A Study of the Dirac Monopole

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1 Introduction

This project is a detailed study of the Dirac monopole from two different perspectives, roughly the physical and mathematical perspectives. The Dirac monopole is a hypothesized magnetic monopole, the magnetic equivalent of the electric point charge, as defined and studied by Paul Alain Maurice Dirac in [2], in 1931. As of yet, no magnetic monopole has been discovered in nature, so it remains a purely theoretical object. Nonetheless, its theory is important both in physics and mathematics. The implications of the existence of a *single* magnetic monopole are aesthetically pleasing, to say the least. This would make Maxwell's equations completely symmetric, which is quite elegant, but this is not all! In his 1931 paper, Dirac showed that the existence of a magnetic monopole would force electric charge to be quantized. This is the physical perspective, which we take up in §2. Taking the mathematical point of view, we note that the Dirac monopole is just an example of a more general notion. It is arguably the simplest nontrivial example of a connection on a principal fibre bundle. We will describe these objects and show exactly how the monopole fits the description in §3. The study of the Dirac monopole from this dual perspective is a stepping stone towards a deeper understanding of both physics and geometry.

2 Magnetic Monopole and Dirac Quantization

In this section, we are working in Euclidean space $\mathbb{R}^3$, endowed with all of its familiar structure: a standard basis and inner product. Elements of $\mathbb{R}^3$ will be written in bold font. $T\mathbb{R}^3$ denotes the tangent bundle of $\mathbb{R}^3$ and $T^*\mathbb{R}^3$, the cotangent bundle. This last vector space is also denoted by $\Omega^1(\mathbb{R}^3)$ when we are thinking of cotangent vectors as differential 1-forms, i.e. in the context of exterior algebra, with a wedge product and exterior derivative; these functors are described more generally for manifolds in §3.1. There is a natural isomorphism between $\mathbb{R}^3 \times \mathbb{R}^3$ and $T\mathbb{R}^3$, and there is a natural isomorphism between $T\mathbb{R}^3$ and $\Omega^1(\mathbb{R}^3)$. These are known as the "musical isomorphisms". Whereas elements of $\mathbb{R}^3$ will be denoted by letters written in bold font, their musically isomorphic copies will be denoted by the same letters without the bold font.

A magnetic monopole generates (and in fact is described by) a magnetic field $\mathbf{B}$ on $\mathbb{R}^3 \setminus \{0\}$ given by

$$\mathbf{B} = \frac{g}{\rho^2}\mathbf{e}_\rho,$$

where g is a constant called the strength of the monopole, ρ is the distance from the origin and $\mathbf{e}_\rho$ is a unit vector pointing away from the origin. Using the traditional Cartesian coordinates,

$$\mathbf{B} = \frac{g}{\rho^3}(x, y, z).$$

Under the natural isomorphism, this vector maps to the 1-form

$$\overline{B} = \frac{g}{\rho^3}\left(x\ dx + y\ dy + z\ dz\right).$$

Now $\mathbf{B}$ is described as a pseudovector in physics. This means that $\mathbf{B}$ changes orientation when $\mathbb{R}^3$ is acted on by an element of $O(3) \setminus SO(3)$, and this happens because the magnetic field is obtained from a cross product. We take this to mean that $\mathbf{B}$ should really be a 2-form. On $\mathbb{R}^n$, the isomorphism between k-forms and $(n-k)$-forms is given by the Hodge star operator. If $n = 3$, the isomorphism is

$$\star dx = dy \wedge dz, \qquad \star dy = dz \wedge dx, \qquad \star dz = dx \wedge dy$$

We write then $B = \star\overline{B}$, and record the appropriate expression for the magnetic field of the monopole:

$$B = \frac{g}{\rho^3}\left(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy\right). \tag{1}$$

Before pursuing further the study of the magnetic monopole, let us quickly recall Maxwell's equations and rewrite them using differential forms.

2.1 Electromagnetism using Differential Forms

Maxwell's equations are classically written as a collection of 4 facts about the curl and divergence of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. These are

$$\nabla \times \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{E} = \rho$$

$$\nabla \times \mathbf{B} = \mathbf{J} + \tfrac{\partial \mathbf{E}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0$$

where ρ is charge per unit volume and $\mathbf{J}$ is current per unit area. Since $\mathbf{E}$ is integrated over paths, it can be seen as a 1-form E and, similarly, $\mathbf{B}$ can be replaced by a 2-form B. Then dE is a 2-form corresponding to $\nabla \times \mathbf{E}$ and dB is a 3-form corresponding to $\nabla \cdot \mathbf{B}$. In order to write the divergence of $\mathbf{E}$ and the curl of $\mathbf{B}$ using the differential operator d, we must make use of the Hodge star $\star$. Then, Maxwell's equations take the form

$$dE = -\tfrac{\partial B}{\partial t} \qquad d \star E = \rho$$

$$d \star B = J + \tfrac{\partial \star E}{\partial t} \qquad dB = 0$$

In the above equations, J, the current density, is a 2-form.

This process of synthesis can be taken one step further by viewing the electric and magnetic fields as 2-forms on Minkowski spacetime and combining them into one single 2-form. If $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ is a vector potential for $\mathbf{B}$, i.e. if $\mathbf{B} = \nabla \times \mathbf{A}$, and if ϕ is a potential for $\mathbf{E}$, we may simplify Maxwell's equations further by introducing the Faraday tensor: first, define the 1-form $A = \phi dt + \mathbf{A}_\mu dx^\mu$ (note that we are using the Einstein summation convention here, so $\mathbf{A}_\mu dx^\mu = \mathbf{A}_1 dx^1 + \mathbf{A}_2 dx^2 + \mathbf{A}_3 dx^3$). We agree that $dx^0 = dt$ and $A_0 = \phi$, and we define

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA.$$

(Again, the Einstein summation convention is used above). Componentwise, with the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

This last equation is perhaps the one most commonly encountered in physics. It is immediate from it that the field strngth F is invariant under a gauge transformation $A \mapsto A + df$, for any $f \in C^\infty(\mathbb{R}^4)$. In matrix form, the Faraday tensor is

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

and Maxwell's equations become

$$dF = 0$$
$$d \star F = j,$$

where the current j is now a 3-form.

