

For simplicity let us assume that both Dirichlet forms above are strongly local. For any $\phi, f \in \mathcal{F} \cap L^\infty, \lambda > 0$, we have by the Leibniz rule

$$\Gamma(\phi \cos(\lambda f), \phi \cos(\lambda f)) + \Gamma(\phi \sin(\lambda f), \phi \sin(\lambda f)) = \Gamma(\phi, \phi) + \lambda^2 \phi^2 \Gamma(f, f).$$

Hence by computing the above energy measures on X and using the assumption, we have

$$\mathcal{E}(\phi,\phi)+\int_X \lambda^2\phi^2\,d\Gamma(f,f)\leqslant C\widetilde{\mathcal{E}}(\phi,\phi)+C\int_X \lambda^2\phi^2\,d\widetilde{\Gamma}(f,f)$$

for all $\phi, f\in\mathcal{F}\cap L^\infty, \lambda>0$. Dividing by λ^2 on both sides and letting $\lambda\to\infty$, we obtain

$$\int_X \phi^2\,d\Gamma(f,f)\leqslant C\int_X \phi^2\,d\widetilde{\Gamma}(f,f),\quad \text{for all } \phi, f\in\mathcal{F}\cap L^\infty.$$

This along with the regularity of $(\mathcal{E},\mathcal{F})$ implies the desired inequality.

This shows that if we have two strongly local regular Dirichlet forms such that $\mathcal{E}\asymp\widetilde{\mathcal{E}}$, then the energy measures are also comparable. This means that functional inequalities such as the Poincaré inequality and cutoff energy inequality are stable under such perturbations.

Roughly speaking Poincaré inequality implies that the space cannot have too many bottlenecks and there is a large collection of curves joining pair of points. Using large collections of curves joining pair of points it is possible to obtain Poincaré inequality. We sketch this in the case of $\mathbb{R}^n$ but this approach also works in many other spaces.

Proposition 3.30. *There exists $A, C>1$ such that for all $f\in C_c^1(\mathbb{R}^n), x\in\mathbb{R}^n, r>0$,*

$$\int_{B(x,r)}\left|f(y)-f_{B(x,r)}\right|^2 dy\leqslant\int_{B(x,Ar)}|\nabla f(y)|^2\,dy.$$

Proof. Let m denote the Lebesgue measure on $\mathbb{R}^n$. Using the identity

$$\int_{B(x,r)}\left|f(y)-f_{B(x,r)}\right|^2 dy=\frac{1}{2m(B(x,r))}\int_{B(x,r)}\int_{B(x,r)}(f(y)-f(z))^2\,dy\,dz,$$

it suffices to estimate $|f(y)-f(z)|$ for any $y,z\in B(x,r)$ in terms of ∇f. For any piecewise smooth curve γ from y to z, we have

$$|f(y)-f(z)|^2\leqslant\left(\int_\gamma|\nabla f(\gamma(s))|\,ds\right)^2\leqslant L(\gamma)\int_\gamma|\nabla f(\gamma(s))|^2\,ds,$$

where ds denotes the arc-length measure of γ and $L(\gamma)$ is the length of γ. We now pick γ at random according to measure $\mathbb{P}_{y,z}$ which is supported on curves of union of two straight lines from y to p and from p to z, where p is chosen uniformly among all points such that

$$|p-y|=|p-z|\leqslant|y-z|.$$

Hence the length of any curve γ picked according to $\mathbb{P}_{y,z}$ is of length at most $2|y-z|\leqslant 4r$ and any point $q\in\gamma$ satisfies $|q-y|\wedge|q-z|\leqslant|y-z|\leqslant 2r$. This random collection of curves is sometimes called a *pencil of curves* between y and z. Combining these we obtain for any $y,z\in B(x,r)$

$$|f(y)-f(z)|^2\leqslant\int L(\gamma)\int_\gamma|\nabla f(\gamma(s))|^2\,ds\,\mathbb{P}_{y,z}(d\gamma)$$

$$\lesssim r \int_{B(x,3r)} |\nabla f(w)|^2 \left(|y-w|^{-n+1} + |z-w|^{-n+1}\right) dw$$

Hence we have

$$\begin{aligned}
&\int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^2 dy \\
&\quad = \frac{1}{2m(B(x,r))} \int_{B(x,r)} \int_{B(x,r)} (f(y)-f(z))^2 \, dy \, dz \\
&\quad \lesssim \frac{r}{m(B(x,r))} \int_{B(x,3r)} |\nabla f(w)|^2 \int_{B(x,r)} \int_{B(x,r)} \left(|y-w|^{-n+1} + |z-w|^{-n+1}\right) dy \, dz \, dw \\
&\quad \asymp r \int_{B(x,3r)} |\nabla f(w)|^2 \int_{B(x,r)} |y-w|^{-n+1} \, dy \, dw \\
&\quad \lesssim r^2 \int_{B(x,3r)} |\nabla f(w)|^2 \, dw.
\end{aligned}$$

□

The same pencil of curves approach to prove Poincaré inequality can be adapted to prove the following theorem.

Theorem 3.31. *Let $V, \Psi : [0,\infty) \to [0,\infty)$ be homeomorphisms such that Ψ is a scale function and there exists $\alpha > 0$ such that*

$$\frac{V(R)}{V(r)} \lesssim \frac{R^\alpha}{r^\alpha}, \quad \text{for all } 0 < r < R. \tag{3.40}$$

Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an unbounded (infinite diameter) MMD space that satisfies the full sub-Gaussian kernel estimate $\mathrm{HKE_f}(\Psi)$. Furthermore, suppose that exists $C_1 > 1$ satisfying

$$C_1^{-1} V(r) \leqslant m(B(x,r)) \leqslant C_1 V(r), \quad \text{for all } x \in X, r > 0. \tag{3.41}$$

Then there exists $C \in (1,\infty)$ such that

$$C^{-1} \frac{R^2}{r^2} \leqslant \frac{\Psi(R)}{\Psi(r)} \leqslant C \frac{RV(R)}{rV(r)}, \quad \text{for all } 0 < r \leqslant R. \tag{3.42}$$

Conversely, if V, Ψ satisfy (3.42) and (3.40), then there exist an unbounded (infinite diameter) MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ that satisfies (3.41) and the full sub-Gaussian kernel estimate $\mathrm{HKE_f}(\Psi)$.

In the special case when $V(r) = r^\alpha, \Psi(r) = r^\beta$, the estimate (3.42) is equivalent to $2 \leqslant \beta \leqslant \alpha + 1$. This special case was shown by Barlow in the discrete time setting of random walks on graphs [Bar04].

We now sketch the proof of Theorem 3.31 and refer to [Mur25+] for a details and further background. Let $R > r > 0$. Let n be the smallest positive integer such that

$B(x, A_1R)\backslash B(x,R) \supseteq \cup_{j=0}^{n-1} B(x, R+(j+1)r)\backslash B(x, R+jr)$, where $A_1 \in (1,\infty)$ is the constant in cap(Ψ). By the triangle inequality

$$n \gtrsim \frac{R}{r}. \tag{3.43}$$

By a covering argument (by covering the annulus $B(x, R+(j+1)r)\backslash B(x, R+jr)$ with balls of radii comparable to r), we obtain cutoff function ψ_j for $B(x, R+jr) \subset B(x, R+(j+1)r)$ such that

$$\mathcal{E}(\psi_j, \psi_j) \lesssim \frac{m(B(x, R+(j+1)r))\backslash B(x, R+jr)}{\Psi(r)}. \tag{3.44}$$

Let N be a $A_1^{-1}r/2$-net (maximal r-separated subset) of $B(x, R+(j+1)r)$, where A_1 is the constant in cap(Ψ). For each $n \in N$, we choose a function $\phi_n \in C_c(X)$ a cutoff function for $B(n, A_1^{-1}r/2) \subset B(n, r/2)$ such that $\mathcal{E}(\phi_n, \phi_n) \lesssim m(B(n, r/2))/\Psi(r)$. Choosing $\phi = \max_{n\in N} \phi_n$ yields the bound (3.44) due the following properties of energy measure:

$$\Gamma(\phi, \phi) \leqslant \sum_{n\in N} \Gamma(\phi_n, \phi_n)$$

and the property that if $f \in \mathcal{F} \cap C_C(X)$ is constant in an open set U, then $\Gamma(f,f)(U) = 0$ (due to strong locality). The first estimate above is due to the following description of energy measure in terms of the associated process (Y_t):

$$\int_X f\, d\Gamma(\phi,\phi) = \lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}_{f\cdot m}\left((\phi(Y_t) - \phi(Y_0))^2\right), \quad \text{for all } f \in C_c(X) \text{ with } f \geqslant 0.$$

By considering the function $\psi = \frac{1}{n}\sum_{j=0}^{n-1} \psi_j$, by strong locality and cap(Ψ)

$$\begin{aligned}\frac{V(R)}{\Psi(R)} &\overset{\text{cap}(\Psi)_{\geqslant},(3.41)}{\lesssim} \operatorname{Cap}(B(x,R), B(x,A_1R)^c) \\ &\leqslant \mathcal{E}(\psi,\psi) = \frac{1}{n^2}\sum_{j=0}^{n-1} \mathcal{E}(\psi_j,\psi_j) \overset{(3.43),(3.44)}{\lesssim} \frac{r^2}{R^2}\frac{\mu(B(x,AR))\backslash B(x,R)}{\Psi(r)} \overset{(3.41)}{\lesssim} \frac{r^2}{R^2}\frac{V(R)}{\Psi(r)}\end{aligned}$$

which implies the necessity of the first estimate in (3.42).

For the necessity of the second estimate in (3.42), we use the result from [Mur20] that the full sub-Gaussian kernel estimate $\mathrm{HKE_f}(\Psi)$ implies that the underlying metric space (X,d) satisfies the chain condition. Let $R > r > 0$ and let ϕ be a function for $\mathcal{E}(\phi,\phi) \leqslant 2\operatorname{Cap}(B(x,R), B(x,A_1R)^c)$ such that $\phi \equiv 1$ on $B(x,R)$ and $\phi \equiv 0$ on $B(x,A_1R)^c$ almost everywhere, where A_1 is the constant in cap(Ψ)$_{\leqslant}$. By the chain condition, we can find a sequence of balls $B_0, \dots, B_n$ where $B_i = B(x_i, r), i = 0, 1, \dots, n$ of radii r and $K \in (1,\infty)$ such that $K^{-1}B_i = B(x_i, K^{-1}r)$ are pairwise disjoint, $B_{i+1} \subset KB_i = B(x_i, Kr)$ for all $i = 0, \dots, n-1$, $B_0 \subset B(x, A_1R)^c, B_n \subset B(x,R)$, and $n \lesssim \frac{R}{r}$. Let

$$\phi_{B_i} := \frac{1}{m(B_i)} \int_{B_i} \phi\, dm.$$

By the above properties of B_i, we have $\phi_{B_0} = 0, \phi_{B_n} = 1$. Let $A_P \in [1,\infty)$ denote the constant in PI(Ψ). We obtain the desired bound as follows:

$$
\begin{aligned}
1 &= \left(\sum_{i=0}^{n-1} \phi_{B_{i+1}} - \phi_{B_i}\right)^2 \leqslant n \sum_{i=0}^{n-1} \left(\phi_{B_{i+1}} - \phi_{B_i}\right)^2 \quad \text{(by Cauchy-Schwarz)} \\
&\lesssim \frac{R}{r} \sum_{i=0}^{n-1} \left(\phi_{B_{i+1}} - \phi_{B_i}\right)^2 \quad (\text{since } n \lesssim R/r) \\
&\lesssim \frac{R}{r} \sum_{i=0}^{n-1} \frac{1}{m(B_i)m(B_{i+1})} \int_{B_{i+1}} \int_{B_i} (\phi(x) - \phi(y))^2 \, m(dy)\, m(dx) \quad \text{(by Jensen's inequality)} \\
&\lesssim \frac{R}{r} \frac{1}{V(r)} \sum_{i=0}^{n-1} \int_{KB_i} |\phi - \phi_{KB_i}|^2 \, dm \quad (\text{by volume doubling, (3.41)}, \ B_{i+1} \subset KB_i) \\
&\lesssim \frac{R}{r} \frac{\Psi(r)}{V(r)} \sum_{i=0}^{n-1} \int_{A_P KB_i} d\Gamma(\phi,\phi) \quad (\text{by PI}(\Psi) \text{ and bounded overlap } (A_P KB_i)_{0 \leqslant i \leqslant n}) \\
&\lesssim \frac{R}{r} \frac{\Psi(r)}{V(r)} \mathcal{E}(\phi,\phi) \lesssim \frac{R}{r} \frac{\Psi(r)}{V(r)} \operatorname{Cap}(B(x,R), B(x,AR)^c) \\
&\lesssim \frac{R}{r} \frac{\Psi(r)}{V(r)} \frac{V(R)}{\Psi(R)} \quad (\text{by cap}(\Psi)_{\leqslant} \text{ and (3.41)}).
\end{aligned}
$$

The proof of sufficiency of (3.42) involves constructing a new family of diffusions. We explain the construction in the Gaussian case $\Psi(r) = r^2$ as the general case is similar. The basic idea behind the construction is to follow an idea of Laakso [Laa] by gluing (typically infinitely many) copies of $[0,\infty)$ together. The different copies of $[0,\infty)$ are indexed by an ultrametric space. We start with a description of the ultrametric space determined by $\mathbf{g} : \mathbb{Z} \to \mathbb{Z}$ that satisfies $1 \leqslant \inf_{\mathbb{Z}} \mathbf{g} \leqslant \sup_{\mathbb{Z}} \mathbf{g} < \infty$. This sequence defines an ultrametric space on the set

$$
\mathcal{U}(\mathbf{g}) := \left\{ \mathbf{s} : \mathbb{Z} \to \mathbb{Z} \mid \mathbf{s}(k) \in [\![0, \mathbf{g}(k) - 1]\!] \text{ for all } k \in \mathbb{Z} \text{ and } \lim_{k \to \infty} \mathbf{s}(k) = 0 \right\}, \tag{3.45}
$$

equipped with the metric

$$
\mathsf{d}_{\mathcal{U}(\mathbf{g})}(\mathbf{s},\mathbf{t}) = 2^{\inf\{k \in \mathbb{Z} \mid \mathbf{s}(l) = \mathbf{t}(l), \text{ for all } l \geqslant k\}}, \quad \text{for all } \mathbf{s},\mathbf{t} \in \mathcal{U}(\mathbf{g}). \tag{3.46}
$$

We denote the open and closed balls centered at $\mathbf{s} \in \mathcal{U}(\mathbf{g})$ with radius $r > 0$ as $B_{\mathcal{U}(\mathbf{g})}(\mathbf{s}, r)$ and $\overline{B}_{\mathcal{U}(\mathbf{g})}(\mathbf{s}, r)$.

Next, we describe a Borel measure $\mathsf{m}_{\mathcal{U}(\mathbf{g})}$ on $(\mathcal{U}(\mathbf{g}), \mathsf{d}_{\mathcal{U}(\mathbf{g})})$. To define this measure, it would be convenient to view $\mathcal{U}(\mathbf{g})$ as a product of two sets $\mathcal{U}(\mathbf{g}, -\infty, 0) \times \mathcal{U}(\mathbf{g}, 1, \infty)$, where

$$
\mathcal{U}(\mathbf{g}, -\infty, 0) := \left\{ \mathbf{s}\big|_{[\![-\infty,0]\!]} \mid \mathbf{s} \in \mathcal{U}(\mathbf{g}) \right\}, \quad \mathcal{U}(\mathbf{g}, 1, \infty) := \left\{ \mathbf{s}\big|_{[\![1,\infty]\!]} \mid \mathbf{s} \in \mathcal{U}(\mathbf{g}) \right\}.
$$

We note that $\mathcal{U}(\mathbf{g}, -\infty, 0)$ can be naturally identified with the product space $\prod_{\substack{j \in \mathbb{Z}, \\ j \leqslant 0}} [\![0, \mathbf{g}(k) - 1]\!]$. Using this we define the measure $\mathsf{m}_{\mathcal{U}(\mathbf{g}, -\infty, 0)}$ on $\mathcal{U}(\mathbf{g}, -\infty, 0)$ as the product measure $\prod_{\substack{j \in \mathbb{Z}, \\ j \leqslant 0}} \mathsf{m}_{\mathbf{g}(k)}$, where $\mathsf{m}_{\mathbf{g}(k)}$ denote the uniform probability measure on the finite set

$[\![0, \mathbf{g}(k) - 1]\!]$. We define $\mathsf{m}_{\mathcal{U}(\mathbf{g},1,\infty)}$ as the counting measure on $\mathcal{U}(\mathbf{g}, 1, \infty)$ (note that $\mathcal{U}(\mathbf{g}, 1, \infty)$ is at most countable).