2.2 Topology of the Monopole

We use de Rham cohomology to discuss the topology of the monopole. Given a manifold M, its k^{th} de Rham cohomology group is denoted $H^k_{dR}(M)$. The main facts that we need are that the cohomology groups are in fact finite-dimensional vector spaces, hence determined by their dimension, and that de Rham cohomology is homotopy-invariant, i.e. that manifolds of the same homotopy type have the same cohomology groups. It will turn out that all of the manifolds we consider here have the homotopy type of a sphere. We record then these cohomology groups:

$$H^i_{dR}(S^n) \simeq \begin{cases} \mathbb{R} & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Let us now return to the monopole, with magnetic field given by (1). It is easy to see that B satisfies Maxwell's equations:

$$dB = g\left(\frac{\partial}{\partial x}\left(\frac{x}{\rho^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\rho^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\rho^3}\right)\right) dx \wedge dy \wedge dz = 0$$

However, such a magnetic field cannot be the curl of a smooth vector field $\mathbf{A}$ defined on $\mathbb{R}^3 \setminus \{0\}$. In terms of differential forms, there is no smooth globally defined 1-form A on $\mathbb{R}^3 \setminus \{0\}$ such that $B = dA$. To see this, proceed by contradiction: integrate B over the surface of a ball of radius r centred at the origin. On the one hand, since B is radially symmetric,

$$\int_{\partial B_r(0)} B = \frac{g}{r^2} \cdot 4\pi r^2 = 4\pi g.$$

On the other hand, applying Stokes' theorem,

$$\int_{\partial B_r(0)} B = \int_{\partial B_r(0)} dA = \int_{\varnothing} A = 0.$$

In other words, B is a closed and non-exact 2-form, i.e. an element of the vector space $H^2_{dR}(\mathbb{R}^3 \setminus \{0\})$. What is this vector space? $\mathbb{R}^3 \setminus \{0\}$ has the homotopy type of S^2, seen by the retraction along radial lines emanating from the origin. Hence $H^2_{dR}(\mathbb{R}^3 \setminus \{0\}) \simeq \mathbb{R}$ and B represents a generator for this vector space. It is thus due to the topology of $\mathbb{R}^3 \setminus \{0\}$ that B is not exact. The field $\mathbf{B}$ is singular at $\rho = 0$, which makes it topologically interesting. It is possible to write B as dA locally, for a 1-form A defined on a subspace of $\mathbb{R}^3$ which is homotopy equivalent to a point. For example, on the sets $U_+ = \mathbb{R}^3 \setminus \{(0,0,z) \in \mathbb{R}^3 : z \leq 0\}$ and $U_- = \mathbb{R}^3 \setminus \{(0,0,z) \in \mathbb{R}^3 : z \geq 0\}$, we may define corresponding 1-forms

$$\begin{aligned} A_+ &= \frac{g}{\rho(z+\rho)}(-y\ dx + x\ dy) \\ A_- &= \frac{g}{\rho(z-\rho)}(-y\ dx + x\ dy) \end{aligned}$$

The above 1-forms are well defined and smooth on their respective domains, because the singularities of the forms on their domains are removable: $1/(\rho(z+\rho)) = 1/2z^2$ if $x = y = 0$ and $z > 0$ and $1/(\rho(z-\rho)) = 1/2z^2$ if $x = y = 0$ and $z < 0$. It is easy to compute the exterior derivatives of the above forms to arrive at

$$\begin{aligned} dA_+ &= g\left(\frac{2\rho(z+\rho) - (z\frac{y^2}{\rho} + 2y^2) - (z\frac{x^2}{\rho} + 2x^2)}{\rho^2(z+\rho)^2}\right) dx \wedge dy \\ &\quad + g\left(\frac{y(\rho + \frac{z^2}{\rho} + 2z)}{\rho^2(z+\rho)^2}\right) dy \wedge dz + g\left(\frac{x(\rho + \frac{z^2}{\rho} + 2z)}{\rho^2(z+\rho)^2}\right) dz \wedge dx \\ &= g\left(\frac{z}{\rho^3} dx \wedge dy + \frac{x}{\rho^3} dy \wedge dz + \frac{y}{\rho^3} dz \wedge dx\right) \\ &= B \end{aligned}$$

Similarly,

$$dA_- = g\left(\frac{z}{\rho^3} dx \wedge dy + \frac{x}{\rho^3} dy \wedge dz + \frac{y}{\rho^3} dz \wedge dx\right) = B$$

On the overlap, we have

$$A_+ - A_- = \frac{g}{\rho}\left(\frac{-2\rho}{z^2 - \rho^2}\right)(-y\ dx + x\ dy) = 2g\left(\frac{-y\ dx + x\ dy}{x^2 + y^2}\right).$$

Clearly, $A_+ - A_-$ is closed. Is it exact? We can proceed as above, and integrate these forms over a path, say the unit circle in the xy-plane. Call it S^1. This circle is the boundary of two hemispheres of $S^2 \subset \mathbb{R}^3$, which we denote by H^+ and H^-. $H^+ \subset U_+$, so A_+ is defined everywhere on H^+. Similarly, A_- is globally defined on H^-. Suppose now that $A_+ - A_- = df$, for some $f : U_+ \cap U_- \to \mathbb{R}$. Applying Stokes' theorem, we have

$$\int_{S^1} A_+ - A_- = \int_{S^1} df = \int_{\partial S^1} f = 0.$$

But we must also have

$$\int_{S^1} A_+ - A_- = \int_{\partial H^+} A_+ - \int_{\partial H^-} A_- = \int_{H^+} dA_+ + \int_{H^-} dA_- = \int_{S^2} B = 4\pi g.$$

Hence there is no f such that $A_+ - A_- = df$, and the 1-form $A_+ - A_-$ is a generator of $H^1_{dR}(U_+ \cap U_-) \simeq H^1_{dR}(S^1) = \mathbb{R}$. It is, however, locally exact. If we define $\theta = \arctan\left(\frac{y}{x}\right)$, then

$$A_+ - A_- = 2g\ d\theta.$$

The above formula is well defined for $\theta \neq 0$, i.e. for $y \neq 0$.

It is important to note at this point that the forms $A_\pm$ are independent of ρ. In terms of the polar coordinates (ρ, φ, θ), where ρ and θ are as above and $\varphi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$,

$$\begin{aligned} A_+ &= g(1 - \cos\varphi)d\theta \\ A_- &= -g(1 + \cos\varphi)d\theta. \end{aligned}$$

It follows that we can consider $A_\pm$ as 1-forms on S^2. The exterior derivative of these 1-forms is

$$dA_+ = dA_- = g \sin\varphi\ d\varphi \wedge d\theta.$$

Note that this is simply the area form on S^2 (with a factor of g). This is thus the expression for the magnetic field of a monopole restricted to S^2.