We identify $\mathcal{U}(\mathbf{g})$ with $\mathcal{U}(\mathbf{g}, -\infty, 0) \times \mathcal{U}(\mathbf{g}, 1, \infty)$ using the obvious bijection and using this bijection we define the measure $\mathsf{m}_{\mathcal{U}(\mathbf{g})}$ on $\mathcal{U}(\mathbf{g})$ as the product measure

$$\mathsf{m}_{\mathcal{U}(\mathbf{g})} := \mathsf{m}_{\mathcal{U}(\mathbf{g},-\infty,0)} \times \mathsf{m}_{\mathcal{U}(\mathbf{g},1,\infty)}. \tag{3.47}$$

This defines a metric measure space $(\mathcal{U}(\mathbf{g}), \mathsf{d}_{\mathcal{U}(\mathbf{g})}, \mathsf{m}_{\mathcal{U}(\mathbf{g})})$. It is easy to verify that the measure of balls is given by

$$\mathsf{m}_{\mathcal{U}(\mathbf{g})}\left(B_{\mathcal{U}(\mathbf{g})}(\mathbf{s}, r)\right) = \begin{cases} \left(\prod_{\substack{0 \leqslant k \leqslant n \\ k \in \mathbb{Z}}} \mathbf{g}(k)\right)^{-1} & \text{if } 2^{n-1} < r \leqslant 2^n, n \in \mathbb{Z}, n \leqslant 0, \\ 1 & \text{if } 1 < r \leqslant 2, \\ \prod_{k=1}^{n-1} \mathbf{g}(k) & \text{if } 2^{n-1} < r \leqslant 2^n, n \in \mathbb{Z}, n \geqslant 2, \end{cases} \tag{3.48}$$

for any $\mathbf{s} \in \mathcal{U}(\mathbf{g})$ and $r > 0$. Since $\sup_{\mathbb{Z}} \mathbf{g} < \infty$, we note that $\mathsf{m}_{\mathcal{U}(\mathbf{g})}$ is a doubling measure on $(\mathcal{U}(\mathbf{g}), \mathsf{d}_{\mathcal{U}(\mathbf{g})})$. Since the volume growth does not depend on the center, we introduce the abbreviated notation

$$V_{\mathbf{g}}(r) := \mathsf{m}_{\mathcal{U}(\mathbf{g})}\left(B_{\mathcal{U}(\mathbf{g})}(\mathbf{s}, r)\right) \quad \text{for all } r > 0. \tag{3.49}$$

For each $n \in \mathbb{Z}$, we define level-n wormholes $\mathcal{W}_n$ is the set $\{(2k-1)2^n : k \in \mathbb{N}\} \subset [0, \infty)$. We define the *Laakso-type space* $\mathcal{L}(\mathbf{g})$ as the quotient of the product space $\mathcal{U}(\mathbf{g}) \times [0, \infty)$ with respect to the smallest equivalence relation that identifies points (s, x) and $(\widetilde{s}, \widetilde{x})$ in $\mathcal{U}(\mathbf{g}) \times [0, \infty)$ if and only if $x = \widetilde{x} \in \mathcal{W}_n$ for some $n \in \mathbb{Z}$ and $s\big|_{\mathbb{Z}\setminus\{n\}} = \widetilde{s}\big|_{\mathbb{Z}\setminus\{n\}}$. In other words, there are $\mathbf{g}(n)$ copies of $[0, \infty)$ identified (or glued) together when the projection to $[0, \infty)$ belongs to a level n-wormhole. The metric is the shortest path metric that agrees with the Euclidean metric on each copy of $[0, \infty)$.

The process on the Laakso-type space $\mathcal{L}(\mathbf{g})$ is defined using the theory of Dirichlet forms and can be described heuristically as follows: the projection of the process to the second ($[0, \infty)$) component is the reflected Brownian motion on $[0, \infty)$ (has law equal to the absolute value of one dimensional Brownian motion). Whenever the process hits a point corresponding to a wormhole in the second component, the process is equally likely to proceed in one of the copies. This process is symmetric with respect to the measure $m_{\mathcal{L}(g)}$ defined as the push-forward of $\mathsf{m}_{\mathcal{U}(\mathbf{g})} \times \lambda$ under the quotient map above, where λ is the Lebesgue measure on $[0, \infty)$. The Dirichlet form can be viewed as $\mathcal{E}(f, f) = \int (f')^2 \, dm_{\mathcal{L}(\mathbf{g})}$, where f' is the derivative in the component corresponding to $[0, \infty)$. A probabilistic construction of such a process is due to Barlow and Evans [BE]. The resulting MMD space satisfies volume growth estimate $r \mapsto rV_{\mathbf{g}}(r)$ and satisfies Gaussian heat kernel bounds. The proof of a Poincaré inequality using the pencil of curves approach sketched for the Euclidean case earlier. The idea here is that the wormholes provide a way for the random curves to spread to different copies of $[0, \infty)$ between any pair of points.

By varying $\mathbf{g} : \mathbb{Z} \to \mathbb{N}$, one can achieve a wide range of possible volume growth. For instance for any $\alpha \geqslant 1$, it is possible to have $rV_{\mathbf{g}}(r) \asymp r^{\alpha}$. The basic idea behind the proof of Poincare inequality is that wormholes can be used to create a pencil of curves

joining and pair of points by creating curves that choose to travel in different copies of $[0,\infty)$. For non-Gaussian Ψ, we glue different copies of a carefully chosen $\mathbb{R}$-tree instead of $[0,\infty)$; see [Mur25+].

As a sharp contrast to Theorem 3.31, Carron and Tewodrose show the following rigidity result [CT].

Theorem 3.32. *Let $(X,d,m,\mathcal{E},\mathcal{F})$ be a strongly local, regular metric measure Dirichlet space and let $\alpha \in [1,\infty)$ be such that the corresponding diffusion process has the Euclidean-like heat kernel*

$$p_t(x,y) = \frac{1}{(2\pi t)^{\alpha/2}} \exp\left(-\frac{d(x,y)^2}{2t}\right) \quad \text{for all } x,y \in X \text{ and } t>0.$$

Then $\alpha = n$ for some $n \in \mathbb{N}$, $X = \mathbb{R}^n$ equipped with Lebesgue measure m, Euclidean metric d and the diffusion process is the Brownian motion on $\mathbb{R}^n$.

A remarkable aspect of this theorem is that α is not assumed to be an integer. The proof uses the characterization of Gaussian heat kernel bounds using doubling and Poincare inequality. Under these assumptions, a result of Colding and Minicozzi states that the space of harmonic functions with atmost linear growth is finite dimensional [CM]. A well-chosen basis of harmonic functions of this space is used to obtain an isometry to $\mathbb{R}^n$. The authors of [CT] also obtain a similar rigidity result for Brownain motion on the sphere $\mathbf{S}^n$ and conjecture[4] a similar rigidity of the heat kernel for hyperbolic space.

3.4 Applications to reflected and killed diffusions

The theory of Dirichlet forms can be used to define and analyze new Markov processes using old ones. We discuss two such transformations now: reflection and killing. We will see how the general characterizations of heat kernel estimates discussed earlier is used to analyze reflected and killed diffusions.

We start with reflected diffusions. There are several approaches based on SDE and Dirichlet forms for defining the reflected Brownian motion on $\mathbb{R}^n$. For a smooth domain U in $\mathbb{R}^n$, the SDE approach involves solving the stochastic differential equation (called the Skorohod equation, 1962)

$$Y(t) = Y(0) + B(t) + \int_0^t \vec{n}(Y(s))\, dL_s,$$

where $B(t)$ is the standard Brownian motion on $\mathbb{R}^n$, L_s is the 'boundary local time' of the process $Y(s)$ and $\vec{n}(x)$ is the inward pointing unit normal vector at $x \in \partial U$. Heuristically, the last term is responsible for 'pushing the diffusion $Y(t)$ back into the domain' when it hits the boundary so that it stays in $\overline{U}$. Given a smooth domain U, the Dirichlet form approach involves the bilinear form

$$\mathcal{E}_U(f,f) := \frac{1}{2}\int_U |\nabla f|^2(x)\, dx,$$

[4] personal communication with G. Carron

for all $f \in W^{1,2}(U)$, where ∇f denotes the distributional gradient of f and $W^{1,2}(U)$ denotes the subspace of functions in $L^2(U)$ whose distributional first order partial derivatives are also in $L^2(U)$. Using the theory of Dirichlet forms, Fukushima (1967) constructs a Markov process with continuous sample paths in some abstract closure of U (called the Martin-Kuramochi compactification). If U is a smooth domain then this compactification coincides with $\overline{U}$ but in general $\overline{U}$ is not a suitable state space for the reflected diffusion as can be seen using slit domains (for example, $\mathbb{R}^2 \backslash \{(x,0) : x \leqslant 0\}$). Yet another approach involves defining the transition density of the reflected diffusion process as the fundamental solution to the heat equation with Neumann boundary condition. The advantage of SDE approach is that it is better suited for generalization to diffusions with oblique reflection along the boundary (as opposed to normal reflection discussed above). Such processes are not symmetric in general and is not suited for the Dirichlet form approach. On the other hand, the Dirichlet form approach is better suited to handle domains with non-smooth boundaries or when the underlying ambient space is not smooth. We refer the reader to [Che93] for a more complete list of references and a more detailed account of these approaches.

The general definition of a reflected diffusion corresponding to a strongly local regular Dirichlet form is as follows.

Definition 3.33 (Local Dirichlet space and its energy measure)**.** For an open set $U \subset X$ of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$, we define the local Dirichlet space $\mathcal{F}_{\text{loc}}(U)$ as

$$\mathcal{F}_{\text{loc}}(U) := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable} \\ \text{functions on } U \text{ such that } f\mathbb{1}_V = f^{\#}\mathbb{1}_V \ m\text{-a.e. for some} \\ f^{\#} \in \mathcal{F} \text{ for each relatively compact open subset } V \text{ of } U \end{array} \right\} \tag{3.50}$$

and the energy measure $\Gamma_U(f,f)$ of $f \in \mathcal{F}_{\text{loc}}(U)$ associated with $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined as the unique Borel measure on U such that $\Gamma_U(f,f)(A) = \Gamma(f^{\#}, f^{\#})(A)$ for any relatively compact Borel subset A of U and any $V, f^{\#}$ as in (3.50) with $A \subset V$; note that $\Gamma(f^{\#}, f^{\#})(A)$ is independent of a particular choice of such $V, f^{\#}$. We define

$$\mathcal{F}(U) := \{f \in \mathcal{F}_{\text{loc}}(U) : \int_U f^2 \, dm + \int_U \Gamma_U(f,f) < \infty\}, \tag{3.51}$$

and the bilinear form $(\mathcal{E}_U, \mathcal{F}(U))$ as

$$\mathcal{E}_U(f,f) = \int_U \Gamma_U(f,f), \quad \text{for all } f \in \mathcal{F}(U). \tag{3.52}$$

The bilinear form $(\mathcal{E}_U, \mathcal{F}(U))$ is not a regular Dirichlet form on $L^2(\overline{U}, m)$ in general. However if we restrict to a class of 'nice' domain we obtain a regular Dirichlet form.

Definition 3.34. A connected, non-empty, proper open set $U \subsetneq X$ is said to be a A-**uniform domain** if for every pair of points $x, y \in U$, there exists a curve γ in U from x to y such that its diameter $\text{diam}(\gamma) \leqslant Ad(x,y)$ and for all $z \in \gamma$,

$$\delta_U(z) \geqslant A^{-1} \min\left(d(x,z), d(y,z)\right),$$

where $\delta_U(z) := \text{dist}(z, U^c)$. Such a curve γ is called a A-*uniform curve*.

Note that the definition of uniform domains allows for a rich family of curves joining any pair of points as the curves in the definition of uniform domain tend to stay away from the boundary.

A large family of uniform domains is due to a construction of T. Rajala [Raj]. We say that a metric space (X, d) is *quasiconvex* if there exists $C_q \in (1, \infty)$ such that for any $x, y \in X$, there is a curve γ connecting x, y such that $\ell(\gamma) \leqslant C_q d(x, y)$. For any quasiconvex, doubling metric space (X, d), for any bounded domain $\Omega \subset X$ and for any $\epsilon > 0$, there exist *uniform* domains Ω_i and Ω_o such that $\Omega_i \subset \Omega \subset \Omega_o$ and

$$\Omega_o \subset [\Omega]_\epsilon, \quad \Omega_i^c \subset [\Omega^c]_\epsilon, \quad \text{where } [A]_\epsilon \text{ denotes the } \epsilon\text{-neighborhood of } A.$$

In other words, every bounded domain can be ϵ-approximated by uniform domains from outside and inside for any $\epsilon > 0$. Examples of domains that *fail* to be uniform include domains with inward or outward pointing cusps, or domains with slits or the strip $(0, 1) \times \mathbb{R} \subset \mathbb{R}^2$.

A natural question is whether reflected diffusion on a domain inherits similar heat kernel estimates as the diffusion on the whole space. The following theorem addresses this question for unfiorm domains.

Theorem 3.35 (Heat kernel estimate for reflected diffusion)**.** *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the heat kernel estimate* HKE(Ψ) *for some scale function Ψ and let m be a doubling measure. Then for any uniform domain U, the bi-linear form $(\mathcal{E}_U, \mathcal{F}(U))$ is a strongly-local regular Dirichlet form on $L^2(\overline{U}, m)$. Moreover, the corresponding MMD space $(\overline{U}, d, m, \mathcal{E}_U, \mathcal{F}(U))$ satisfies the heat kernel estimate* HKE(Ψ)*.*

The Gaussian case $\Psi(r) = r^2$ of the above result was obtained by [GyS] and the general case was handled in [Mur24]. The setting of [GyS] is more general as they allow for *inner uniform* domains which are uniform domains with respect to the intrinsic metric (distance between points of U is the infimal length of the curves in U joining them). Such domains can contain slits. While the methods of [GyS] can be used to obtain Poincaré inequality for general space-time scalings, the main difficulty is to obtain cutoff energy inequality. The proof in [Mur24] for cutoff energy inequality does not work for inner uniform domains. We conjecture that Theorem 3.35 should also be true if we replace uniform domains with inner uniform domain equipped with the intrinsic metric mentioned above. N. Kajino (unpublished work) observed that a positive resolution of the conjecture that sub-Gaussian heat kernel estimates is equivalent to VD, PI(Ψ) and cap(Ψ) would imply a positive answer to the above conjecture concerning reflected diffusion on inner uniform domains (see [Mur24, §6.3]). The difficulty of proving a cutoff energy inequality does not arise in the Gaussian case as it can be characterized only using the volume doubling property and Poincaré inequality.

The key tool behind the proof of Theorem 3.35 is the following extension theorem in [Mur24]. Similar extension theorem for Sobolev spaces on uniform domains was obtained by P. Jones [Jon81] which generalized earlier results be Calderon and Stein for Lipschitz domains.

Theorem 3.36 (Extension property)**.** *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the heat kernel estimate* HKE(Ψ) *for some scale function Ψ and let m be a doubling measure. Let U be a uniform domain U and let $(\mathcal{E}_U, \mathcal{F}(U))$ denote the bi-linear form in Definition 3.33. There is a linear operator $E : \mathcal{F}(U) \to \mathcal{F}$ such that the restriction of $E(f)$ to U is f for all $f \in \mathcal{F}$ (that is, E is an extension operator). Furthermore, there exist $C, K \in (1, \infty), c \in (0, 1)$ such that for all $x \in \overline{U}$, and $f \in \mathcal{F}(U)$, we have*

$$\Gamma(E(f), E(f))(B(x, r)) \leqslant C\Gamma_U(f, f)(B_U(x, Kr)), \quad \text{for all } 0 < r < c\operatorname{diam}(U); \tag{3.53}$$

$$\int_{B(x,r)} |E(f)|^2 \, dm \leqslant C \int_{B_U(x,Kr)} f^2 \, dm \quad \text{for all } r > 0; \tag{3.54}$$

$$\mathcal{E}(E(f), E(f)) \leqslant C \left(\mathcal{E}_U(f, f) + \frac{1}{\Psi(\operatorname{diam}(U))} \int_U f^2 \, dm \right); \tag{3.55}$$

$$\int_X |Ef|^2 \, dm \leqslant C \int_U f^2 \, dm. \tag{3.56}$$

Here Γ, Γ_U denote the energy measures of $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}_U, \mathcal{F}(U))$ respectively. Furthermore, if $f \in \mathcal{F}(U)$ is a version that belongs to $C_c(\overline{U})$, then the extension $E(f)$ can be chosen to be a pointwise well-defined version in $C_c(X)$.

In (3.55) above, we interpret $\frac{1}{\Psi(\infty)} = 0$ in case $\operatorname{diam}(U) = \infty$.

We now sketch the Theorem 3.35 using the extension property above. Let us check that $(\mathcal{E}_U, \mathcal{F}(U))$ is a closed quadratic form. We equip $\mathcal{F}(U), \mathcal{F}$ with the corresponding inner products $\mathcal{E}_U(\cdot, \cdot) + \langle \cdot, \cdot \rangle_{L^2(U,m)}$, and $\mathcal{E}(\cdot, \cdot) + \langle \cdot, \cdot \rangle_{L^2(X,m)}$ respectively. Let $E : \mathcal{F}(U) \to \mathcal{F}$ be an extension operator as given above. If f_n is a Cauchy sequence in $\mathcal{F}(U)$, then $E(f_n)$ is a Cauchy sequence in $\mathcal{F}$ (since E is a bounded operator). Since $\mathcal{F}$ is complete, $E(f_n)$ converges to a limit, say $g \in \mathcal{F}$. The restriction of g to U yields the desired limit of f_n in $\mathcal{F}(U)$.

The regularity of $(\mathcal{E}_U, \mathcal{F}(U))$ is an easy consequence of the regularity of $(\mathcal{E}, \mathcal{F})$ and the extension property. To this end, note that function $f \in \mathcal{F}(U)$ has an extension $E(f) \in \mathcal{F}$ which is a limit of functions in $C_c(X) \cap \mathcal{F}$ and hence by restricting this sequence of functions to $\overline{U}$, we obtain that $C_c(\overline{U}) \cap \mathcal{F}(U)$ is dense in $\mathcal{F}(U)$. Since any function in $C_c(\overline{U})$ can be extended to a function on $C_c(X)$, by the same argument as above, we obtain that any function on $C_c(\overline{U})$ is a limit (with respect to the uniform norm) of functions in $C_c(\overline{U}) \cap \mathcal{F}(U)$.

We will see how to prove Poincaré inequality for the MMD space $(\overline{U}, d, m|_{\overline{U}}, \mathcal{E}_U, \mathcal{F}(U))$ corresponding to the reflected diffusion using that for the ambient diffusion $(X, d, m, \mathcal{E}, \mathcal{F})$. For any $f \in \mathcal{F}(U), x \in \overline{U}, 0 < r < \operatorname{diam}(\overline{U}, d)/A_2$

$$\begin{aligned}
\inf_{\alpha \in \mathbb{R}} \int_{\overline{U} \cap B(x,r)} |f - \alpha|^2 \, dm &\leqslant \inf_{\alpha \in \mathbb{R}} \int_{B(x,r)} |E(f) - \alpha|^2 \, dm \\
&\lesssim r^\beta \int_{B(x,A_1 r)} d\Gamma(E(f), E(f)) \\
&\lesssim r^\beta \int_{U \cap B(x,KA_1 r)} d\Gamma_U(f, f).
\end{aligned}$$

The proof of cutoff energy inequality for the reflected diffusion using the extension map is similar to the proof of Poincare inequality above.

Next, we defined the diffusion killed upon exiting a domain U. The *part process* of a diffusion $\{Y_t\}_{t\geqslant 0}$ killed upon exiting U is denoted by $\{Y_t^U\}_{t\geqslant 0}$ is defined as

$$Y_t^U := \begin{cases} Y_t & \text{if } t < \tau_U, \\ \Delta & \text{if } t \geqslant \tau_U, \end{cases} \qquad t \in [0, \infty],$$

where $\tau_U = T_{U^c}$ is the exit time of U and Δ denotes the cemetery state. It is an $m|_U$-symmetric diffusion process on U, its Dirichlet form $(\mathcal{E}^U, \mathcal{F}^0(U))$ is a strongly local regular symmetric Dirichlet form on $L^2(U, m|_U)$ and identified as the *part Dirichlet form* of $(\mathcal{E}, \mathcal{F})$ on U given by

$$\mathcal{F}^0(U) = \{f \in \mathcal{F} \mid \widetilde{f} = 0 \ \mathcal{E}\text{-q.e. on } X\backslash U\} \qquad \text{and} \qquad \mathcal{E}^U = \mathcal{E}|_{\mathcal{F}^0(U)\times\mathcal{F}^0(U)},$$

where $\widetilde{f}$ denotes a quasi-continuous version of f [FOT, Theorems 4.4.2 and 4.4.3]. Note that the above definition is natural by recalling our convention that the value of any function is extended by 0 to the cemetery state. In this case, we view U^c as part of the cemetery state.

There are two important tools to obtain heat kernel bounds for diffusions killed upon exiting a uniform domain: Doob h-transform and the boundary Harnack principle. Let h be a positive harmonic function on U with Dirichlet boundary condition along U.

More generally, we are interested in harmonic functions on an open set V with zero (or Dirichlet) boundary condition "along the boundary of a larger open set U" as defined below.

Definition 3.37 (Function with Dirichlet boundary condition)**.** Let $(X, m, \mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet space, and let $V \subset U$ be open subsets of X. We define

$$\mathcal{F}^0_{\text{loc}}(U, V) := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable func-} \\ \text{tions on } V \text{ such that } f = f^{\#} \ m\text{-a.e. on } A \text{ for some } f^{\#} \in \mathcal{F}^0(U) \\ \text{for each open subset } A \text{ of } V \text{ with } \overline{A} \text{ compact and } \overline{A} \cap \overline{U\backslash V} = \varnothing \end{array} \right\}, \tag{3.57}$$

so that $\mathcal{F}^0_{\text{loc}}(U, V)$ is a linear subspace of $\mathcal{F}_{\text{loc}}(V)$, and call each $u \in \mathcal{F}^0_{\text{loc}}(U, V)$ a *function on V with Dirichlet boundary condition relative to U*. Each $u \in \mathcal{F}^0_{\text{loc}}(U, V)$ that is $\mathcal{E}$-harmonic on V is called an *$\mathcal{E}$-harmonic function on V with Dirichlet boundary condition relative to U*.

Consider an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ and let U be an unbounded uniform domain. Assume that there exists $h \in \mathcal{F}^0_{\text{loc}}(U, U)$ such that $h > 0$ m-a.e. on U and harmonic on U. Consider the unitary operator $M_h : L^2(U, h^2 \cdot m) \to L^2(U, m)$, where M_h denotes the multiplication by h

$$M_h(f) = hf, \quad \text{for all } f \in L^2(U, h^2 \cdot m).$$

Let $(\mathcal{E}^U, \mathcal{F}^0(U))$ denote the Dirichlet form on $L^2(U, m\big|_U)$ corresponding to the process Y_t^U killed upon exiting U. We define a closed quadratic form $(\mathcal{E}_h^U, \mathcal{F}_h^U)$ on $L^2(U, h^2 \cdot m)$

by $\mathcal{F}_h^U = M_h^{-1}(\mathcal{F}^0(U))$ and

$$\mathcal{E}_h^U(f,f) = \mathcal{E}(M_h(f), M_h(f)), \quad \text{for all } f \in \mathcal{F}_h^U.$$

If $(P_t^U)_{t>0}$ is the semigroup corresponding to the process (Y_t^U), then it is easy to verify that

$$P_t^{U,h} := M_h^{-1} \circ P_t^U \circ M_h, \quad \text{for all } t > 0,$$

is the (strongly continuous, contraction) semigroup associated with the closed quadratic form $(\mathcal{E}_h^U, \mathcal{F}_h^U)$ on $L^2(U, h^2 \cdot m)$. Using the harmonicity of h on U, a result due to Hunt implies that $P_t^U(h) \leqslant h$ and hence $P_t^{U,h}\mathbb{1}_U \leqslant \mathbb{1}_U$. This implies that the semigroup $(P_t^{U,h})$ is a Markovian semigroup. This transformation of the semigroup is called the *Doob h-transform* (we refer to [Doo58] for the original reference and [CW, Chapter 11] and [Doo84, Part 2, Chp. X] for more on h-transforms). If the semigroup $(P_t^{U,h})$ admits a heat kernel $p_t^{U,h}(\cdot,\cdot)$, then it is easy (Exercise) to verify that the semigroup (P_t^U) also admits a heat kernel $p_t^U(\cdot,\cdot)$ given by

$$p_t^U(x,y) = h(x)h(y)p_t^{U,h}(x,y). \tag{3.58}$$

Often, it is easier to analyze the process corresponding to the semigroup $(P_t^{U,h})$ (for example, using Theorem 3.27) and this can be used to understand the semigroup (P_t^U) corresponding to the process (Y_t^U) killed upon exiting U. In order to do this, we first need to construct a harmonic function h with zero boundary conditions.

Let us explain this construction of such a harmonic function for the Brownian motion as the general case is similar. Let U be an unbounded uniform domain in $\mathbb{R}^n$ and let $g_U(x,y), x, y \in D$ denote the Green function on a domain D. That is, $\Delta_y g_U(x,y) = -\delta_x$ in the sense of distributions and $g_U(x,\cdot)$ has boundary condition 0 on ∂U. We consider a sequence of points $x_n \to \infty$ such that $x_n \in U$. Fix some base point $x_0 \in D_1$. Then the sequence of Green functions $\frac{g_{D_n}(x_n,\cdot)}{g_{D_n}(x_n,x_0)}$ normalized to be 1 at x_0 admits a subsequence converges uniformly and compact subsets (by the Arzela-Ascoli theorem) to a limiting function $h : U \to (0,\infty)$ that is positive and harmonic on U with Dirichlet boundary conditions on U [GyS, §4.3.1.]; that is, $h \in \mathcal{F}^0_{\mathrm{loc}}(U,U)$. The uniqueness of such a harmonic function (up to a constant multiplicative factor) follows from the boundary Harnack principle which in the Euclidean case is due to Aikawa [Aik01]. Aikawa's argument works for any uniform domain in a MMD space satisfying sub-Gaussian heat kernel bounds [GyS, Lie15, BM19, Che25].

Definition 3.38 (Boundary Harnack principle (BHP))**.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let U be an open subset of X. We say that U satisfies the **(scale-invariant) boundary Harnack principle**, abbreviated as BHP, if there exist $A_0, A_1, C_1 \in (1,\infty)$ such that for all $\xi \in \partial U$, all $r \in (0, \mathrm{diam}(U)/A_1)$ and for any two non-negative $\mathcal{E}$-harmonic functions u, v on $U \cap B(\xi, A_0 r)$ with Dirichlet boundary condition relative to U such that $v > 0$ m-a.e. on $U \cap B(\xi, r)$, we have

$$\operatorname*{ess\,sup}_{x \in U \cap B(\xi,r)} \frac{u(x)}{v(x)} \leqslant C_1 \operatorname*{ess\,inf}_{x \in U \cap B(\xi,r)} \frac{u(x)}{v(x)}. \tag{BHP}$$

The proof of uniqueness follows from obtaining a version of Moser's oscillation lemma for the ratio of two positive harmonic functions (with the same proof as in the case of Moser's oscillation lemma). Then the proof of Liouville property under elliptic Harnack inequality can be adapted to show that the ratio of any two such harmonic functions is a constant.

The boundary Harnack principle also provides useful estimates on the harmonic function above which leads to proof of doubling property, Poincaré and cutoff energy inequalities for the h-transformed Dirichlet space $(\mathcal{E}_h^U, \mathcal{F}_h^U)$ on $L^2(U, h^2 \cdot m)$ and leads to the following theorem [GyS, Lie22].

Theorem 3.39. *If $(X, d, m, \mathcal{E}, \mathcal{F})$ is a MMD space and let $\Psi : [0, \infty) \to [0, \infty)$ be a scale function such that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the parabolic Harnack inequality* PHI(Ψ)*. Let U be an unbounded uniform domain and let $h \in \mathcal{F}^0_{\mathrm{loc}}(U, U)$ the unique (up to a multiplicative constant) positive harmonic function with Dirichlet boundary condition on U. Then the MMD space $(U, d, h^2 \cdot m\big|_U, \mathcal{E}_h^U, \mathcal{F}_h^U)$ also satisfies the the parabolic Harnack inequality* PHI(Ψ)*.*

The above result along with (3.58) and Theorem 3.27 leads to heat kernel bounds for process killed upon exit a uniform domain. In case of bounded uniform domains, one replaces the harmonic function with the first Dirichlet eigenfunction in the Doob transform (sometimes called the ground state transform) and similar arguments apply in this case as well. The proof of the above theorem relies on obtaining Poincaré and cutoff energy inequalities for the Doob h-transformed space $(U, d, h^2 \cdot m\big|_U, \mathcal{E}_h^U, \mathcal{F}_h^U)$.

Example 3.40. If we consider the killed Brownian motion on the domain $U = (0, \infty) \subset X = \mathbb{R}$, we obtain the harmonic function $h(x) = x$ on U. The process obtained after the Doob h-transform is the three dimensional Bessel process (see Exercise 2.50).

More generally, if $U = (0, \infty)^n \subset \mathbb{R}^n$ $h(x_1, \ldots, x_n) = \prod_{i=1}^n x_i$. If $U = \mathbb{R}^n \backslash \overline{B(0, 1)}$ is the exterior of the closed unit disk where $n \geqslant 2$. Then $h(x) = 1 - |x|^{2-n}$ if $n \geqslant 3$ and $h(x) = \log(|x|)$ if $n = 2$. In general, it is not always easy to estimate or compute h. In two dimensions conformal maps are useful to compute h but much less is known in higher dimensions. Getting good estimates on h or first Dirichlet eigenfunctions for bounded domains is a challenging problem in general; see [BS-C] for a recent work in this direction.

The proof of Poincaré inequalities for both reflected and killed diffusion and also the construction of the extension map in Theorem 3.36 rely on the notion of Whitney cover.

Definition 3.41. Let $\epsilon \in (0, 1/2)$ and $U \subsetneq X$. We say a collection of balls $\{B(x_i, r_i) : x_i \in U, r_i > 0, i \in I\}$ is an ϵ-Whitney cover if it satisfies the following properties:

(i) The collection of sets $\{B(x_i, r_i), i \in I\}$ are pairwise disjoint.

(ii) The radii r_i satisfy
$$r_i = \frac{\epsilon}{1 + \epsilon} \mathrm{dist}(x_i, U^c), \quad \text{for all } i \in I.$$

(iii) $\bigcup_{i\in I} B_U(x_i, K_\varepsilon r_i) = U$, where $K_\epsilon = 2(1+\epsilon) \in (2,3)$.

Note that since $K_\epsilon < 3$ by (iii), we have

$$\bigcup_{i\in I} B(x_i, 3r_i) = U.$$

The existence of such a Whitney cover follows from Zorn's lemma by picking a maximal collection of balls that satisfy properties (i) and (ii) above.

Exercise 3.42. Use the above statement to prove the existence of ϵ-Whitney cover. If (X,d) admits a volume doubling measure m, then show that there exists C depending only on the constant in doubling property and ϵ such that for any ϵ-Whitney cover $\mathcal{W}(U)$ of a domain U, we have

$$\sum_{B(x,r)\in\mathcal{W}(U)} \mathbb{1}_{B(x,\varepsilon^{-1}r)} \leqslant C\mathbb{1}_U.$$

We briefly sketch the construction of the extension map $E : \mathcal{F}(U) \to \mathcal{F}$ in Theorem 3.36 which follows an idea due to Jones [Jon81]. The basic idea is to pick two Whitney covers for the domains U and $V := (\overline{U})^c$, say $\mathcal{W}(U)$ and $\mathcal{W}(V)$ respectively. We pick a subset $\widetilde{\mathcal{W}(V)}$ of $\mathcal{W}(V)$ that consist of all balls of radii less than $c\mathrm{diam}(U)$ for some $c \in (0,1)$. This allows us to define a 'reflection map' $R : \widetilde{\mathcal{W}(V)} \to \mathcal{W}(U)$ such that each ball $B(x_i, r_i) \in \widetilde{\mathcal{W}(V)}$ is mapped to $Q(B(x_i, r_i)) = B(y_i, s_i)$ such that $d(x_i, y_i) \asymp r_i \asymp s_i$.

Using upper bounds on capacity of annuli, there exist a partition of unity of V indexed by $\mathcal{W}(V)$ such that for each $B(x_i, r_i) \in \mathcal{W}(V)$, there exists a function $\phi^{B(x_i,r_i)}$ supported in $B(x_i, Kr_i) \subset V$ and such that $\mathcal{E}(\phi^{B(x_i,r_i)}\phi^{B(x_i,r_i)}) \asymp \frac{m(B(x_i,r_i))}{\Psi(r_i)}$. The extension $E(f)$ is then defined as

$$E(f) = f(\cdot)\mathbb{1}_U(\cdot) + \sum_{B(x,r)\in\widetilde{\mathcal{W}(V)}} f_{Q(B(x,r))}\phi^{B(x,r)}(\cdot),$$

where $f_{Q(B(x,r))}$ is the m-average of f in the reflected ball $Q(B(x,r))$. The above construction of the extension map is essentially same as that of P. Jones [Jon81]. However the proof that the extension map is bounded requires new ideas. In the case of Sobolev spaces on $\mathbb{R}^n$, Jones uses pointwise estimates on the gradient to bound Sobolev norms. However in our setting pointwise estimates on gradient could be meaningless as the energy measures could be singular [KM20]. Our approach is based on a result of Korevaar and Schoen [KoSc, Theorem 1.6.2] which implies that for any $f \in W^{1,2}(\mathbb{R}^n)$, the Dirichlet energy $\int_{\mathbb{R}^n} |\nabla f|^2(x)dx$ is comparable to

$$\limsup_{r\downarrow 0} \int_{\mathbb{R}^n} \frac{1}{r^{n+2}} \int_{\{y:|y-x|<r\}} |f(x)-f(y)|^2\, dy\, dx.$$

An extension of Korevaar-Schoen theorem for the setting of Dirichlet forms satsifying sub-Gaussian heat kernel estimates due to Grigor'yan, Hu and Lau [GHL03]. The extension of these results also hold for energy measures [Mur24] and is an important ingredient in the proof of extension theorem.