Remark. It may seem bothersome that the above discussion of the magnetic monopole uses the specific vector potentials A_+ and A_-. These are ansätze and, therefore seem to not describe the monopole with appropriate generality. Are there other interesting vector potentials which we are neglecting by working with A_+ and A_-? We remark that it is indeed a general description of the magnetic field of a monopole and its vector potentials and that any other choice of vector potentials would yield the same results. Let $a, b \subset \mathbb{R}^3$ be Dirac strings, i.e. nonintersecting (and not selfintersecting) paths in $\mathbb{R}^3$ starting at 0 and extending to infinity, and let $U_a = \mathbb{R}^3 \setminus a$, $U_b = \mathbb{R}^3 \setminus b$. On U_a and U_b there exist 1-forms A_a and A_b such that $dA_a = dA_b = B$. This follows from topological considerations. The de Rham cohomology is homotopy invariant, in the sense that manifolds with the same homotopy type have the same de Rham cohomology. We have

$$H^2_{dR}(U_a) = H^2_{dR}(point) = 0,$$

so, since B is closed, its restriction to U_a or U_b must also be exact. This gives the existence of A_a and A_b. Now, on the overlap $U_a \cap U_b$, $d(A_a - A_b) = 0$. The overlap is homotopy equivalent to S^1, so $H^1_{dR}(U_a \cap U_b) = \mathbb{R}$, a 1-dimensional vector space. Therefore the 1-form $A_a - A_b$ differs from $A_+ - A_-$ by a scalar multiple. This shows that there aren't many "different" vector potentials satisfying $dA = B$.

We further remark that we do not lose generality by considering 1-forms locally defined on S^2. If $U \subset S^2$ and $A \in \Omega^1(U)$ is such that $dA = \iota^* B$, where $\iota : S^2 \hookrightarrow \mathbb{R}^3$ is the inclusion map, then we might as well extend A on a larger subspace of $\mathbb{R}^3 \setminus 0$. For example, if $\pi : \mathbb{R}^3 - 0 \to S^2$ is the retraction given by $\pi(x) = x/\|x\|$, then we may extend A to the space $\pi^{-1}(U)$.

2.3 Quantization

To finalize our study of the monopole, let us derive the Dirac quantization condition. This was originally done by Dirac in [2], but we follow the work of Wu & Yang [7], who do it in more modern terms, i.e. using gauge theory. These two approaches are equivalent [1]. The paper by Wu & Yang contains a very readable account of gauge theory, the Aharanov-Bohm effect and the Dirac monopole.

To understand the derivation of the quantization condition in the paper by Wu & Yang, it is only necessary to understand the concept of gauge transformations. Maxwell's equations are invariant under the transformation $A \mapsto A' = A + d\alpha$, where A is the electromagnetic 1-form from equation (2) above and α is any 0-form. This is a gauge transformation and a field of charge e transforms accordingly:

$$\psi \mapsto \psi' = e^{-ie\alpha}\psi$$

We turn back to the 1-forms A_+ and A_- defined above. These new forms are related by a gauge transformation with gauge $S = e^{-2ige\theta}$:

$$A_a = A_b + 2gd\theta = A_- + \frac{i}{e}S^{-1}dS.$$

Such a gauge transformation is allowed iff S is single-valued [7], i.e. iff $2ge \in \mathbb{Z}$. Thus we arrive at the Dirac quantization condition.

3 Geometry

The monopole magnetic field is defined on $\mathbb{R}^3 \setminus \{0\}$, or, if we want to work in spacetime, it is defined on $\mathbb{R}^4$ with the worldline of the monopole removed. This space is homotopy-equivalent to S^2. This motivation for studying circle bundles over S^2 is to be found in Trautman [6]. To the author's knowledge, this is the first paper in which a general construction of solutions to the Yang-Mills equations is given in geometric terms. To begin understanding the construction, we develop these "geometric terms". $S^1 \subset \mathbb{C}$ is diffeomorphic to the Lie group $U(1)$. A manifold with the additional structure that it is locally homeomorphic (or diffeomorphic) to a Cartesian product with $U(1)$ is an example of a principal fibre bundle. We now turn to the theory of principal fibre bundles and connections on them.

3.1 Preliminaries

We need first to fix some notation.

3.1.1 Manifolds

In this section, we work with smooth manifolds. The reference is Naber's textbook [3]. As with $\mathbb{R}^n$, T_pM denotes the tangent space of a manifold M at a point $p \in M$, TM denotes the tangent bundle of M and $\Omega^k(M)$ denotes the $C^\infty(M)$-module of k-forms. We point out that T_pM is both the vector space of derivations at p, as well as the collection of differentiable curves $\gamma : \mathbb{R} \to M$ satisfying $\gamma(0) = p$ (to be precise, T_pM is the set of *equivalence classes* of curves passing through p at $t = 0$, where $\gamma \sim \sigma$ if $\gamma'(0) = \sigma'(0)$). We will use both the notion of a linear map and that of a differentiable curve when dealing with tangent vectors on a manifold. $\Gamma(TM)$ is the collection of smooth sections of TM, i.e. it is the set of vector fields on M. If $X \in \Gamma(TM)$ is a vector field, we will use the notation $X|_p$ or X_p to denote the element $X(p) \in T_pM$.

An example of a smooth manifold, which is the only example we need here, is S^n, the n-sphere. S^n is embedded in $\mathbb{R}^{n+1}$. It is standard to describe S^n using polar coordinates in the following fashion: if $(x^1, \dots, x^{n+1}) \in S^n \subset \mathbb{R}^{n+1}$, then we put

$$\begin{aligned}
x^1 &= \cos(\varphi_1)\\
x^2 &= \cos(\varphi_2)\sin(\varphi_1)\\
&\dots\\
x^n &= \cos(\varphi_n)\sin(\varphi_{n-1})\dots\sin(\varphi_1)\\
x^{n+1} &= \sin(\varphi_n)\dots\sin(\varphi_1),
\end{aligned}$$

where $\varphi_1 \in [0, 2\pi)$ and $\varphi_j \in (0, \pi)$, for $j \geq 2$.

Another description of S^n is via stereographic projection. Let $x \in S^n \hookrightarrow \mathbb{R}^{n+1}$ be a point which is not $(0, \dots, 0, 1)$ (we call this the north pole). Map it to $\mathbb{R}^n$ via

$$x = (x^1, \dots, x^{n+1}) \mapsto \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}} \right).$$

The inverse map takes a point $x = (x^1, \dots x^n) \in R^n$ and maps it to $S^n \hookrightarrow \mathbb{R}^{n+1}$ via

$$x \mapsto \frac{1}{1 + \|x\|^2} \left(2x^1, \dots, 2x^n, \|x\|^2 - 1 \right).$$

It is easy to check that the above maps are inverses of each other and that they are diffeomorphisms between $\mathbb{R}^n$ and S^n with the north pole removed. We may also use the stereographic projection to identify S^n with the one-point compactification of $\mathbb{R}^n$. We will require this identification in the case $n = 2$.

3.1.2 Lie Groups

A Lie group G is a group and a smooth manifold such that the group operations are smooth with respect to the differentiable structure. Common examples include the general linear groups (denoted by $GL_n(F)$ for $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), as well as the orthogonal groups and special orthogonal groups. In this paper, Lie groups will be matrix Lie groups, i.e. subgroups of $GL_n(F)$. If $g \in G$, we let L_g be the map $x \mapsto gx$ and R_g, the map $x \mapsto xg$. L_g and R_g are called, respectively, left and right translation by g. To every Lie group$G$, we can associate an algebra called its Lie algebra. This is the vector space of left-invariant vector fields, together with the Lie bracket operation, and it is denoted by $\mathfrak{g}$. More precisely, $X \in \mathfrak{g}$ is a vector field such that, for all $g \in G$,

$$X_g = (L_g)_* x_e.$$

3.1.3 Bundles

Definition 1. A Principal Fibre Bundle (also referred to as a PFB) is a quadruple (P, B, π, G), where P and B are smooth manifolds, $\pi : P \to B$ is a smooth surjective map, G is a Lie group acting smoothly on P from the right, and the following conditions are satisfied:

1. The action preserves fibres: for all $p \in P$ and $g \in G$,

   $$\pi(p \cdot g) = \pi(p).$$

   This means that for every $x \in B$, the fibre $\pi^{-1}(\{x\})$ contains the orbit of x under G.
2. The bundle is locally trivial: for every $x \in B$, there is a neighbourhood V of x and a diffeomorphism $\Psi : \pi^{-1}(V) \to V \times G$ of the form $\Psi(p) = (\pi(p), \psi(p))$. Such a pair (V, Ψ) is called a (local) trivialization of the principal bundle at x.
3. Furthermore, the group action is compatible with the bundle structure: if $x \in B$, (V, Ψ) is a trivialization at x, then, with the same notation as above,

   $$\psi(p \cdot g) = \psi(p) \cdot g$$

   for all $g \in G$.

If the rest is clear from context, we denote a principal fibre bundle (P, B, π, G) by $G \to P \to B$. (The first arrow in this notation denotes the action of G on P, whereas the second is the projection π).

Condition 3. in the above definition implies that we may, in a sense, identify the fibre of a PFB with the group G:

Lemma 1. Let (P, B, π, G) be a PFB and let $p \in P$. The fibre containing p is the orbit of p under the action of G. This is seen by working with trivializations and using property (3) of the definition.

The main example of a prinicipal fibre bundle is the Hopf bundle $S^1 \to S^3 \to S^2$, which will be described in detail below. It is clear that S^3 is not homeomorphic to $S^2 \times S^1$. One way to prove this cleanly is by noticing that the fundamental group of S^3 is trivial, whereas the fundamental group of $S^2 \times S^1$ is isomorphic to $\mathbb{Z}$. To try to capture the geometry of this bundle, we need an object which will describe the manner in which the fibres S^1 are glued together to form S^3, very loosely speaking. This object is the connection on a principal fibre bundle, and it is defined below, using also the notion of fundamental vector field.

Definition 2. Let G be a Lie group with identity e. Denote its Lie algebra by $\mathfrak{g}$. Recall that (a) $\mathfrak{g}$ denotes the set of left-invariant vector fields on G and (b) the Lie algebra of a Lie group is isomorphic to the tangent space to G at the identity. $L_g : G \to G$ is the map $x \mapsto gx$. (a) says that

$$\mathfrak{g} = \{X \in \Gamma(TG) : \forall g \in G,\ X|_g = (L_g)_* X|_e\}.$$

Incidentally, this also provides us with the idea for an isomorphism between T_eG and $\mathfrak{g}$ (recall that L_g is an isomorphsim). We write this as $\mathfrak{g} \simeq T_eG$. Suppose P is a smooth manifold and $\sigma : P \times G \to P$ is a smooth right-action on P given by $\sigma(p, g) = p{\cdot}g$. Define $\sigma_p : G \to P$ by $\sigma_p(g) = \sigma(p, g) = p{\cdot}g$. Given $A \in \mathfrak{g}$, define the fundamental vector field corresponding to A to be $A^{\#} \in \Gamma(TP)$ given by $A^{\#}_p = (\sigma_p)_{*e}(A_e)$, where $A_e = A(e)$. We define the map $Ad_g : G \to G$ by $h \mapsto ghg^{-1}$ and the map $ad_g : \mathfrak{g} \to \mathfrak{g}$ by $ad_g(A) = (Ad_g)_{*e}(A_e)$. In our case, every Lie group is a matrix Lie group, i.e. $G \subset GL_n(F)$ for $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Remark. We note that the notation used for the adjoint representation is non-standard. It is however the one which appears in [3].

Definition 3. Let (P, B, π, G) be a principal bundle. Let $\sigma : P \times G \to P$ denote the (right) action and let $\sigma_p : G \to P$ be as in the above definition amd $\sigma_g : P \to P$ be defined by $\sigma_g(p) = \sigma(p, g)$. A connection on a principal fibre bundle, is a $\mathfrak{g}$-valued 1-form ω on P such that

$$\begin{aligned} &(\sigma_g)^*\omega = ad_{g^{-1}} \circ \omega \\ &\omega(A^{\#}) = A \end{aligned} \tag{3}$$

Remark. (1) In practice, we use the identification of $\mathfrak{g} \simeq T_eG$, so ω_p is a map from T_pP into T_eG and the second equation becomes $\omega(A^{\#}) = A_e$.