4 Time change of Markov processes

Let us illustrate time change of symmetric Markov process in a simple case by revisiting Examples 2.2-(iii) and 2.9. Let $X = \{1, \dots, n\}$ and $c : X \times X \to [0, \infty)$ be such that $c(x, y) = c(y, x) \geqslant 0$ for all $x, y \in X$. Let $m : X \to [0, \infty)$ be

$$m(x) := \sum_{y \in X} c(x, y).$$

Assume that $m(x) > 0$ for all $x \in X$. This defines a *discrete time* Markov chain $(Z_n)_{n \in \mathbb{N} \cup \{0\}}$ with transition probabilities given by

$$P(x, y) := \frac{c(x, y)}{m(x)}, \quad \text{for all } x, y \in X.$$

Let us assume that the corresponding Markov chain is irreducible and let $\lambda : X \to (0, \infty)$ be a function determining a continuous time process as follows: at each state $x \in X$, the process waits (independently of other waiting times and transitions) an exponential time with rate $\lambda(x)$ before jumping to a new state y with probability $P(x, y)$. Call this process $(Y_t^\lambda)_{t \geqslant 0}$ on X. Recall that the generator L^λ of the semigroup $(P_t^\lambda)_{t>0}$ generator corresponding to the process $(Y_t^\lambda)_{t \geqslant 0}$ is

$$L^\lambda f(x) = \lim_{t \downarrow 0} \frac{P_t^\lambda f(x) - f(x)}{t} = -\lambda(x) f(x) + \lambda(x) \sum_{y \in X} P(x, y) f(y) = -\lambda(x)(I - Q) f(x).$$

It follows from (2.3) that L^λ is $\tilde{m}$-symmetric with $\tilde{m}(x) = \lambda(x)^{-1} m(x)$ and the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by the following expression: for all $f \in \mathcal{F} = \mathbb{R}^X$, we have

$$\begin{aligned}
\mathcal{E}(f, f) &= \langle -L^\lambda f, f \rangle_{L^2(X, \tilde{m})} = \langle (I - Q) f, f \rangle_{L^2(m)} \\
&= \sum_{x, y \in X} (f(x) - f(y)) P(x, y) m(x) = \sum_{x, y \in X} (f(x) - f(y)) c(x, y) \\
&= \frac{1}{2} \sum_{x, y \in X} (f(x) - f(y))^2 c(x, y) \quad (\text{since } c(x, y) = c(y, x)). \qquad (4.1)
\end{aligned}$$

In particular, the Dirichlet form does not depend on the choice of λ.

If $\lambda \equiv 1$, we drop λ from the notation; that is the corresponding process, semigroup and generator are denoted by $(Y_t), (P_t), L$ respectively. Let us show a construction of (Y_t^λ) as a time-change of the process (Y_t). We define a process $(A_t^\lambda)_{t \geqslant 0}$ as

$$A_t^\lambda = \int_0^t \lambda(Y_s)^{-1} \, ds.$$

We note that if $(\mathcal{F}_t)$ and $(\theta_t)_{t \geqslant 0}$ are the filtration and shift-operators corresponding to the Markov process (Y_t), we have the following properties.

1. (A_t^λ) is $[0,\infty)$ valued, non-decreasing[5] $(\mathcal{F}_t)$-adapted process with $A_0^\lambda = 0$.

2. $A_{t+s}^\lambda(\omega) = A_t^\lambda(\omega) + A_s^\lambda \circ \theta_t(\omega)$ for all $t, s \geqslant 0$.

Let (τ_t^λ) denote the right continuous inverse of (A_t^λ) defined by

$$\tau_t^\lambda := \inf\{s \geqslant 0 | A_s > t\}.$$

Then the process $(Y_{\tau_t^\lambda})$ has the same law as the process (Y_t^λ). To see this note that if the process (Y_t) spends a time T_1 at state x_1 before jumping to the next state, followed by a duration T_2 at state x_2 (that is, $Y_t = x_2$ for $t \in [T_1, T_1 + T_2)$ before jumping to the next state, and so on. Then $(Y_{\tau_t^\lambda})$ spends time $T_1/\lambda(x_1)$ at state x_1 before jumping to state x_2 and spending $T_2/\lambda(x_2)$ at state x_2 before jumping to the next state. This proves our claim and $(Y_{\tau_t^\lambda})$ is also a Markov process (recall that if $T \overset{(d)}{=} \text{Exp}(1)$, the $T/\lambda \overset{(d)}{=} \text{Exp}(\lambda)$).

Let us consider the case when some of the states are allowed to have instantaneous jumps (that is, $\lambda(x) = \infty$ for some $x \in X$). In this case, the corresponding symmetric measure $\widetilde{m}$ does not have full support. Let $F = \text{supp}(\widetilde{m})$ denote the support of the measure $\widetilde{m}$. Assume that $F \neq \varnothing$ and $X \backslash F \neq \varnothing$.

Our discussion above suggests that $(Y_{\tau_t^\lambda})$ is $\widetilde{m}$-symmetric Markov process on F (note that the process $(Y_{\tau_t^\lambda})$ doesn't spend any time at $X \backslash F$). Let us now compute its Dirichlet form. Let T_F denote the hitting time of the set F for the process (Y_t); that is

$$T_F = \inf\{t \geqslant 0 | Y_t \in F\}.$$

For any $g \in \mathbb{R}^F$, we define $H_F(g) \in \mathbb{R}^X$ as

$$H_F(g)(x) := \mathbb{E}_x[g(Y_{T_F})].$$

The following exercise described equivalent descriptions of $H_F(g)$. It can be viewed as a solution to the Dirichlet problem for L-harmonic function with prescribed boundary value g on F. That is, $H_F(g)$ is the harmonic extension of g. We saw such extensions in the construction of Dirichlet form for the Brownian motion on the Sierpinski gasket.

Exercise 4.1. Let $X = \{1, 2, \ldots, n\}$, $\lambda : X \to (0, \infty]$, $\widetilde{m}(x) = m(x)/\lambda(x)$ for all $x \in X$, $c : X \times X \to (0, \infty), L : \mathbb{R}^X \to \mathbb{R}^X$ be as above. Let (Y_t) denote the m-symmetric process corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ given by (4.1). Let $g \in \mathbb{R}^F$ and let $\widetilde{g} \in \mathbb{R}^X$ be an extension of g (that is, $\widetilde{g}\big|_F = g$). Then the following are equivalent:

(a) $\widetilde{g}(x) = H_F(g)(x) = \mathbb{E}_x[g(T_F)]$ for all $x \in X$.

(b) (harmonic in $X \backslash F$ in the strong sense) $L(\widetilde{g})\big|_{X \backslash F} \equiv 0$.

(c) For any $\phi \in \mathbb{R}^X$ such that $\text{supp}(\phi) \subset X \backslash F$, we have (weak formulation of harmonicity)

$$\mathcal{E}(\widetilde{g}, \phi) = 0.$$

[5] strictly increasing in this case but we will soon allow λ to take infinite value

(d) (variational formulation of harmonicity) $\mathcal{E}(\widetilde{g},\widetilde{g}) = \min_{\substack{h\in\mathbb{R}^X,\\ h|_F\equiv g}} \mathcal{E}(h,h)$.

Similar to Example 2.9, we can compute the semigroup corresponding to $(Y_{\tau_t^\lambda})$ up to $O(t^2)$ error as

$$P_t^\lambda g(x) = \mathbb{E}_x[g(Y_t^\lambda)] = e^{-\lambda(x)t}g(x) + \sum_{y\in X}(1-e^{-\lambda(x)t})P(x,y)(H_F(g))(y) + O(t^2),$$

for all $g \in \mathbb{R}^F, t \in (0,1), x \in F$. By the same computation as Exercise 2.9, we have the generator

$$L^\lambda g(x) := -\lambda(x)(I-Q)(H_F(g))(x), \quad \text{for all } g \in \mathbb{R}^F,\ x \in F,$$

and the Dirichlet form

$$\check{\mathcal{E}}(g,g) = \mathcal{E}(H_F(g), H_F(g)), \quad \text{for all } g \in \mathbb{R}^F,$$

where $\mathcal{E}$ is the Dirichlet form corresponding to (Y_t) given in (4.1). We note that this formula gives a probabilistic interpretation of the construction of Dirichlet form on Sierpiński gasket using graph approximations.

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(X,m)$. Let (Y_t) be the corresponding m-symmetric Markov process with state space X with time-shift operators $\theta_t : \Omega \to \Omega$. Let $(\mathcal{F}_t)_{t\in[0,\infty]}$ denote the corresponding filtration. A collection $A = \{A_t\}_{t\in[0,\infty)}$ of $[0,\infty]$-valued random variables on Ω is called a *positive continuous additive functional* (*PCAF* for short) of (Y_t), if the following three conditions hold:

(i) A_t is $\mathcal{F}_t$-measurable for any $t \in [0,\infty)$.

(ii) There exist $\Lambda \in \mathcal{F}_\infty$ and a properly exceptional set $\mathcal{N} \subset X$ for (Y_t) such that $\mathbb{P}_x(\Lambda) = 1$ for any $x \in X\backslash\mathcal{N}$ and $\theta_t(\Lambda) \subset \Lambda$ for any $t \in [0,\infty)$.

(iii) For any $\omega \in \Lambda$, $[0,\infty) \ni t \mapsto A_t(\omega)$ is a $[0,\infty]$-valued continuous function with $A_0(\omega) = 0$ such that for any $s,t \in [0,\infty)$, $A_t(\omega) < \infty$ if $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ if $t \geqslant \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$.

The sets Λ and $\mathcal{N}$ are referred to as a *defining set* and an *exceptional set*, respectively, of the PCAF A. If $\mathcal{N}$ can be taken to be the empty set $\varnothing$, then we say that A is a *PCAF in the strict sense* of (Y_t). the *support* F of A defined by

$$F := \{x \in X\backslash\mathcal{N} \mid \mathbb{P}_x(R=0) = 1\}, \quad \text{where } R := \inf\{t \in (0,\infty) \mid A_t > 0\},$$

and $\mathcal{N}$ is an exceptional set of the PCAF. Given a PCAF A, there is a unique Borel measure μ on X such that

$$\int_X f\,d\mu = \lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}_m\left[\int_0^t f(Y_s)\,dA_s\right],$$

for any non-negative Borel measurable function $f : X \to [0,\infty)$. This measure μ does not charge $\mathcal{E}$-polar set and is called the *Revuz measure* of the PCAF A. Conversely, for any

Radon measure μ on X such that $\mu(A) = 0$ for any $\mathcal{E}$-polar Borel set A, there exists a PCAF whose Revuz measure is A. The condition that Revuz measure does not charge any polar set is natural as probability of hitting a polar set is zero from almost every starting point. So we would like such sets to remain polar after the time change.

Example 4.2. 1. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$ and let (Y_t) be the associated process. If $g : X \to (0, \infty)$ is a strictly positive function such that $g \cdot m$ is a Radon measure on X. Then the PCAF A whose Revuz measure is μ is given by

$$A_t := \int_0^t g(Y_s)\, ds.$$

2. For the Dirichlet form corresponding to one-dimensional Brownian motion (B_t), the Dirac measure $\mu = \delta_x$ for $x \in \mathbb{R}$ is the Revuz measure of the PCAF (L_t^x), where L_t^x is the local time at x. Recall that L_t^x can be viewed as the occupation density of the Brownian motion at x as

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{B_s \in (x-\epsilon, x+\epsilon)\}}\, ds.$$

In this case, for any Radon measure μ, the corresponding PCAF A^μ is given by

$$A_t^\mu = \int_{\mathbb{R}} L_t^x\, \mu(dx).$$

Hence PCAF can be interpreted as an integrated version of local time. For higher dimensional Brownian motion the Dirac measure is not a suitable Revuz measure for a PCAF as it assigns positive measure to a polar set. Nevertheless, it is useful to think of PCAFs as integrated (or averaged) local times.

We will only consider the case when A is a PCAF in the strict sense and the support of A coincides with the topological support of the Revuz measure μ[6]. In this case, let us denote the support of A (and μ) by F. The time-changed process corresponding to the PCAF A supported on F with Revuz measure μ is given by

$$\check{Y}_t := Y_{\tau_t}, \quad \tau_t := \inf\{s > 0 | A_s > t\}.$$

Similar to the discrete setting, it turns out that $(\check{Y}_t)$ is a μ-symmetric process on F, whose Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is given by

$$\check{\mathcal{F}} = \{\widetilde{f}\Big|_F : f \in \mathcal{F}_e, \widetilde{f}\Big|_F \in L^2(F, \mu)\},$$

where $\widetilde{f}$ denotes a quasi-continuous version of $f \in \mathcal{F}_e$, and

$$\check{\mathcal{E}}(u, u) := \mathcal{E}(H_F(\widetilde{u}), H_F(\widetilde{u}))$$

[6]Although this condition is typically satisfied, it is not always true as observed by Sturm [Stu92]. In general, the support of the PCAF is contained in the topological support of the Revuz measure.

where H_F denotes the harmonic extension of ϕ defined by

$$H_F\widetilde{u}(x) := \mathbb{E}_x\big[\widetilde{u}(Y_{T_F})\mathbb{1}_{\{T_F<\infty\}}\big], \quad T_F = \inf\{t \in (0,\infty) \mid Y_t \in F\}.$$

We refer to [FOT, Theorem 6.2.1] for the proof of identification of the Dirichlet form corresponding to the time-changed process $(\check{Y}_t)$. It is possible for the $m(F) = 0$ and hence $f\big|_F$ is not well-defined if $f \in L^2(m)$. On the other hand, since quasi-continuous versions are well-defined up to q.e. equivalence and μ does not charge sets of capacity zero, $\widetilde{f}\Big|_F$ is well-defined up to μ-a.e. equivalence if $\widetilde{f}$ is quasi-continuous. This explains the need for choosing quasi-continuous versions in the above definitions.

Perhaps the most well-known example of time change concerns the conformal invariance of two dimensional Brownian motion.

Example 4.3. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form $L^2(U,m)$ corresponding to a process (Y_t). Let $\Phi : U \to V$ be a bijection. We could like to compute the Dirichlet form corresponding to the process $\widetilde{Y}_t := \Phi(Y_t)$ on V. Let $f \in L^2(V,\widetilde{m})$, where $\widetilde{m} := \Phi_*(m)$, so that $f\circ\Phi \in L^2(U,m)$. Let (P_t) and $(\widetilde{P}_t)$ denote the semigroups corresponding to (Y_t) and $(\widetilde{Y}_t)$. Then

$$\widetilde{P}_t f(y) = \widetilde{E}_y(f(\widetilde{Y}_t)) = E_{\Phi^{-1}(y)}(f\circ\Phi(Y_t))) = P_t(f\circ\Phi)(\Phi^{-1}(y)), \quad \text{for all } y \in V, f \in L^2(V,\widetilde{m}).$$

The following properties are easily verified from the above expression of the semigroup (Exercise).

(i) $(\widetilde{P}_t)$ is a $\widetilde{m}$-symmetric (strongly continuous) Markovian semigroup on $L^2(V,\widetilde{m})$.