Remark. (2) If the Lie group G is a matrix Lie group, then the map ad_g is conjugation by the element g. Explicitly, if G is a subgroup of $GL_n(F)$, $\mathfrak{g}$ is its Lie algebra, $g \in G$ and $X \in \mathfrak{g}$, then

$$ad_g(X) = gXg^{-1}.$$

This can be seen by thinking of tangent vectors as differentiable curves, so that

$$ad_g(X) = \frac{d}{dt}\left(Ad_g(\exp(tX_e))\right)\bigg|_{t=0},$$

where exp is the usual exponential map of a Lie algebra into its Lie group. We have $Ad_g(\exp(tX_e)) = \exp(tgX_eg^{-1})$. Taking the derivative yields the result.

3.2 The Hopf Bundle $S^1 \to S^3 \to S^2$

We will see now that the Dirac monopole is just an instance of the more general idea of a connection on a principal fibre bundle.

Consider the PFB $(S^3, S^2, \pi, U(1))$, where $S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}$, $U(1) = \{g \in \mathbb{C} : |g| = 1\}$, and the action $\sigma : S^3 \times U(1) \to S^3$ is given as follows:

$$\sigma((z^1, z^2), g) = (z^1 g, z^2 g),$$

for all $(z^1, z^2) \in S^3$ and $g \in U(1)$.

Definition 4. This principal fibre bundle is called a Hopf bundle.

The Dirac monopole is an instance of a connection in the following (almost) precise sense:

Theorem 1. Let U_N, U_S denote the sphere S^2 with the north and south pole removed, respectively (the north pole is the point $(0,0,1)$ and the south pole is $(0,0,-1)$). There is a connection ω on the PFB $S^1 \to S^3 \xrightarrow{\pi} S^2$, and there are sections $s_N : U_N \to S^3$, $s_S : U_S \to R^3$, which are in some sense natural, and such that

$$s_N^*\omega = A_+ \qquad s_S^*\omega = A_-,$$

where $A_\pm$ are the 1-forms on S^2 which define the Dirac monopole, and are defined in section 3.2. The sense in which the sections s_N, s_S are natural will be made precise in the following discussion of the bundle.

This is the main theorem of this project. We will give the proof in the following subsections.

3.2.1 The Hopf Fibration $S^3 \xrightarrow{\pi} S^2$

The space S^2 is obtained from S^3 by taking the quotient by the action of $U(1)$, which yields the complex projective space $\mathbb{CP}^1$ (this is isomorphic to S^2), and identifying $\mathbb{CP}^1$ with S^2. We will give this map explicitly, in three different guises, but first, we establish the isomorphism between $S^2, \mathbb{C} \cup \{\infty\}$ and $\mathbb{CP}^1$.

Lemma 2. $S^2 \simeq \mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$.

Proof. First, let $\sigma : \mathbb{C} \cup \{\infty\} \to \mathbb{CP}^1$ be given by

$$\sigma(z) = \begin{cases} [z,1] & \text{if } z \neq \infty \\ [1,0] & \text{if } z = \infty \end{cases}.$$

Note: $[z,1] = [z(|z|^2+1)^{-1/2}, (|z|^2+1)^{-1/2}]$. This definition of σ is preferable if we want the coset to be represented by an element of S^3. σ^{-1} is obviously

$$\sigma^{-1}([z^1, z^2] = \begin{cases} \frac{z^1}{z^2} & \text{if } z^2 \neq 0 \\ \infty & \text{if } z^2 = 0 \end{cases}$$

σ^{-1} is well defined because, if $[z^1, z^2] = [w^1, w^2]$, then there is some $\lambda \in \mathbb{C}$ such that $z^1 = w^1\lambda$ and $z^2 = w^2\lambda$, hence $z^1/z^2 = w^2/w^2$.

Thus $\mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$. The isomorphism $S^2 \xrightarrow{\rho} \mathbb{C} \cup \{\infty\}$ is obtained by extending the stereographic projection of S^2 (with a point removed) onto $\mathbb{R}^2 \simeq \mathbb{C}$ to a map of the whole sphere onto $\mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$. This is (see section (2.1)):

$$\rho(x^1, x^2, x^3) = \begin{cases} \frac{1}{1-x^3}(x^1 + ix^2) & \text{if } x^3 \neq 1 \\ \infty & \text{if } x^3 = 1 \end{cases}$$

For $z \in \mathbb{C} \cup \{\infty\}$, the inverse is

$$\rho^{-1}(z) = \begin{cases} \frac{1}{1+|z|^2}(z + \overline{z}, i(\overline{z} - z), |z|^2 - 1) & \text{if } z \neq \infty \\ (0,0,1) \text{ if } z = \infty \end{cases}$$

Finally, let's compose these maps to obtain the explicit isomorphism $\tau : S^2 \to \mathbb{CP}^1$:

$$\tau(x^1, x^2, x^3) = (\sigma \circ \rho)(x^1, x^2, x^3) = \begin{cases} \left[\frac{x^1 + ix^2}{\sqrt{2(1-x^3)}}, \frac{1-x^3}{\sqrt{2(1-x^3)}} \right] & \text{if } x^3 \neq 1 \\ [1,0] & \text{if } x^3 = 1 \end{cases}$$

The inverse is $\tau^{-1} = \rho^{-1} \circ \sigma^{-1}$:

$$\tau^{-1}([z^1, z^2]) = \begin{cases} \frac{1}{|z^1|^2 + |z^2|^2}(\overline{z}^1 z^2 + z^1 \overline{z}^2, i(\overline{z}^1 z^2 - z^1 \overline{z}^2), |z^1|^2 - |z^2|^2) & \text{if } z^2 \neq 0 \\ (0,0,1) & \text{if } z^2 = 0 \end{cases}$$

τ^{-1} is well-defined because σ^{-1} is. $\square$

Note that we can also see τ^{-1} as a map from C^2 into S^2. In this case, $\tau^{-1}|_{S^3} = \pi$, where π is the Hopf fibration. We look at this map more closely now.