(ii) The Dirichlet form $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}})$ on $L^2(V,\widetilde{m})$ corresponding to the semigroup $(\widetilde{P}_t)$ is given by

$$\widetilde{\mathcal{F}} = \{f \in L^2(V,\widetilde{m}) : f\circ\Phi \in \mathcal{F}\}, \quad \widetilde{\mathcal{E}}(f,f) = \mathcal{E}(f\circ\Phi, f\circ\Phi).$$

Now, let us specialize to case when U and V are domains in $\mathbb{R}^2$ and $\Phi : U \to V$ is an orientation preserving, C^1-diffeomorphism. That is Φ and $\Phi^{-1} : V \to U$ are continuously differentiable bijections and $\det(D\Phi) > 0$ on U. Let $(\mathcal{E},\mathcal{F}) := (\mathcal{E}^U, \mathcal{F}^0(U))$ denote the Dirichlet form corresponding to Brownian motion killed upon exiting U on $L^2(U,m)$, where m denotes the Lebesgue measure on $\mathbb{R}^2$ and

$$D\Phi := \begin{bmatrix} \frac{\partial\Phi_1}{\partial x_1} & \frac{\partial\Phi_2}{\partial x_1} \\ \frac{\partial\Phi_1}{\partial x_2} & \frac{\partial\Phi_2}{\partial x_2} \end{bmatrix}$$

denotes the derivative of $\Phi = (\Phi_1,\Phi_2)$. Then the Dirichlet form $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}})$ on $L^2(V,\widetilde{m})$ is given by

$$\begin{aligned}\widetilde{\mathcal{E}}(f,f) = \mathcal{E}(f\circ\Phi, f\circ\Phi) &= \int_U \nabla(f\circ\Phi)(y)^2\,dy \\ &= \int_V \det(D\Phi(\Phi^{-1}(x)))^{-1}\nabla f(x)^T D\Phi(\Phi^{-1}(x))^T D\Phi(\Phi^{-1}(x))\nabla f(x)\,dx\end{aligned}$$

$$= \int_V \nabla f(x)^T \mathcal{A}(x) \nabla f(x)\, dx$$

for all $f \in \widetilde{\mathcal{F}}$, where $\mathcal{A}(x) := \det(D\Phi(\Phi^{-1}(x)))^{-1} D\Phi(\Phi^{-1}(x))^T D\Phi(\Phi^{-1}(x))$ and $\widetilde{m} = \Phi_*(m)$. It is an easy exercise to verify that $\mathcal{A}(x) \equiv \mathrm{Id}$ if and only if Φ satisfies the Cauchy-Riemann equations or equivalently, that Φ is conformal. Hence the image of a absorbing Brownian motion on U under Φ is a time-change of the absorbing Brownian motion on V if and only if $\Phi : U \to V$ is conformal.

The positive-definite matrix valued function $\mathcal{A} : V \to \mathbb{R}^{2\times 2}$ defines a uniformly-elliptic operator if and only if there exists $K \geqslant 1$ such that

$$\|D\Phi(y)\|^2 \leqslant K \det(D\Phi(y)), \quad \text{for all } y \in U,$$

where $\|D\Phi(y)\|$ denotes the largest singular value (operator norm) of the matrix $\|D\Phi(y)\|$. The above inequality is essentially the definition of quasiconformal maps (except for the caveat that this condition and the condition that Φ is differentiable is only required almost everywhere. We refer to [Ahl, LV] for the theory of quasiconformal mappings). Geometrically, under a quasiconformal map image of infinitesimal circles are ellipses with bounded eccentricity K. In summary, the image of an absorbing Brownian motion on U under Φ is a time-change of the absorbing diffusion generator by an uniformly elliptic operator on V if and only if $\Phi : U \to V$ is quasi-conformal.

Using the above observation and a quasi-conformal mapping due to Beurling and Ahlfors [BA], Caffarelli, Fabes and Kenig construct an example of an uniformly elliptic operator whose associated diffusion has harmonic measure singular with respect to the surface area measure on unit disk or upper half-space [CFK].

The following example is of interest as we will learn in the mini-course this week.

Example 4.4. Let $(\mathcal{E}, \mathcal{F})$ denote the Dirichlet form for the two dimensional Brownian motion on $L^2(\mathbb{R}^2, m)$, where m is the Lebesgue measure. Let M the Liouville quantum gravity measure which is formally defined as $e^{\gamma X}\, dx$, where X is the massive Gaussian free field on $\mathbb{R}^2$. The measure M almost surely is a Radon measure that does not charge any $\mathcal{E}$-polar set for the Brownian motion and the corresponding PCAF A in the strict sense has full support [GRV14, Lemmas 1.5 and 1.6]. The corresponding time-change process is called the *Liouville Brownian motion* [GRV14, GRV16, Ber].

The next example due to Kigami illustrates that a drastic difference in heat kernel behavior is possible under a time-change.

Example 4.5. Let K denote the (standard two-dimensional) Sierpinski gasket equipped with Euclidean metric d and let m denote the self-similar probability measure (normalized Hausdorff measure). Let $(\mathcal{E}, \mathcal{F})$ denote the Dirichlet form for the Brownian motion on Sierpinski gasket described in Proposition 3.9. Consider harmonic functions $h_1, h_2 \in \mathcal{F}$ $\mathcal{E}$-harmonic on $K \backslash V_0$, with

$$(h_1(q_1), h_1(q_2), h_1(q_3)) = \frac{\sqrt{2}}{3}(1, -1/2, -1/2), \quad (h_2(q_1), h_2(q_2), h_2(q_3)) = \frac{1}{\sqrt{6}}(0, 1, -1).$$

The harmonic functions h_1, h_2 have energy 1 and are orthogonal in the sense that $\mathcal{E}(h_1, h_2) = 0$ (any pair of such functions would work for this construction). The Kusuoka measure $\mu = \Gamma(h_1, h_1) + \Gamma(h_2, h_2)$ is the sum of the two energy measures. The Kusuoka measure μ is the Revuz measure of a PCAF in the strict sense with full support K. The corresponding time-changed process has Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$. Define the intrinsic metric with respect to μ as

$$d^{\mu}_{\text{int}}(x, y) := \sup\{f(x) - f(y) | f \in \mathcal{F} \cap C(K), \Gamma(f, f) \leqslant \mu\}.$$

While the MMD space $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfies sub-Gaussian heat kernel bounds with walk dimensions $\frac{\log 5}{\log 2}$ [BP], the time-changed MMD space $(K, d^{\mu}_{\text{int}}, \mu, \mathcal{E}, \mathcal{F})$ satisfies Gaussian heat kernel bounds [Kig08].

4.1 Boundary trace of reflected diffusions

Unlike the above three examples, time-change of a diffusion process can lead to a process with jumps. This is often the case if the corresponding PCAF does not have full support. In such case, in order to analyze the time-changed process, it is useful to compute the Beurling-Deny decomposition of the trace Dirichlet form. Let us illustrate this for reflected Brownian motion restricted to the boundary.

Example 4.6. Let us compute the Dirichlet form corresponding to the boundary trace process for reflected Brownian motion in a bounded smooth domain. That is, the PCAF (and the Revuz) measure have support ∂U. For instance, this is the case if we chose the Revuz measure to be either the surface area measure on ∂U or the harmonic measure $\omega^U_{x_0}$ (law of B_{τ_U} with $B_0 = x_0 \in U$, where B_t is the Brownian motion). The Green's function of a domain U is for $x, y \in U$ is defined by

$$\begin{cases} \Delta_y g_U(x, y) = -\delta_x, & \text{if } y \in U \\ g_U(x, y) = 0, & \text{if } y \in \partial U \end{cases}$$

For a function $f : \partial U \to \mathbb{R}$, let $H_{\partial U}(f) : \overline{U} \to \mathbb{R}$ denote the harmonic function on U (that is $\Delta H_{\partial U}(f) = 0$ on U) with Dirichlet boundary value $f = H_{\partial U}(f)\big|_{\partial U}$. By the theory described earlier, the Dirichlet form on ∂U is given by $f \mapsto \int_U |\nabla(H_{\partial U}(f))|^2 \, dx$. We would like to describe this more explicitly by computing its Beurling-Deny decomposition. As we will see this is a pure-jump Dirichlet form which is not at all clear from the above expression.

Next, we apply the Green's second identity

$$\int_U (v_2(x)\Delta v_1(x) - v_1(x)\Delta v_2(x)) \, dx = \int_{\partial U} \left(v_1 \frac{\partial v_2(\xi)}{\partial \vec{n_\xi}}(\xi) - v_2(\xi) \frac{\partial v_1}{\partial \vec{n_\xi}}(\xi) \right) \sigma(d\xi),$$

where $\vec{n_\xi}$ is the inward pointing normal derivative and σ is the surface measure on ∂U. Substituting $v_1 = H_{\partial U}(f), v_2 = g_U(x, \cdot)$, we obtain

$$H_{\partial U}(f)(x) = \int_{\partial U} f(\xi) \frac{\partial g(x, \xi)}{d\vec{n_\xi}} \, \sigma(d\xi),$$

where $\vec{n_\xi}$ is the inward pointing normal vector at $\xi \in \partial U$. In other words, the harmonic measure ω_x^U on ∂U for Brownian motion started at x (the exit distribution of a Brownian motion started at x) is given by

$$\omega_x^U(d\xi) = \frac{\partial g(x,\xi)}{d\vec{n_\xi}} \cdot \sigma(d\xi). \tag{4.2}$$

Next, we would like to compute the Dirichlet energy of $H_{\partial U}(f)$ in terms of the boundary values f. We apply Green's first identity

$$\int_U \nabla v_1 \cdot \nabla v_2 \, dx = -\int_U v_1 \Delta v_2 \, dx - \int_{\partial U} v_1 \frac{\partial v_2}{\partial n_\xi} \, \sigma(d\xi),$$

where $\vec{n_\xi}$ is the inward point normal vector at $\xi \in \partial U$, to $v_1 = v_2 = H_{\partial U}(f)$, to obtain

$$\int_U |\nabla(H_{\partial U}(f))|^2 \, dx = -\int_{\partial U} f(\xi) \frac{\partial H_{\partial U}(f)}{\partial \vec{n_\xi}} \, \sigma(d\xi) = -\int_{\partial U}\int_{\partial U} f(\xi) f(\eta) \frac{\partial^2 g_U(\eta,\xi)}{\partial \vec{n_\eta} \partial \vec{n_\xi}} \, \sigma(d\eta) \, \sigma(d\xi).$$

Since $\int_{\partial U} \frac{\partial g(x,\xi)}{d\vec{n_\xi}} \cdot \sigma(d\xi) = 1$ for all $x \in U$, we have

$$\int_{\partial U} \frac{\partial^2 g_U(\eta,\xi)}{\partial \vec{n_\eta} \partial \vec{n_\xi}} \, \sigma(d\xi) = 0.$$

Hence we have (using the symmetry $g_U(x,y) = g_U(y,x)$ for all $x, y \in U$ and the above identity)

$$\begin{aligned} \int_U |\nabla(H_{\partial U}(f))|^2 \, dx &= -\int_{\partial U}\int_{\partial U} f(\xi) f(\eta) \frac{\partial^2 g_U(\eta,\xi)}{\partial \vec{n_\eta} \partial \vec{n_\xi}} \, \sigma(d\xi) \\ &= \frac{1}{2} \int_{\partial U}\int_{\partial U} (f(\xi) - f(\eta))^2 \frac{\partial^2 g_U(\eta,\xi)}{\partial \vec{n_\eta} \partial \vec{n_\xi}} \, \sigma(d\xi). \end{aligned} \tag{4.3}$$

This expression is the Beurling-Deny decomposition of the boundary trace process, and gives the interpretation of

$$\frac{\partial^2 g_U(\eta,\xi)}{\partial \vec{n_\eta} \partial \vec{n_\xi}} \, \sigma(d\eta) \, \sigma(d\xi) \tag{4.4}$$

as the jumping measure for the boundary trace process. The above expression also shows that the killing and strongly local parts of the trace process are zero.

Recently, there are a number of works on equations involving fractional Laplace operator by using the fact that it arises as a boundary operator corresponding to some local (diffusion) operator. For $\alpha \in (0,2)$, the fractional Laplace operator $-(-\Delta)^{\alpha/2} f$ on $\mathbb{R}^n$ can be defined using the spectral theorem and functional calculus. Alternately, it can be defined using the Fourier transform $\widehat{(-\Delta)^{\alpha/2} f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$ or as a singular integral

$$-(-\Delta)^{\alpha/2} f(x) = \lim_{r \downarrow 0} c_{n,\alpha} \int_{\mathbb{R}^n \setminus B(x,r)} \frac{f(y) - f(x)}{|x-y|^{n+\alpha}} \, dy,$$

where

$$c_{n,\alpha} = \frac{2^\alpha \Gamma((n+\alpha)/2)}{\pi^{n/2}\Gamma(-\alpha/2)}.$$

The fractional Laplace operator $-(-\Delta)^{\alpha/2}$ is the generator of the symmetric α-stable process for all $\alpha \in (0,2)$. We refer to [Kwa] for equivalence of different definitions of the fractional Laplace operator.

Example 4.7. Molchanov and Ostrowski [MO] discovered that α-stable processes can be obtained as a boundary trace process of a reflected diffusion on the closed upper half space. This was later independently revisited by Caffarelli and Silvestre in an equivalent form [CS].

We consider the Molchanov–Ostrowski diffusion [MO] on the closed upper half-space $X = \{(x,y) : x \in \mathbb{R}^n,\, y \in [0,\infty)\} = \mathbb{R}^n \times [0,\infty)$ is induced by the Dirichlet form $(\mathcal{E},\mathcal{F})$ given by

$$\mathcal{E}(u,u) := \int_{\mathbb{R}^n}\int_0^\infty |\nabla u|^2(x,y)|y|^{1-\alpha}\,dy\,dx$$

on $L^2(\mathbb{R}^n \times [0,\infty), |y|^{1-\alpha}\,dy\,dx)$, where $\alpha \in (0,2)$. This corresponds to the Brownain motion in $\mathbb{R}^n$ direction and an independent $(2-\alpha)$-dimensional Bessel process (recall Exercise 2.50) in the last $[0,\infty)$ direction.

Let us identify the boundary of $\mathbb{H}^{n+1} := \mathbb{R}^n \times [0,\infty)$ with $\mathbb{R}^n$. Consider a function $f \in C_c^\infty(\mathbb{R}^n)$. Since the generator of $(\mathcal{E},\mathcal{F})$ is given by

$$Lu = y^{\alpha-1}\mathrm{div}\,(y^{1-\alpha}\nabla u) = \Delta_x u + \frac{1-\alpha}{y}u_y + u_{yy}.$$

If $H_{\partial\mathbb{H}^{n+1}} f = u$, then u satisfies the equation $Lu = 0$ on $\mathbb{R}^n \times (0,\infty)$ with boundary condition f on $\mathbb{R}^n$. In order solve this equation, we take Fourier transform with respect to the x-variable. Defining

$$\widehat{u}(\xi,y) := \int u(x,y)e^{-ix\cdot\xi}\,dx,$$

the equation $Lu = 0$ can be written as

$$-|\xi|^2\widehat{u}(\xi,y) + \frac{1-\alpha}{y}\widehat{u}_y(\xi,y) + \widehat{u}_{yy}(\xi,y),$$

with boundary condition

$$\widehat{u}(\xi,0) = \widehat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}\,dx, \quad \text{for all } \xi \in \mathbb{R}^n.$$

Note that the PDE has been transformed to a ODE in the variable y. Using the homogeneity and scaling properties of this ODE, solution is given by $\widehat{u}(\xi,y) = \widehat{f}(\xi)\phi(|\xi|y)$, where $\phi : [0,\infty) \to \mathbb{R}$ solves the ODE

$$-\phi + \frac{1-\alpha}{y}\phi_y + \phi_{yy} = 0, \quad \text{on } (0,\infty) \text{ with } \phi(0) = 1.)$$

This is the Bessel differential equation and has a unique solution such that $\lim_{y\to\infty}\phi(y) = 0$. We would like to express the energy $\int_{\mathbb{R}^n}\int_0^\infty |\nabla u|^2(x,y)|y|^{1-\alpha}\,dy\,dx$ in terms of the boundary condition f. Ignoring multiplicative constants and using Parseval's formula, Fubini's theorem and the solution $\widehat{u}(\xi,y) = \widehat{f}(\xi)\phi(|\xi|y)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_0^\infty |\nabla u|^2(x,y)|y|^{1-\alpha}\,dy\,dx &= c\int_0^\infty\int_{\mathbb{R}^n}(|\xi|^2|\widehat{u}(\xi,y)|^2 + |\widehat{u}_y(\xi,y)|^2)|y|^{1-\alpha}\,d\xi\,dy \\
&= c\int_0^\infty\int_{\mathbb{R}^n}|\xi|^2\Big|\widehat{f}(\xi)^2\Big|(|\phi(|\xi|y)|^2 + |\phi'(|\xi|y)|^2)|y|^{1-\alpha}\,d\xi\,dy \\
&= c\int_0^\infty\int_{\mathbb{R}^n}|\xi|^\alpha\Big|\widehat{f}(\xi)^2\Big|(|\phi(z)|^2 + |\phi'(z)|^2)|z|^{1-\alpha}\,d\xi\,dz \\
&= c'\int_{\mathbb{R}^n}|\xi|^\alpha\Big|\widehat{f}(\xi)^2\Big|\,d\xi \\
&= c''\langle(-\Delta)^{\alpha/2}f, f\rangle = c'''\int_{bR^n}\int_{bR^n}\frac{(f(x)-f(y))^2}{|x-y|^{n+\alpha}}\,dy\,dx.
\end{aligned} \tag{4.5}$$

which corresponds to the Dirichlet form for symmetric α-stable process (up to identifying the multiplicative constant). Using Harnack inequality for the diffusion process (or equivalently, the generator L), Caffarelli and Silvestre were able to obtain Harnack inequality for the α-stable process (or equivalently, the fractional Laplacian) [CS].