First, the quotient of S^3 by the action of $U(1)$ is given by $(z^1, z^2) \mapsto [z^1, z^2] \in \mathbb{CP}^1$. This is easy to see: on the one hand, for all $z^1, z^2 \in \mathbb{C}$ and $g \in U(1)$, $(z^1, z^2) \cdot g \mapsto [z^1 g, z^2 g] = [z^1, z^2]$, i.e. elements of the same orbit under the action of $U(1)$ map to the same point in $\mathbb{CP}^1$; on the other hand, if $[z^1, z^2] = [w^1, w^2] \in \mathbb{CP}^1$ and $(z^1, z^2), (w^1, w^2) \in S^3$, then there must be an element $\lambda \in \mathbb{C}$ such that $(w^1, w^2) = (z^1, z^2) \cdot g$. It is clear that λ must belong to $U(1)$ so that (w^1, w^2) is in the orbit of (z^1, z^2). Hence $S^3/\!\sim$ is isomorphic to $\mathbb{CP}^1$.

Second, we apply the isomorphism $\mathbb{CP}^1 \to S^2$. Composing these maps, we obtain the Hopf map $\pi : S^3 \to S^2$.

1. If $S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}$, then
$$\pi(z^1, z^2) = (z^1 \overline{z}^2 + \overline{z}^1 z^2, -iz^1\overline{z}^2 + i\overline{z}^1 z^2, |z^1|^2 - |z^2|^2)$$
2. If $S^3 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$, then
$$\pi(x^1, x^2, x^3, x^4) = (2x^1x^3 + 2x^2x^4, 2x^2x^3 - 2x^1x^4, (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)$$
3. Finally, we may express S^3 using spherical coordinates: let $z^1, z^2 \in \mathbb{C}$ be such that $|z^1|^2 + |z^2|^2 = 1$. If we write $z^1 = r_1 e^{i\xi_1}$ and $z^2 = r_2 e^{i\xi_2}$, then $r_1^2 + r_2^2 = 1$, so there is an angle ϕ such that $r_1 = \cos\frac{\phi}{2}$ and $r_2 = \sin\frac{\phi}{2}$. For cosmetic reasons, we use the variables $\chi = \xi_1 + \xi_2$ and $\theta = \xi_1 - \xi_2$. Then $S^3 = \{(\cos\frac{\phi}{2} e^{\frac{i}{2}(\chi+\theta)}, \sin\frac{\phi}{2} e^{\frac{i}{2}(\chi-\theta)}) \in \mathbb{C}^2 : \phi \in [0, \pi], \theta, \chi \in \mathbb{R}\}$, and
$$\pi(\phi, \theta, \chi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

This last expression of the Hopf fibration is especially revealing: given spherical coordinates (ϕ, θ, χ) on S^3, we may consider the first two as describing the base manifold, that is S^2, and the third as giving the angle in a circle which is a fibre of π.

3.2.2 An Explicit Connection

Let us now find a very concrete example of a connection on the Hopf bundle. To do this, let ω' be a $\mathcal{U}(1)$-valued 1-form on $\mathbb{R}^4$, so $\omega' = i(\omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3 + \omega_4 dx^4)$, and consider the form $\omega = \iota^*\omega'$, where $\iota : S^3 \hookrightarrow \mathbb{R}^3$ is the inclusion map. Let $p \in S^3$ (so $p = (p^1, p^2) \in \mathbb{C}^2$ is such that $|p^1|^2 + |p^2|^2 = 1$), and let $X_p \in T_pS^3$. Since S^3 is a submanifold of $\mathbb{C}^2$, T_pS^3 is in fact a vector subspace of $T_p\mathbb{C}^2$, which is canonically identified with $\mathbb{C}^2$. Additionally, the tangent space of a product manifold is isomorphic to the product of the tangent space of the manifolds (see Trautman 1970). Hence we identify X_p with a pair $(X^1_{p^1}, X^2_{p^2}) \in T_{p^1}\mathbb{C} \times T_{p^2}\mathbb{C}$, or with a pair $(X^1_{p^1}, X^2_{p^2}) \subset \mathbb{C}^2$. The use of the same letters should not cause confusion. If ω is a connection on S^3, we must have
$$\omega_{p\cdot g}((\sigma_g)_* X_p) = ad_{g^{-1}}(\omega_p(X_p)) = g^{-1}\omega_p(X_p)g = \omega_p(X_p), \qquad \text{for all } g \in U(1).$$

Lemma 3. $(\sigma_g)_{*p}(X^1_{p^1}, X^2_{p^2}) = (X^1_{p^1} g, X^2_{p^2} g)$.

Proof. □

Finally, to compute very concretely, let's write $g = e^{it} = \cos t + i\sin t$ for some $t \in \mathbb{R}$, $X^1_{p^1} = v^1 + iw^1$, $X^2_{p^2} = v^2 + iw^2$, and $p^1 = q^1 + ir^1$, $p^2 = q^2 + ir^2$, for some real numbers $v^1, w^1, v^1, w^2, q^1, r^1, q^2, r^2$. Then, by the previous lemma,
$$\begin{aligned}(\sigma_g)_{*p}(X^1_{p^1}, X^2_{p^2}) &= (X^1_{p^1} \cdot g, X^2_{p^2} \cdot g) \\ &= ((v^1\cos t - w^1\sin t) + i(v^1\sin t + w^1\cos t), (v^2\cos t - w^2\sin t) + i(v^2\sin t + w^2\cos t)),\end{aligned}$$
so the equation
$$\omega_{p\cdot g}((\sigma_g)_{*p}(X^1_{p^1}, X^2_{p^2}) = \omega_p(X^1_{p^1}, X^2_{p^2})$$

is equivalent to

$$\begin{aligned}\omega_1|_{p\cdot g}(v^1\cos t - w^1\sin t) &+ \omega_2|_{p\cdot g}(v^1\sin t + w^1\cos t)\\ &+\omega_3|_{p\cdot g}(v^2\cos t - w^2\sin t) + \omega_4|_{p\cdot g}(v^2\sin t + w^2\cos t)\\ &= \omega_1|_p v^1 + \omega_2|_p w^1 + \omega_3|_p v^2 + \omega_4|_p w^2\end{aligned}$$

With some thought and reorganisation, the following ansatz seems like it might solve the above:

$$\omega = i(-x^2dx^1 + x^1dx^2 - x^4dx^3 + x^3dx^4).$$

This is indeed the case. An easy calculation shows that, with this 1-form, the left hand side is:

$$\begin{aligned} & -(q^1\sin t + r^1\cos t)(v^1\cos t - w^1\sin t) + (q^1\cos t - r^1\sin t)(v^1\sin t + w^1\cos t)\\ & -(q^2\sin t + r^2\cos t)(v^2\cos t - w^2\sin t) + (q^2\cos t - r^2\sin t)(v^2\sin t + w^2\cos t)\\ =\ & (-q^1\sin t\cos t - r^1\cos^2 t + q^1\cos t\sin t - r^1\sin^2 t)v^1\\ & + (q^1\sin^2 t + r^1\cos t\sin t + q^1\cos^2 t - r^1\sin t\cos t)w^1\\ & + (-q^2\sin t\cos t - r^2\cos^2 t + q^2\cos t\sin t - r^2\sin^2 t)v^2\\ & + (q^2\sin^2 t + r^2\cos t\sin t + q^2\cos^2 t - r^2\sin t\cos t)w^2\\ =\ & -r^1v^1 + q^1w^1 - r^2v^2 + q^2w^2\\ =\ & \omega_1|_p v^1 + \omega_2|_p w^1 + \omega_3|_p v^2 + \omega_4|_p w^2\end{aligned}$$

What we have just shown is that the connection ω is right-equivariant, i.e. that $\sigma_g^*\omega = ad_{g^{-1}}\circ\omega$. This is thus a good candidate for a connection. What is left to prove is that it acts trivially on fundamental vector fields. First note that, equivalently, we may express the connection in terms of the complex variables as

$$\omega = i\cdot Im(\overline{z}^1dz^1 + \overline{z^2}dz^2),$$

since, if we put $z^1 = x^1 + ix^2$ and $z^2 = x^3 + ix^4$, we obtain the same 1-form:

$$\overline{z}^1dz^1 + \overline{z^2}dz^2 = (x^1dx^1 + x^2dx^2 + x^3dx^3 + x^4dx^4) + i(-x^2dx^1 + x^1dx^2 - x^4dx^3 + x^3dx^4).$$

Let now $A_e = i\alpha \in i\mathbb{R} \simeq T_eU(1) \simeq \mathcal{U}(1)$. The fundamental vector field $A^\# \in \Gamma(TS^3)$ is defined by

$$A_p^\#(f) = (\sigma_p)_{*e}A_e(f) = A_e(f\circ\sigma_p).$$

Now, a differential curve corresponding to A_e is $\gamma : \mathbb{R}\to S^1$ given by $\gamma(t) = e^{i\alpha t}$ (it can be easily checked that $\gamma(0) = e$ and $\gamma'(0) = i\alpha$). Thus, if $p = (p^1, p^2) \in S^3$ and $p^1 = v^1 + iw^1$, $p^2 = v^2 + iw^2$, as above, then

$$\begin{aligned}A_p^\#(f) &= \frac{d}{dt}(f\circ\sigma_p\circ\gamma)(t)\bigg|_{t=0} = \frac{d}{dt}f(p^1e^{i\alpha t}, p^2e^{i\alpha t})\bigg|_{t=0}\\ &= \frac{\partial}{\partial z^1}f(p^1)\cdot i\alpha p^1 + \frac{\partial}{\partial z^2}f(p^2)\cdot i\alpha p^2,\end{aligned}$$

where $\frac{\partial}{\partial z^i}f$ is the partial derivative of f with respect to the i^{th} complex coordinate. The last step is simply an application of the chain rule. We have therefore that

$$A_p^\# = i\alpha\left(p^1\frac{\partial}{\partial z^1}\bigg|_{p^1} + p^2\frac{\partial}{\partial z^2}\bigg|_{p^2}\right).$$

We apply the 1-form ω to this tangent vector to obtain

$$\omega_p(A_p^\#) = i\cdot Im(i\alpha(\overline{p}^1p^1 + \overline{p}^2p^2)) = i\cdot Im(i\alpha(|p^1|^2 + |p^2|^2)) = i\alpha = A.$$

In the last step we have used the fact that $p = (p^1, p^2) \in S^3$. Thus ω satisfies both conditions required in the definition of a connection. Now that we have a connection, we can do concrete computations. Additionally, we now know that our whole discussion isn't void, since at least one connection exists. ω is known in the literature as the natural connection form on the Hopf bundle. (Trautman 1977)

3.2.3 Canonical Cross-sections

One last notion we require before achieving the geometric description of the monopole is that of cross-section:

Definition 5. Given a PFB $G \to P \xrightarrow{\pi} B$ and a trivialization (V, Ψ), we define the canonical cross-section associated to V to be the function $s : V \to \pi^{-1}(V)$ given by $s(x) = \Psi^{-1}(x, e)$.

Note that canonical cross-sections are indeed sections, since they are right-inverses of the projection: $\pi \circ s = id_V$.

These will turn out to be the natural sections on the Hopf bundle, alluded to in Theorem 1 above.

In the example of the bundle $S^1 \to S^3 \xrightarrow{\pi} S^2$, the canonical cross-sections are obtained by first trivializing S^2 on the neighbourhoods $U_N = S^2 - \{(0,0,1)\}$ and $U_S = S^2 - \{(0,0,-1)\}$. It is best done by identifying S^2 with $\mathbb{CP}^1$ and working in the projective space. The isomorphism τ maps the charts U_N, U_S to charts on $\mathbb{CP}^1$:

$$\tau(U_N) = V_2 = \{[z^1, z^2] \in \mathbb{CP}^1 : z^2 \neq 0\}$$
$$\tau(U_S) = V_1 = \{[z^1, z^2] \in \mathbb{CP}^1 : z^1 \neq 0\}$$