Doob [Doo62] found a remarkable extension of (4.3) to domains that are not necessarily smooth. He stated the result under an abstract potential theoretic setting of (locally Euclidean) Green spaces[7] in the sense of Brelot and Choquet [BC], in which the boundary values of the harmonic functions are prescribed on the *Martin boundary* $\partial_M D$ of the domain D. To describe Doob's result, we recall the *Naïm kernel* $\Theta^D_{x_0}(\cdot,\cdot)$ defined by

$$\Theta^D_{x_0}(\xi,\eta) = \lim_{x\to\xi}\lim_{y\to\eta}\frac{g_D(x,y)}{g_D(x_0,x)g_D(x_0,y)} \qquad \text{for } \xi,\eta\in\partial_M D,\ \xi\neq\eta, \tag{4.6}$$

where the limits are with respect to the *fine topology*, $x_0\in D$ is an arbitrary base point, and $g_D(\cdot,\cdot)$ is the Green function on D as before. The existence of the above limits in the setting of Green spaces follows from the fundamental work by L. Naïm [Naï]. Then it was shown in [Doo62, Theorem 9.2] that the *Doob–Naïm formula*

$$\int_D |\nabla u|^2(x)\,dx = \frac{1}{2}\int_{\partial_M D}\int_{\partial_M D}(f(\xi)-f(\eta))^2\Theta^D_{x_0}(\xi,\eta)\,d\omega^D_{x_0}(\xi)\,d\omega^D_{x_0}(\eta) \tag{4.7}$$

holds if u is a harmonic function on the domain D with fine boundary value $f\colon\partial_M D\to\mathbb{R}$, where $\omega^D_{x_0}$ denotes the *harmonic measure*, i.e., the probability distribution of the position of the first hitting to $\partial_M D$, of the Brownian motion on D started at x_0. There is a version

[7] Roughly speaking, a Green space is a locally compact, connected Hausdorff space with a countable base that is *locally Euclidean* of dimension $n\geqslant 2$ and such that a strictly positive nonconstant superharmonic function exists.

of the Doob–Naïm formula for transient symmetric Markov chains on countable state spaces due to M. Silverstein [Sil, Theorem 3.5]. Let us give some heuristic behind (4.7). From the expression (4.4) for the jump measure in the smooth case and the formula (4.2) for the harmonic measure, we can express the jump measure for bounded smooth domains as

$$\frac{\partial^2 g_U(\eta,\xi)}{\partial \vec{n_\eta}\partial \vec{n_\xi}}\,\sigma(d\eta)\,\sigma(d\xi) = \frac{\partial^2 g_U(\eta,\xi)}{\partial \vec{n_\eta}\partial \vec{n_\xi}}\,\frac{\partial g(x_0,\xi)}{d\vec{n_\xi}}^{-1}\,\frac{\partial g(x_0,\eta)}{d\vec{n_\eta}}^{-1}\,\omega^U_{x_0}(d\xi)\,\omega^U_{x_0}(d\eta).$$

This is suggestive of the Doob-Naïm formula (if we 'cancel' the terms $\partial\vec{n_\eta}$ and $\partial\vec{n_\xi}$).

We describe an extension of Doob-Naïm formula that was obtained in a joint work with N. Kajino [KM24]. Unlike the original version due to Doob, our version applies to diffusions generated by uniformly elliptic operators or the Molchanov-Ostrowski diffusion. In particular, our result shows that the results of [MO, CS] establishing α-stable process as the boundary trace process can be viewed as a special case of our version of the Doob-Naïm formula. In order to define the Naïm kernel first we recall the existence and properties of Green functions.

For the remainder of §4.1, we make the following assumption. Let $(X,d,m,\mathcal{E},\mathcal{F})$ be an MMD space that satisfies full sub-Gaussian heat kernel bound $\mathrm{HKE_f}(\Psi)$, where $\Psi:[0,\infty)\to[0,\infty)$ is a scale function. To simplify some technicalities, we assume that (X,d) is a geodesic space. We assume that U is a uniform domain satisfying the *capacity density condition*:

$$\mathrm{Cap}(B(\xi,r)B(\xi,2r)^c) \lesssim \mathrm{Cap}(B(\xi,r)\backslash U, B(\xi,2r)^c) \quad \text{for all } \xi\in\partial U,\ 0<r\lesssim \mathrm{diam}(U). \tag{4.8}$$

This class includes all bounded smooth domains, Lipschitz domains and non-tangetially accessible domains in the sense of Jerison and Kenig. In particular, this class includes domains with fractal boundary such as the von Koch snowflake domain. A major advantage of this class of domains is that Aikawa and Hirata obtained sharp bounds on the harmonic measure [AH] which in particular implies that the harmonic measure has full support on ∂U. More precisely, there exist $C, A\geqslant 1$ such that for all $x_0\in U, \xi\in\partial U, 0<r<d(x_0,\xi)/A$

$$\omega^U_{x_0}(B(\xi,r)\cap\partial U) \asymp g_U(x_0,\xi_r)\,\mathrm{Cap}(B(\xi,r),B(\xi,2r)^c), \tag{4.9}$$

where $\xi_r\in U$ is such that $d(\xi,\xi_r)\asymp d(\xi_r,\partial U)\asymp r$ and g_U denotes the Green function whose properties are recalled below in Proposition 4.8. The existence of such a ξ_r follows from the definition of uniform domain (cf. [Mur24, Lemma 3.4]). Aikawa and Hirata's argument also applies to the above general setting of MMD space satisfying sub-Gaussian heat kernel bounds; see [KM24, Theorem 4.6] or [CC].

We collect some basic properties of Green function in this setting. We refer to [BCM, Theorem 4.4] for a proof in a more general setting.

Proposition 4.8. *Let $(X,d,m,\mathcal{E},\mathcal{F})$ be an MMD space satisfying sub-Gaussian heat kernel bound $\mathrm{HKE_f}(\Psi)$. Let (Y_t) denote the corresponding diffusion process. Let D be a domain such that Dirichlet form $(\mathcal{E}^D,\mathcal{F}^0(D))$ corresponding to killed diffusion process (Y^D_t) is transient. Then there exists a function $g_D: D\times D\to[0,\infty]$ and a properly exceptional set $\mathcal{N}$ for (Y_t) such that the following hold:*

1. *(symmetry)* $g_D(x,y) = g_D(y,x)$ *for all* $(x,y) \in D \times D$.

2. *(continuity)* $g_D(\cdot,\cdot)$ *is* $[0,\infty)$*-valued and continuous on* $D \times D \backslash \operatorname{diag}_D$.

3. *(harmonicity) For each* $x \in D$*, we have* $g_D(x,\cdot) \in \mathcal{F}^0(D, D\backslash\{x\})$ *and that* $g_D(x,\cdot)$ *is* $\mathcal{E}$*-harmonic on* $D\backslash\{x\}$.

4. *(occupation density formula) For any Borel measurable function* $f : D \to [0,\infty)$,

$$G_D f(x) := \begin{cases} \int_D g_D(x,y) f(y)\, m(dy) & \text{if } x \in D, \\ 0 & \text{if } x \notin D. \end{cases}$$

satisfies

$$\mathbb{E}_x\left[\int_0^{\tau_D} f(Y_s) ds\right] = G_D f(x) \quad \text{for every } x \in D\backslash\mathcal{N}.$$

Furthermore if for a Borel measurable function $f : D \to [0,\infty)$*, if* $\int_D f G_D(f)\, dm < \infty$*, then* $G_D(f) \in \mathcal{F}^0(D)_e$ *and satisfies*

$$\mathcal{E}^D(G^D(f), G^D(f)) = \int_D f G_D(f)\, dm.$$

More generally, for any $u \in \mathcal{F}^0(D)_e$*, we have*

$$\mathcal{E}^D(G^D(f), u) = \int_D f u\, dm.$$

The following lemma provides a suitable generalization of the formula for harmonic measure that $\omega_x^U(d\xi) = \frac{\partial g(x,\xi)}{d\vec{n_\xi}} \cdot \sigma(d\xi)$ obtained in (4.2) for Brownian motion on smooth domains. This result can be paraphrased by saying that the harmonic measure is the distributional Laplacian (or more generally, the generator) of the Green function.

Lemma 4.9. *For all* $x \in U, u \in \mathcal{F}(U) \cap L^\infty(U)$ *such that* $x \notin \operatorname{supp}_{\overline{U}}[u]$*, we have*

$$\mathcal{E}_U(g_U(x,\cdot), u) = -\int_{\partial U} \widetilde{u} d\omega_x^U.$$

(Recall that since $g_U(x,\cdot)) \in \mathcal{F}^0_{\text{loc}}(U, U\backslash\{x_0\}$ *the expression* $\mathcal{E}_U(g_U(x,\cdot), u)$ *is defined).*

Proof. Due to transience of the killed process, there exists a non-negative function $f \in L^1(U,m)$ with $\int_U f G_U(f)\, dm < \infty$. To see the existence of such a function, we pick some $f_1 \in L^1 \cap L^\infty$ with $f_1 > 0$ m-a.e. By the transience we have $G(f_1) < \infty$ m-a.e. Then $f := \frac{f_1}{G(f_1)\vee 1}$ satisfies the desired property. Now pick $r > 0$ small enough that $\operatorname{supp}_{\overline{U}}[u] \subset B(x,r)^c$. Replacing f with $f\mathbb{1}_{B(x,r)}$ and multiplying by a constant, we may assume that $\operatorname{supp}_m[f] \subset B(x,r)$ and $\int_U f\, dm = 1$.

Since $G_U(f) \in \mathcal{F}^0(U)_e$ (due to $\int_U f G_U(f)\, dm < \infty$) and $H_{\partial U}(\widetilde{u}))$ is a harmonic function on U, we have $\mathcal{E}_U(G_U(f), H_{\partial U}(\widetilde{u})) = 0$. Then we have

$$\mathcal{E}_U(G_U(f), u) = \mathcal{E}_U(G_U f, u) - \mathcal{E}_U(G_U(f), H_{\partial U}(\widetilde{u}))$$

$$= \mathcal{E}_U(G_U(f), u - H_{\partial U}(\widetilde{u})) = \int_U f(u - H_{\partial U}(\widetilde{u}))\, dm = -\int_U f H_{\partial U}(\widetilde{u})\, dm.$$

Letting $r \downarrow 0$, we obtain the desired result as $G_U(f) \to g_U(x, \cdot)$ and $\int_U f H_{\partial U}(\widetilde{u})\, dm \to H_{\partial U}(\widetilde{u})(x) = \int_{\partial U} \widetilde{u}\, d\omega_x^U$. □

The following lemma concerns the equilibrium measures for capacity between two sets.

Lemma 4.10. *([KM24, Lemma 2.11]) Let $A, B \subset X$ be disjoint closed sets such that A and $\overline{B^c}$ are compact. Let $e_{A,B} \in \mathcal{F}_e \cap L^\infty$ such that $\widetilde{e_{A,B}} = 1$ q.e. on A and $\widetilde{e_{A,B}} = 0$ q.e. on B and $e_{A,B}$ is harmonic on $(A \cup B)^c$. Then there exists finite Borel measure $\lambda^1_{A,B}$ and $\lambda^0_{A,B}$ charging no $\mathcal{E}$-polar set such that $\lambda^1_{A,B}((\partial A)^c) = \lambda^0_{A,B}((\partial B)^c) = 0$ and $\lambda^1_{A,B}(X) = \lambda^0_{A,B}(X) = \mathcal{E}(e_{A,B}, e_{A,B})$. For any $u \in L^\infty \cap \mathcal{F}_e$, we have*

$$\mathcal{E}(e_{A,B}, u) = \int_{\partial A} \widetilde{u}\, d\lambda^1_{A,B} - \int_{\partial B} \widetilde{u}\, d\lambda^0_{A,B}.$$

Remark 4.11. The function $e_{A,B}$ above is called the *equilibrium potential* and $\lambda^1_{A,B}$ and $\lambda^0_{A,B}$ are called the equilibrium measures. In terms of the corresponding process (Y_t) we can describe $e_{A,B}$ as

$$e_{A,B}(x) = \mathbb{P}_x[T_A < T_B],$$

where T_A, T_B denote the hitting times of A, B respectively.

Proof Sketch. We only sketch the proof for $\lambda^1_{A,B}$ as $\lambda^0_{A,B}$ is similar. First, we briefly explain the reason for the exitstence of $e_{A,B}$. This is due to the fact that $(\mathcal{F}^0(B^c)_e, \mathcal{E})$ is a Hilbert space (owing to the transience of the process killed upon exiting B^c) and hence the set

$$\mathcal{L}_{A,B} := \{f \in \mathcal{F}^0(B^c)_e | \widetilde{f} \geqslant 1 \text{ q.e. on } A\}$$

is a closed convex subset of the Hilbert space $(\mathcal{F}^0(B^c)_e, \mathcal{E})$ and hence there exists an element $e_{A,B} \in \mathcal{L}_{A,B}$ of minimal energy $\mathcal{E}$. By the Markov property of $\mathcal{E}$, we note that $(0 \vee e_{A,B}) \wedge 1 = e_{A,B}$ and hence $\widetilde{e_{A,B}} = 1$ q.e. on A and $\widetilde{e_{A,B}} = 0$ q.e. on B and $e_{A,B}$ is harmonic on $(A \cup B)^c$.

By the minimality of energy of $e_{A,B}$ and replacing $e_{A,B}$ with $e_{A,B} + tv$ for all $t \geqslant 0$, where $v \in \mathcal{F}^0(B^c)_e$ is such that $v \geqslant 0$ a.e., we conclude

$$\mathcal{E}(e_{A,B}, v) \geqslant 0, \quad \text{for all } v \in \mathcal{F}^0(B^c)_e \text{ with } v \geqslant 0 \ m\text{-a.e.}$$

This along with Riesz representation theorem allows us to conclude the existence of measure $\lambda^1_{A,B}$ such that $\mathcal{E}(e_{A,B}, v) = \int_A \widetilde{v}\, d\lambda^1_{A,B}$ for all $v \in \mathcal{F}^0(B^c)_e$. The fact that this measure is supported in ∂A is due to the strong locality and the assertion that $\widetilde{e_{A,B}} = 1$ q.e. on A. □

Definition 4.12. Let $x_0 \in U$. We recall the notion of *Martin kernel* introduced by R. S. Martin [Mar].

$$K^U_{x_0}(x, \xi) := \begin{cases} \dfrac{g_U(x, \xi)}{g_U(x_0, \xi)} & \text{if } \xi \in U \backslash \{x_0, x\}, \\ \lim\limits_{U \ni y \to \xi} \dfrac{g_U(x, y)}{g_U(x_0, y)} & \text{if } \xi \in \partial U, \end{cases} \tag{4.10}$$

The existence of the above limit follows from Hölder regularity estimate on ratio of harmonic function obtained using boundary Harnack inequality and the Moser's oscillation lemma argument.

Similarly, the Naïm kernel is first defined for $(x, y) \in (U \backslash x_0) \times (U \backslash \{x_0\}) \backslash \operatorname{diag}_{U \backslash \{x_0\}}$ as

$$\Theta_{x_0}^U(x, y) := \frac{g_U(x, y)}{g_U(x_0, x) g_U(x_0, y)} \tag{4.11}$$

and then extended as a continuous function to $(\overline{U} \backslash x_0) \times (\overline{U} \backslash \{x_0\}) \backslash \operatorname{diag}_{\overline{U} \backslash \{x_0\}}$.

The Naïm kernel can be expressed in terms of the Martin kernel as

$$\Theta_{x_0}^U(x, y) = \begin{cases} \dfrac{K_{x_0}^U(x, y)}{g_U(x_0, x)} & \text{if } x \in U, \\ \lim\limits_{U \ni z \to x} \dfrac{K_{x_0}^U(z, y)}{g_U(x_0, z)} & \text{if } x \in \partial U, \end{cases} \tag{4.12}$$

where the existence of the above limit can be justified using the boundary Harnack principle and Moser's oscillation lemma argument.

The Martin kernel is useful to relate harmonic measure at different starting points.

Proposition 4.13. *For all $x_0, x \in U$, we have*

$$\frac{d\omega_x^U}{d\omega_{x_0}^U} = K_{x_0}^U(x, \cdot).$$

Proof Sketch. Let $\xi \in \partial U$ and $r > 0$ be small. Consider the sets $A = \overline{B(\xi, r)} \cap \partial U$ and $B = \overline{U} \cap B(\xi, 2r)^c$. Consider the function $e_{A,B}$ as given in Lemma 4.10 for the reflected diffusion and the corresponding equilibrium measure $\lambda_{A,B}^1$ and $\lambda_{A,B}^0$. Then by Lemma 4.9 and 4.10, we have (denoting by $S(\xi, 2r)$ the sphere of radius $2r$)

$$\begin{aligned} 0 < \int_{\partial U} \widetilde{e}_{A,B} \, d\omega_x^U &= -\mathcal{E}_U(g_U(x, \cdot), e_{A,B}) \\ &= -\left(\int_A g_U(x, y) \, d\lambda_{A,B}^1(y) - \int_{\overline{U} \cap S(\xi, 2r)} g_U(x, y) \, d\lambda_{A,B}^0(y) \right) \\ &= \int_{\overline{U} \cap S(\xi, 2r)} g_U(x, y) \, d\lambda_{A,B}^0(y). \end{aligned} \tag{4.13}$$

Taking ratio of (4.13) for x and for x_0 in place of x, we obtain

$$\begin{aligned} \left| \frac{\int_{\partial U} \widetilde{e}_{A,B} \, d\omega_x^U}{\int_{\partial U} \widetilde{e}_{A,B} \, d\omega_{x_0}^U} - K_{x_0}(x, \xi) \right| &= \left| \frac{\int_{\overline{U} \cap S(\xi, 2r)} g_U(x, y) \, d\lambda_{A,B}^0(y)}{\int_{\overline{U} \cap S(\xi, 2r)} g_U(x_0, y) \, d\lambda_{A,B}^0(y)} - K_{x_0}(x, \xi) \right| \\ &\leqslant \left| \frac{\int_{\overline{U} \cap S(\xi, 2r)} g_U(x_0, y)(K_{x_0}(x, y) - K_{x_0}(x, \xi)) \, d\lambda_{A,B}^0(y)}{\int_{\overline{U} \cap S(\xi, 2r)} g_U(x_0, y) \, d\lambda_{A,B}^0(y)} \right| \end{aligned}$$

$$\leqslant \frac{\int_{\overline{U}\cap S(\xi,2r)} g_U(x_0,y)|K_{x_0}(x,y)-K_{x_0}(x,\xi)|\,d\lambda^0_{A,B}(y)}{\int_{\overline{U}\cap S(\xi,2r)} g_U(x_0,y)\,d\lambda^0_{A,B}(y)}. \quad (4.14)$$

Letting $r \downarrow 0$ and using Lebesgue differentiation theorem and the continuity of Martin kernel on the boundary, we obtain the desired result. □

In order to make the argument in the last step precise, we need the following version of Lebesgue differentiation theorem for pointwise-doubling measures. The pointwise doubling condition (4.15) required for the application of Lebesgue differentiation theorem follows from (4.9) (cf. [HKST, Theorem 3.4.3 and (3.4.10)]).

Lemma 4.14 (Lebesgue's differentiation theorem)**.** *Let (X,d,m) be a metric measure space such that (X,d) is separable, $m(B(x,r))<\infty$ for some $r\in(0,\infty)$ for each $x\in X$, and*

$$\limsup_{r\downarrow 0}\frac{m(B(x,2r))}{m(B(x,r))}<\infty \quad \text{for } m\text{-a.e. } x\in X. \quad (4.15)$$

Then for any locally integrable function $f\colon X\to\mathbb{R}$ almost every point is a Lebesgue point of f; that is,

$$\lim_{r\downarrow 0}\fint_{B(x,r)}|f(y)-f(x)|\,dm(y) \quad (4.16)$$

for m-a.e. $x\in X$. In particular, for any $x\in\operatorname{supp}_X[m]$ satisfying (4.16), if $\varepsilon\in(0,\infty)$ and $\psi_r\colon X\to\mathbb{R}$ is a Borel measurable function satisfying $\mathbb{1}_{B(x,r)}\leqslant\psi_r\leqslant\mathbb{1}_{B(x,2r)}$ for each $r\in(0,\varepsilon)$, then

$$\lim_{r\downarrow 0}\frac{\int_X\psi_r f\,dm}{\int_X\psi_r\,dm}=f(x). \quad (4.17)$$

Next, we show the Doob-Naïm formula for the jumping measure.

Proposition 4.15. *The jump measure of the boundary trace process corresponding to reflected diffusion on a uniform domain U satisfying the capacity density condition is given by the Doob-Naïm formula; that is the jump measure equals $\Theta^U_{x_0}(\xi,\eta)\,d\omega^U_{x_0}(d\xi)\,d\omega^U_{x_0}(d\eta)$.*

Proof Sketch. Let $\xi,\eta\in\partial U$ be distinct and $r<d(\xi,\eta)/4$. Let $A=B(\xi,r)\cap\partial U$, $B=B(\xi,2r)^c\cap\overline{U}$ and $v:=e_{A,B}\in\mathcal{F}^0(B^c)$ denote the equilibrium potential correpsonding to the reflected diffusion on U. Let $u\in\mathcal{F}(U)\cap C_c(\overline{U})$ be any function such that

$$\mathbb{1}_{B(\eta,r)}\leqslant u\leqslant\mathbb{1}_{B(\eta,2r)}.$$

By Lemma 4.10 and $\operatorname{supp}(\omega^U_{x_0})=\partial U$, we have

$$0<\int_{\partial U}\widetilde{e}_{A,B}\,d\omega^U_{x_0}=\int_{\overline{U}\cap\partial B}g_U(x_0,y)\,d\lambda^0_{A,B}(y). \quad (4.18)$$

Since $H_{\partial U}u$ is $\mathcal{E}_U$-harmonic on U and $H_{\partial U}\widetilde{e}_{A,B}=\widetilde{e}_{A,B}$ $\mathcal{E}_U$-q.e. on ∂U, we have

$$\mathcal{E}_U\big(H_{\partial U}u,H_{\partial U}\widetilde{e}_{A,B}\big)=\mathcal{E}_U\big(H_{\partial U}u,\widetilde{e}_{A,B}\big)$$

$$
\begin{aligned}
&= -\int_{\overline{U}\cap S(\xi,2r)} H_{\partial U} u \, d\lambda^0_{A,B} \quad \text{(by Lemma 4.10)} \\
&= -\int_{\overline{U}\cap S(\xi,2r)} \left(\int_{\partial U} u(z) \, d\omega^U_y(z) \right) d\lambda^0_{A,B}(y) \\
&= -\int_{\overline{U}\cap S(\xi,2r)} \left(\int_{\partial U} u(z) K_{x_0}(y,z) \, d\omega^U_{x_0}(z) \right) d\lambda^0_{A,B}(y), \qquad (4.19)
\end{aligned}
$$

where we use Proposition 4.13 to change the base point of the harmonic measure. If J is the jump measure of the Beurling-Deny decomposition of the boundary trace Dirichlet form, we have

$$
\begin{aligned}
\mathcal{E}_U\big(H_{\partial U} u, H_{\partial U} \widetilde{e}_{A,B}\big) &= \frac{1}{2} \int_{(\partial U \times \partial U)\setminus \mathrm{diag}_{\partial U}} (u(x)-u(y))(\widetilde{e}_{A,B}(x) - \widetilde{e}_{A,B}(y)) \, J(dx\, dy) \\
&= -\int_{(\partial U \times \partial U)\setminus \mathrm{diag}_{\partial U}} u(x) \widetilde{e}_{A,B}(y) \, J(dx\, dy). \qquad (4.20)
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&\frac{\int_{(\partial U)^2_{\mathrm{od}}} u(x)\widetilde{e}_{A,B}(y) \, J(dx\, dy)}{\int_{\partial U} u \, d\omega^U_{x_0} \int_{\partial U} \widetilde{e}_{A,B} \, d\omega^U_{x_0}} \\
&\quad = \frac{-\mathcal{E}_U(H_{\partial U} u, H_{\partial U}(\widetilde{e}_{A,B}))}{\int_{\partial U} u \, d\omega^U_{x_0} \int_{\partial U} \widetilde{e}_{A,B} \, d\omega^U_{x_0}} \quad \text{(by (4.20))} \\
&\quad = \frac{\int_{\overline{U}\cap S(\xi,2r)} \left(\int_{\partial U} u(z) K_{x_0}(y,z) \, d\omega^U_{x_0}(z) \right) \, d\lambda^0_{A,B}(y)}{\int_{\partial U} u \, d\omega^U_{x_0} \int_{\overline{U}\cap S(\xi,2r)} g_U(x_0,y) \, d\lambda^0_{A,B}(y)} \quad \text{(by (4.18) and (4.19))} \\
&\quad \overset{(4.12)}{=} \int_{\overline{U}\cap S(\xi,2r)} \int_{\partial U} \Theta^U_{x_0}(y,z) \frac{u(z)}{\int_{\partial U} u \, d\omega^U_{x_0}} \, d\omega^U_{x_0}(z) \frac{g_U(x_0,y)}{\int_{\overline{U}\cap S(\xi,2r)} g_U(x_0,\cdot) \, d\lambda^0_{A,B}} \, d\lambda^0_{A,B}(y).
\end{aligned}
\qquad (4.21)
$$

As $r \downarrow 0$, we note that $(z,y) \to (\xi,\eta)$ and by using the joint continuity of Naïm kernel, we obtain the desired result. □

We omit the calculation of the strongly local part and the killing part of the Beurling-Deny decomposition as they are both zero under (4.8) and refer to [KM24].

We describe our choice for the Revuz measure for PCAF corresponding to a boundary trace process. The use of harmonic measure $\omega^U_{x_0}$ is natural. This requires a choice of a base point x_0. If the uniform domain U is bounded ($\mathrm{diam}(U) < \infty$), then any choice $x_0 \in U$ with $\mathrm{dist}(x_0, U^c) \asymp \mathrm{diam}(U)$ would lead to a doubling measure. Furthermore, by the elliptic Harnack inequality any two such measures assign comparable mass to all measurable sets.

If U is unbounded the following construction provides a canonical version of harmonic measure. It can be viewed as a rescaled harmonic measure from infinity. Such measures were first constructed by Kenig and Toro in Euclidean case [KT, Corollary 3.2].

Lemma 4.16. *Let $x_0 \in U$, and let $\{x_n\}_{n\in\mathbb{N}} \subset U\backslash\{x_0\}$ be any sequence satisfying $\lim_{n\to\infty} d(x_0, x_n) = \infty$. Let $h_{x_0}(\cdot) = \lim_{n\to\infty} K_{x_0}(\cdot, x_n)$ denote the unique harmonic function on U with Dirichlet boundary conditions with $h_{x_0}(x_0) = 1$. Then the sequence of the measures $\nu_n := g_U(x_0, x_n)^{-1}\omega^U_{x_n}\big|_{\overline{U}}$ converges in total variation on any compact subset of $\overline{U}$ to an $\mathcal{E}_U$-smooth Radon measure $\nu^U_{x_0}$ on $\overline{U}$ with $\nu^U_{x_0}(U) = 0$ and*

$$\mathcal{E}_U(h_{x_0}, u) = -\int_{\partial U} \tilde{u}\, d\nu^U_{x_0} \tag{4.22}$$

for all $u \in \mathcal{F}(U) \cap L^\infty(\overline{U}, m|_{\overline{U}})$ such that $\mathrm{supp}_{m|_{\overline{U}}}[u]$ is compact. This measure $\nu^U_{x_0}$ is a doubling measure on ∂U.

Proof sketch. We simply use Lemma 4.9 to obtain

$$g_U(x_0, x_n)^{-1}\mathcal{E}_U(g_U(x_n, \cdot), u) = -\int_{\partial U} \tilde{u} g_U(x_0, x_n)^{-1} d\omega^U_{x_n}.$$

for all $x \in U, u \in \mathcal{F}(U) \cap L^\infty(U)$ such that $x \notin \mathrm{supp}_{\overline{U}}[u]$. By the BHP, we can show that the left hand side above converges to $g_U(x_0, x_n)^{-1}\mathcal{E}_U(g_U(x_n, \cdot), u)$ as $g_U(x_0, x_n)^{-1}g_U(x_n, \cdot)$ converges to h_{x_0}. □

The Doob-Naïm formula along with general characterization of stable-like heat kernel bounds established by Chen, Kumagai and Wang [CKW] imply sharp two-sided bounds on the boundary trace process [KM24]. This can also be obtained without the exact identification of the jump kernel but only two-sided estimates estimates on the jump kernel as shown by [CC].

If U is unbounded uniform domain and ∂U is bounded (for example, complement of a bounded convex set is $\mathbb{R}^n$ is always a uniform domain [GyS]). In this case, we could introduce a weaker version of capacity density condition, where (4.8) is

$$\mathrm{Cap}(B(\xi, r)B(\xi, 2r)^c) \lesssim \mathrm{Cap}(B(\xi, r)\backslash U, B(\xi, 2r)^c) \quad \text{for all } \xi \in \partial U,\ 0 < r \lesssim \mathrm{diam}(\partial U).$$

Our proof of the Doob-Naïm formula also extends to this case but in this case the killing measure need not be zero. The killing measure is zero if and only if the associated reflected diffusion in recurrent. If the reflected diffusion on U is transient, then

$$\mathbb{P}_{x_0}(T_{\partial U} = \infty)h^U_{x_0}(x) = 1 - H_{\partial U}\mathbb{1}_{\partial U}(x) = \mathbb{P}_x(T_{\partial U} = \infty).$$

Note that for any $u \in \mathcal{F}(U) \cap C_c(\overline{U})$, we have

$$\mathcal{E}_U(H_{\partial U}\mathbb{1}, H_{\partial U}(u)) = \int_{\partial U} u\, d\kappa,$$

where κ is the killing measure of the boundary trace process. Hence by (4.22),

$$\kappa = \mathbb{P}_{x_0}(T_{\partial U} = \infty)\nu^U_{x_0}.$$

Hence the killing measure can be viewed as a rescaled limit of harmonic measure at infinity.

4.2 Time change, conformal gauge and Harnack inequalities

Next we will describe how time change can be used to better understand the relationship between elliptic and parabolic Harnack inequalities. There are two useful operations that preserve the elliptic Harnack inequality: time-change by an *admissible* measure and quasisymmetric change of metric.

Let us describe the notion of admissible measure. Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space. A Radon measure μ is said to be admissible if it the Revuz measure of a positive continuous additive functional (PCAF) with full support X. Recall that such a measure μ defines a time change of the process whose associated Dirichlet form is called the trace Dirichlet form given by

$$\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(X, \mu) \qquad \text{and} \qquad \mathcal{E}^\mu(u, v) = \mathcal{E}(u, v) \quad \text{for } u, v \in \mathcal{F}^\mu,$$

and $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is a strongly local, regular symmetric Dirichlet form on $L^2(X, \mu)$ by [FOT, Theorems 5.1.5, 6.2.1 and Exercise 3.1.1]. We also note that by [CF, Theorem 5.2.11],

$$\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) = \mathcal{A}(X, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu), \quad (\mathcal{F}^\mu)_e = \mathcal{F}_e.$$

Since the extended Dirichlet space remains invariant under a time change by an admissible measure, the space of harmonic functions remains unchanged and hence the elliptic Harnack inequality is preserved under time change.

The description of Harnack inequalities rely on balls and hence they depend on the metric. There is a particular change of metrics that preserve the elliptic Harnack inequality.

Definition 4.17. A *distortion function* is a homeomorphism of $[0, \infty)$ onto itself. Let η be a distortion function. A map $f : (X_1, d_1) \to (X_2, d_2)$ between metric spaces is said to be *η-quasisymmetric*, if f is a homeomorphism and

$$\frac{d_2(f(x), f(a))}{d_2(f(x), f(b))} \leqslant \eta\left(\frac{d_1(x, a)}{d_1(x, b)}\right)$$

for all triples of points $x, a, b \in X_1, x \neq b$. We say f is a *quasisymmetry* if it is η-quasisymmetric for some distortion function η. We say that metric spaces (X_1, d_1) and (X_2, d_2) are quasisymmetric, if there exist s a quasisymmetry $f : (X_1, d_1) \to (X_2, d_2)$. We say that metrics d_1 and d_2 on X are *quasisymmetric*, if the identity map $\mathrm{Id} : (X, d_1) \to (\mathcal{X}, d_2)$ is a quasisymmetry. It is an easy exercise to verify that the notion of quasisymmetry defines an equivalence relation among metrics. The *conformal gauge* of a metric space (X, d) is defined as

$$\mathcal{J}(X, d) := \{\theta \colon X \times X \to [0, \infty) \mid \theta \text{ is a metric on } X,\ d \text{ is quasisymmetric to } \theta\}. \quad (4.23)$$

In other words, this is the equivalence class corresponding to the equivalence relation of quasisymmetry.

Quasisymmetric maps are considered as the 'right' analouge of conformal (or quasiconformal) maps on metric spaces. This might seem strange as quasisymmetry is a global condition rather an infinitesimal one. A homeomorphism $f : (X_1, d_1) \to (X_2, d_2)$ is K-quasiconformal (where $K \geqslant 1$), if

$$\limsup_{r \downarrow 0} \frac{\sup\{d_2(f(x), f(z)) : d_1(x, z) \leqslant r\}}{\inf\{d_2(f(x), f(z)) : d_1(x, z) \geqslant r\}} \leqslant K, \quad \text{for all } x \in X_1.$$

It is easy to verify that quasisymmetry implies quasiconformality (with $K = \eta(1)$) while the converse is not true in general. However many metric spaces satsify a *local to global principle*. Gehring (1960) showed that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal (where $n \geqslant 2$), then f is a quasisymmetry. However this fails when $n = 1$ (Exercise: consider $f(x) = x+e^x$ on $\mathbb{R}$). If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is orientation preserving and 1-quasiconformal, then f is conformal. We refer to [HeKo] for a far reaching generalization of Gehring's theorem.