The charts V_1, V_2 cover $\mathbb{CP}^1$. The idea for obtaining the trivialization is to normalize an element $[z^1, z^2]$ of V_k by dividing by z^k. Consider the map $\Phi_k : V_k \times S^1 \to \pi^{-1}(V_k)$ given by

$$\Phi_k([z^1, z^2], g) = (z^1, z^2) \cdot (z^k)^{-1} |z^k| g.$$

This map is well defined. It is straightforward to check that the map $\Psi_k : \pi^{-1}(V_k) \to V_k \times S^1$ given by

$$\Psi_k(z^1, z^2) = ([z^1, z^2], z^k / |z^k|)$$

is the inverse of Φ_k. Since this map is both a left and right inverse and they are both smooth, it follows that (V_k, Ψ_k) is a trivialization of the Hopf bundle (with base manifold $\mathbb{CP}^1$). We can now define the canonical cross-sections associated to these trivializations. Let's first compute $\Psi^{-1}([z^1, z^2], e)$, for $(z^1, z^2) \in S^3$, and use the Euler angles $z^1 = \cos\frac{\phi}{2} e^{\frac{i}{2}(\chi+\theta)}$, $z^2 = \sin\frac{\phi}{2} e^{\frac{i}{2}(\chi-\theta)}$:

$$\Psi_1^{-1}([z^1, z^2], e) = \left(|z^1|, \frac{z^2|z^1|}{z^1}\right) = \left(\cos\frac{\phi}{2}, \sin\frac{\phi}{2} e^{-i\theta}\right)$$
$$\Psi_2^{-1}([z^1, z^2], e) = \left(\frac{z^1|z^2|}{z^2}, |z^2|\right) = \left(\cos\frac{\phi}{2} e^{i\theta}, \sin\frac{\phi}{2}\right)$$

If $(x^1, x^2, x^3) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ is a point on S^2, then $s_N : U_N \to S^3$ and $s_S : U_S \to S^3$ are given by composing Ψ_k^{-1} and τ:

$$s_N(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \Phi_2(\tau(x^1, x^2, x^3), e) = \left(\cos\frac{\phi}{2}, \sin\frac{\phi}{2} e^{-i\theta}\right)$$
$$s_S(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \Phi_1(\tau(x^1, x^2, x^3), e) = \left(\cos\frac{\phi}{2} e^{i\theta}, \sin\frac{\phi}{2}\right)$$

If we want to work on $\mathbb{R}^4$ instead, we compose with the inclusion $\iota : \S^3 \hookrightarrow \mathbb{R}^4$:

$$\iota \circ s_N(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \left(\cos\frac{\phi}{2}, 0, \sin\frac{\phi}{2}\cos\theta, -\sin\frac{\phi}{2}\sin\theta\right)$$
$$\iota \circ s_S(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \left(\cos\frac{\phi}{2}\cos\theta, \cos\frac{\phi}{2}\sin\theta, \sin\frac{\phi}{2}, 0\right)$$

3.2.4 Proof of Theorem 1

We are now in a good position to prove theorem 1.

Proof. With all the pieces laid out, this is now just an exercise in computing pull-backs. Recall that the 1-form we wish to pull back is $\omega = \iota^*(-x^2dx^1 + x^1dx^2 - x^4dx^3 + x^3dx^4)$. (We omitted the factor of i for ease of notation). So we wish to compute $s_N^*(\iota^*\omega) = (\iota \circ s_N)^*\omega$ and $s_S^*(\iota^*\omega) = (\iota \circ s_S)^*\omega$. Let's fix a point (ϕ, θ) on a chart of S^3. Then the pull-back is

$$
\begin{aligned}
&(\iota \circ s_N)^*\omega|_{(\phi,\theta)} \\
&\qquad = (-0)\cdot d(\cos\frac{\phi}{2}) + (\cos\frac{\phi}{2})\cdot d(0) - (-\sin\frac{\phi}{2}\sin\theta)\cdot d(\sin\frac{\phi}{2}\cos\theta) + (\sin\frac{\phi}{2}\cos\theta)\cdot d(-\sin\frac{\phi}{2}\sin\theta) \\
&\qquad = \sin\frac{\phi}{2}\sin\theta\left(\frac{1}{2}\cos\frac{\phi}{2}\cos\theta\ d\phi - \sin\frac{\phi}{2}\sin\theta\ d\theta\right) - \sin\frac{\phi}{2}\cos\theta\left(\frac{1}{2}\cos\frac{\phi}{2}\sin\theta\ d\phi + \sin\frac{\phi}{2}\cos\theta\ d\theta\right) \\
&\qquad = -\frac{1}{2}(1-\cos\phi)d\theta
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(\iota \circ s_S)^*\omega|_{(\phi,\theta)} =& -(\cos\frac{\phi}{2}\sin\theta)\cdot d(\cos\frac{\phi}{2}\cos\theta) + (\cos\frac{\phi}{2}\cos\theta)\cdot d(\cos\frac{\phi}{2}\sin\theta) \\
=& -(\cos\frac{\phi}{2}\sin\theta)\cdot\left(-\frac{1}{2}\sin\frac{\phi}{2}\cos\theta\ d\phi - \cos\frac{\phi}{2}\sin\theta\ d\theta\right) \\
& + (\cos\frac{\phi}{2}\cos\theta)\cdot\left(-\frac{1}{2}\sin\frac{\phi}{2}\sin\theta\ d\phi + \cos\frac{\phi}{2}\cos\theta\ d\theta\right) \\
=& \frac{1}{2}(1+\cos\phi)d\theta
\end{aligned}
$$

These are just the expressions of the vector potential for a Dirac monopole with strength $g = \frac{1}{2}$. This proves Theorem 1.

□

We have achieved our stated purpose of giving a description of the Dirac monopole as the pullback to the base manifold of a connection, that is a 1-form, defined on the total space of a principal fibre bundle.

4 Concluding Remarks

Several remarks are in order. These are mostly paths which still merit to be pursued, for they are interesting extensions of the ideas in this project.

The Hopf bundle described in this paper is not the only one. As hinted, there are also Hopf bundles obtained by considering spheres in F^2, where $F = \mathbb{R}, \mathbb{H}$, or even $\mathbb{O}$, the octonions. By a sphere in F^2, we mean a set $S = \{(q^1, q^2) \in F^2 : \langle q^1, q^2 \rangle = 1\}$, where $\langle \cdot, \cdot \rangle$ is the usual inner product. There is an action of the unit-norm elements of F on S. Taking the quotient by this action we obtain a Hopf bundle for every normed division algebra. Applying the same procedure as in this project to the quaternionic Hopf bundle yields a solution to the Yang-Mills equation known as the BPST instanton, since it was first discovered in a paper by authors Belavin, Polyakov, Schwartz and Tyupkin [3].

In [6], the natural connection on the Hopf bundle is obtained from the Riemann metric on S^3. Trautman finds other solutions to the Yang-Mills equations by looking at the Riemann metric on more general spheres. It would be interesting to understand this procedure more deeply and see how it applies to the even more general settings alluded to at the end of Trautman's paper.

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