The following comparison of annuli follows readily from the definition.

Lemma 4.18. *Let the identity map* $\mathrm{Id} : (X, d_1) \to (X, d_2)$ *be an η-quasisymmetry for some distortion function η. Then for all $A > 1, x \in X, r > 0$, there exists $s > 0$ such that, writing B_i for balls in (X, d_i)*

$$B_2(x, s) \subset B_1(x, r) \subset B_1(x, Ar) \subset B_2(x, \eta(A)s). \tag{4.24}$$

Moreover, for all $A > 1, x \in X, r > 0$, there exists $s > 0$ such that

$$B_1(x, r) \subset B_2(x, s) \subset B_2(x, As) \subset B_1(x, A_1 r), \tag{4.25}$$

where $A_1 = 1/\eta^{-1}(A^{-1})$.

The EHI is also a quasisymmetry invariant as shown in the following easy but important lemma due to J. Kigami (unpublished work).

Lemma 4.19. *Let $(X, d_i, \mu, \mathcal{E}, \mathcal{F}^\mu), i = 1, 2$ be two MMD spaces such that d_1 and d_2 are quasisymmetric. If $(X, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies EHI, then so does $(X, d_1, \mu, \mathcal{E}, \mathcal{F}^\mu)$.*

Proof. Let $C_H, A > 1$ be constants corresponding to EHI for $(X, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$ such that $\sup_{B(x,r)} h \leqslant C_H \inf_{B(x,r)} h$ for all functions h non-negative and harmonic in $B(x, Ar)$. Then by (4.25), we have EHI for $(X, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$ with constants $C_H, A_1 > 1$, where A_1 is as given in Lemma 4.18. □

The following example illustrates how time-change along with quasisymmetric change of metrics could be used to obtain heat kernel bounds.

Example 4.20 (Snowball)**.** Snowballs are fractals that are homeomorphic to $\mathbb{S}^2$ and are defined as limits of polyhedral complexes. Their name stems from the fact that snowballs can be viewed as higher dimensional analogues of the Koch snowflake. We recall the definition of one such fractal below.

Let $(\mathcal{S}_0, d_0)$ denote the surface of the unit cube, equipped with the intrinsic metric d_0. In other words, $(\mathcal{S}_0, d_0)$ can be viewed a polyhedral complex obtained by gluing six unit squares similar to the faces of a cube. We replace each *face* in $\mathcal{S}_0$ by 13 squares with edge length $\frac{1}{3}$ as shown in Figure 1 to obtain a polyhedral complex $(\mathcal{S}_1, d_1)$. More generally, we repeat this construction to obtain a geodesic metric space $(\mathcal{S}_n, d_n)$ from $(\mathcal{S}_{n-1}, d_{n-1})$ $(n \geqslant 1)$ by replacing each square of length $3^{-(n-1)}$ with 13 squares of each with edge length 3^{-n} as shown in Figure 1. The polyhedral complex $(\mathcal{S}_n, d_n)$ is obtained by gluing $6 \times (13)^n$ faces, where each face is isometric to a square with edge length 3^{-n}. It is easy to see that the metric spaces $(\mathcal{S}_n, d_n)$ has a Gromov-Hausdorff limit $(\mathcal{S}, d_{\mathcal{S}})$, which is called the snowball.

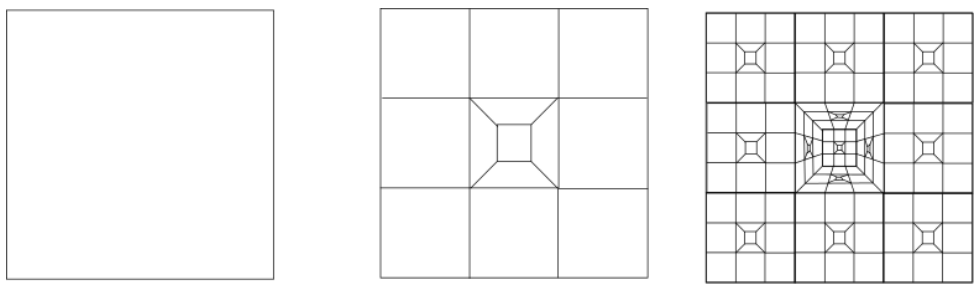

Figure 1: The sequence of graphs viewed from the central square converges to an infinite quadrangulation of the plane with walk dimension $d_w = \log_3(13)$.

We collect some properties of the metric space $(\mathcal{S}, d_{\mathcal{S}})$. Evidently, the spaces $(\mathcal{S}_n, d_n)$ for $n \geqslant 0$, and $(\mathcal{S}, d_{\mathcal{S}})$ are all homeomorphic to $\mathbb{S}^2$. Let $d_{\mathbb{S}^2}$ denote the standard Riemannian metric on $\mathbb{S}^2$, viewed as an embedded surface in $\mathbb{R}^3$. It is known that $(\mathcal{S}, d_{\mathcal{S}})$ and $(\mathbb{S}^2, d_{\mathbb{S}^2})$ are quasisymmetric – see [Mey10].

There exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$, and η-quasisymmetric homeomorphisms $f_n : (\mathcal{S}_n, d_n) \to (\mathbb{S}^2, d_{\mathbb{S}^2})$, $n \geqslant 0$, and $f : (\mathcal{S}, d) \to (\mathbb{S}^2, d_{\mathbb{S}^2})$ satisfying the following properties:

(a) The push-forward metrics $\rho_n : \mathbb{S}^2 \times \mathbb{S}^2 \to [0, \infty)$, $n \geqslant 0$, where $\rho_n(x, y) = d_n(f_n^{-1}(x), f_n^{-1}(y))$, converge uniformly in $\mathbb{S}^2 \times \mathbb{S}^2$ to $\rho : \mathbb{S}^2 \times \mathbb{S}^2 \to [0, \infty)$, where $\rho(x, y) = d_{\mathcal{S}}(f^{-1}(x), f^{-1}(y))$, that is

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{S}^2} |\rho_n(x, y) - \rho(x, y)| = 0.$$

(b) Let $F_i^n, i = 1, 2, \ldots, 6 \cdot (13)^n$ denote the faces of the polyhedral complex $\mathcal{S}_n, n \geqslant 0$. We have

$$\lim_{n\to\infty} \max_{1\leqslant i\leqslant 6(13)^n} \operatorname{diam}(f(F_i^n)) = 0, \tag{4.26}$$

where diam above denotes the diameter in $d_{\mathbb{S}^2}$ metric. For $n \geqslant 1$, and for $i = 1, 2, \ldots, 6 \cdot (13)^n$, there exists $j = 1, \ldots, 6 \cdot (13)^{n-1}$ such that

$$f_n(F_i^n) \subset f_{n-1}(F_j^{n-1}).$$

(c) The maps $f_n : \mathcal{S}_n \to \mathbb{S}^2$ are conformal maps when $\mathcal{S}_n$ and $\mathbb{S}^2$ are viewed as Riemann surfaces (The polyhedral surface $\mathcal{S}_n$ has a canonical Riemann surface structure as explained in [Bea, Section 3.3]. Around a vertex with total angle θ, we use the chart $z \mapsto z^{2\pi/\theta}$ to 'flatten' it.)

If μ_n denotes the uniform probability measure on $\mathcal{S}_n$ such that each face has equal mass, then this defines a canonical diffusion on $\mathcal{S}_n$. By the conformal invariance of Dirichlet energy, the image of the canonical diffusion on $\mathcal{S}_n$ under f_n can be viewed as a time-change of the Brownian motion on $\mathbb{S}^2$ with respect to an admissible measure given by $(f_n)_*(\mu_n)$. The limiting measure μ on $(\mathcal{S}, d_{\mathcal{S}})$ has polynomial volume growth with exponent $\log 13/\log 3$ and μ is such that $f_*(\mu)$ is an admissible measure for $\mathbb{S}^2$. This can be considered as the image of the cannonical diffusion process on (S, d, μ) and is a time change of the Brownian motion on $\mathbb{S}^2$. By Lemma 4.18, we have the elliptic Harnack inequality and matching two sided capacity bounds for the diffusion process on (S, d, μ). This satisfies sub-Gaussian heat kernel bounds with $d_f = d_w = \log 13/\log 3$ by using Theorem 3.27 (characterization using VD, EHI and cap(Ψ)).

We have the easy implication that the elliptic Harnack inequality implies parabolic Harnack inequality. The converse can be easily seem to be false by using a time change with an admissible measure that is not volume doubling as such spaces will fail to satisfy the parabolic Harnack inequality. In order to 'upgrade' from the elliptic to parabolic Harnack inequality it is therefore natural to allow ourselves a time change and/or a quasisymmetric change of metric. This motivates the following definition.

Definition 4.21 (Conformal walk dimension)**.** The *conformal walk dimension* d_{cw} of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined as

$$d_{\mathrm{cw}} := \inf\left\{\beta > 0 \;\middle|\; \begin{array}{l}\text{there exist } \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \text{ and } \theta \in \mathcal{J}(X, d) \text{ such} \\ \text{that } (X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \text{ satisfies PHI}(\beta)\end{array}\right\}, \tag{4.27}$$

where $\inf \varnothing := \infty$ and $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ denotes the time-changed Dirichlet form on $L^2(X, \mu)$.

Since Gaussian space-time scaling is the smallest possible, we have $d_{\mathrm{cw}} \geqslant 2$. The reason for the infimum in the above definition is that it is always possible to increase the walk dimension by a quasisymmetric change of metric. A change of metric from d to d^γ for some $\gamma \in (0, 1)$ results in a change from PHI(β) to PHI(β/γ). On the other hand, it might not be possible to reduce the walk dimension β (for instance, if $\beta = 2$).

We say that a metric space (X, d) is quasi-arc connected if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for any pair of distinct points $x, y \in X$, there is a curve $\gamma_{x,y} : [0, 1] \to X$ such that $\gamma_{x,y}$ is a η-quasisymmetry between $[0, 1]$ and the image $\gamma_{x,y}([0, 1])$.

The following theorem clarifies the relationship between elliptic and parabolic Harnack inequalities and was obtained in [BM18, BCM, KM23].

Theorem 4.22. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space with a quasi-arc connected metric. Then the following are equivalent:*

1. $(X, d, m, \mathcal{E}, \mathcal{F})$ *satisfies the* EHI.

2. $d_{\mathrm{cw}} < \infty$.

3. $d_{\mathrm{cw}} = 2$.

A consequence of the above theorem along with the stability of parabolic Harnack inequality (Theorem 3.27) is that the elliptic Harnack inequality is stable under perturbations. For example, the Laplace-Beltrami operator on a Riemannian manifold satisfies the elliptic Harnack inequality if and only if any uniformly elliptic operator satisfies the elliptic Harnack inequality.

It is trivial that (3) implies (2). The invariance of EHI under quasisymmetric change of metric and time change with respect to an admissible measure, along with the fact that PHI implies EHI shows that (2) implies (1). The result (1) implies (2) was first obtained in [BM18] under slightly restrictive assumptions and generalized in [BCM]. The result (2) implies (3) was obtained in [KM23]. The first evidence towards this implication is the example of Kigami (Example 4.5), where he obtains Gaussian heat kenrel bounds by a time change and quasisymmetric change of metric for the Brownian motion on the Sierpinksi gasket. The infimum in the definition of conformal walk dimension is not always attained. This was shown from higher dimensional Sierpinski gaskets in [KM23] which is in contrast with the two-dimensional Sierpinksi gasket.

We now provide some heuristics behind the proof of (2) implies (3). Assume that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies PHI(γ) for some $\gamma > 2$. Let us try construct a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the corresponding MMD space $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies PHI(2) (Note that this is not always possible as the infimum need not be attained as mentioned above).

Consider the annulus $B_d(x, 2r) \backslash B_d(x, r)$. Discretize X s at a smaller scale s (say $s \leqslant r/100$). This is done by choosing an s-net N (maximal s-separated subset) of X as the set of vertices of a graph and two vertices $x_1, x_2 \in N$ are adjacent if $B_d(x_1, \lambda s) \cap B_d(x_2, \lambda s) \neq \varnothing$, where $\lambda > 2$ is fixed. If such θ, μ exist then they satisfy certain properties. For $v \in N$ such that $B_d(v, s) \cap B_d(x, 2r) \neq \varnothing$, we define the *relative diameter* as

$$\rho(v) := \frac{\mathrm{diam}_\theta(B_d(v, s))}{\mathrm{diam}_\theta(B_d(x, r))}$$

(I) (No shortcuts) $\sum_{v \in \gamma} \rho(v) \gtrsim 1$ for any path γ in the graph joining $B_d(x, r)$ to $B_d(x, 2r)^c$; that is, the first vertex v in the path satisfies $B_d(v, s) \subset B_d(x, r)$ and all the vertices except the last one belongs to $B_d(x, 2r)$ and the last vertex w satisfies $B_d(w, s) \subset B_d(x, 2r)^c$.

This condition follows from triangle inequality and quasisymmetry as

$$\sum_{v \in \gamma} \mathrm{diam}_\theta(B_d(v, s)) \geqslant \mathrm{dist}_\theta(B_d(x, r), B_d(x, 2r)^c) \asymp \mathrm{diam}_\theta(B_d(x, r)).$$

(II) (Mass bound) Let $N(B_d(x, 2r)) := \{v \in N | B_d(x, 2r) \cap B_d(v, \lambda s) \neq \varnothing\}$. By the doubling property of μ, quasisymmetry and the capacity bounds in Theorem 3.27,

we have

$$\mu(B_d(x,r)) \asymp \mu(B_d(x,2r)) \asymp \sum_{v\in N(B_d(x,2r))} \mu(B_d(v,\lambda s))$$
$$\asymp \sum_{v\in N(B_d(x,2r))} \operatorname{diam}_\theta(B_d(v,\lambda s))^2 \operatorname{Cap}(B_d(v,s), B_d(v,2s)^c)$$

Since $\mu(B_d(x,r)) \asymp \operatorname{diam}_\theta(B_d(x,r))^2 \operatorname{Cap}(B_d(x,r), B_d(x,2r)^c)$, the above estimate implies

$$\sum_{v\in N(B_d(x,2r))} \rho(v)^2 \operatorname{Cap}(B_d(v,r), B_d(v,2r)^c) \asymp \operatorname{Cap}(B_d(x,r), B_d(x,2r)^c).$$

We will sketch a construction of ρ satisfying the no shortcuts condition and the mass bound above. We denote by $B_v := B_d(v,s), 2B_v = B_d(v,2s)$ for any $v \in N$. Consider a function $u \in C_c(X) \cap \mathcal{F}$ that is almost minimizer for the capacity across the annulus $\operatorname{Cap}(B_d(x,r), B_d(x,2r)^c) \asymp \mathcal{E}(u,u)$. Define

$$\widetilde{\rho}(v) := \sum_{w\sim v} \left| \fint_{B_v} u\, dm - \fint_{B_w} u\, dm \right|.$$

By Poincaré inequality PI(γ), we have

$$|\widetilde{\rho}(v)|^2 \lesssim \frac{s^\gamma}{m(B_v)} \int_{A\cdot B_v} d\Gamma(u,u) \asymp \frac{1}{\operatorname{Cap}(B_v,(2B_v)^c)} \int_{A\cdot B_v} d\Gamma(u,u), \quad \text{for all } v \in N(B_d(x,2r)).$$

Summing over v and using bounded overlap of $(A\cdot B_v)$, we obtain

$$\sum_{v\in N(B_d(x,2r))} \widetilde{\rho}(v)^2 \operatorname{Cap}(B_v,(2B_v)^c) \lesssim \sum_{v\in N(B_d(x,2r))} \int_{A\cdot B_v} d\Gamma(u,u) \asymp \mathcal{E}(u,u) \operatorname{Cap}(B_d(v,r), B_d(v,2r)^c).$$

This is the mass (upper) bound.

The no shortcut condition follows from triangle inequality as the first ball in the path has average one and the last ball has average 0. Since increasing $\widetilde{\rho}$ preserves the no shortcut condition, it is always possible to obtain matching mass lower bound by increasing $\widetilde{\rho}$ if needed.

The main message here is that the discrete gradient of energy minimizers across annuli can be used to deform the metric on a sequence of scales. The proof of (2) implies (3) relies heavily on hyperbolic spaces as a metric space viewed simultaneously at a sequence of scales has the structure of a Gromov hyperbolic space (see [GH90] for an introduction to Gromov hyperbolic spaces). More precisely, a result of Bourdon and Pajot shows that every (nice) metric space can be viewed as the boundary of a hyperbolic space [BP]. A result of Bonk and Schramm [BoSc] implies that metrics in the conformal gauge of the boundary of a hyperbolic space can be obtained by a bi-Lipschitz change of this Gromov hyperbolic space. We adapt arguments of Carrasco [Car13] where such techniques were developed for a different notion of conformal dimension. The question of whether or not a given MMD space attains the infimum in the definition of conformal walk dimension remains a challenging problem in many examples (see [KM23]).

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