From Seifert Fibrations to Geometry

Mihai Marian

January 5, 2019

1 Introduction

We know that Seifert fibre spaces, which are 3-manifolds foliated by circles, make up 6 of the 8 Thurston geometries. The point of this project is to answer the following question: how does the foliation of a 3-manifold by circles give information about its geometry?

At first I thought that a Seifert fibration should be used to define a Riemannian metric carrying the geometric information. Of course, I misunderstood a key lesson of Thurston (and Klein): the geometry of a space is determined by the group of isometries acting on that space; if we understand the group, then we understand the geometry. So instead we should use the foliation to describe the isometries.

To achieve the stated goal, I attempt to explain the geometrization of the closed and orientable Seifert manifolds. The precise theorem is the following

Theorem 1. Let M be a closed (i.e. compact and without boundary) 3-manifold. M admits a geometric structure modelled on one of $S^2 \times \mathbb{R}, S^3, \mathbb{R}^3, \mathrm{Nil}, \mathrm{H}^2 \times \mathbb{R}$ or $\widetilde{SL_2}$ iff M is a Seifert fibre space. Moreover, there are two combinatorial invariants of Seifert fibre spaces, e and χ, which separate Seifert fibred spaces into six classes such that the six model geometries correspond to the classes according to the following table:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathrm{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{SL_2}$

I will explain the backward direction in the orientable case, i.e. the following statement: if M is Seifert fibred and orientable, then it admits the appropriate geometric structure.

Remark. An explanation of the terms:

(1) Having a geometric structure modelled on X essentially means being a quotient of X by some discrete subgroup of $\mathrm{Isom}(X)$. The manifolds $\mathbb{R}^3, \mathrm{Nil}, \mathrm{H}^2 \times \mathbb{R}, \widetilde{SL_2}$ are all simply connected and homeomorphic to $\mathbb{R}^3$; their geometry is contained in their specified isometry groups. The full definition is more detailed and can be found in chapter 3.8 of [7].

(2) Seifert fibre spaces are 3-manifolds with a foliation by circles. Equivalently, they are almost S^1-bundles over a surface in a sense which we will make precise later on. An example to keep in mind is S^3 foliated by the fibres of the Hopf fibration.

The standard reference for understanding the non-hyperbolic geometries on 3-manifolds is [5]. This is the main source of the results presented here. There are also other books and articles which describe Thurston's program, such as [3] and, from the fountainhead, [7]. As an introduction to 3-manifolds in general, there is [1], which has beautiful illustrations, and [2], which presents the more basic results, such as the prime decomposition and JSJ decomposition theorems.

In order to prove Theorem 1, we need to know the classification of Seifert fibre spaces, the geometrization of the so-called good 2 dimensional orbifolds and the fundamental groups of Seifert fibre spaces. Thus in part 2 we introduce these spaces and give their basic properties, one of which is the orbifold structure of the base space. In part 3 we give the necessary facts on orbifolds and state how they can be given a geometric structure, in part 4 we return to Seifert fibre spaces to give their classification and describe their fundamental groups. Finally, in part 5 we pull it all together to show how to obtain a given Seifert fibre space as a quotient of the appropriate geometry.

2 Introduction to Seifert fibre spaces

The definitions given by Scott start by describing the simplest Seifert fibre spaces in terms of foliations, but we are in fact only considering very nice examples of foliations (the leaves are always either copies of S^1 or part of a product foliation by copies of $\mathbb{R}$), so there is no need to have more than a vague understanding of what a foliation is in order to understand the rest. The reason this concept is used is that Seifert fibre spaces are not S^1-bundles: they decompose as a disjoint union of circles, finitely many of which do not have a neighbourhood of circles that is isomorphic to $D^2 \times S^1$. We will say more about this below.

Definition 1. A **trivial fibred solid torus** is $D^2 \times S^1$ with the product foliation by circles (this is the same as the trivial S^1-bundle over D^2). A **fibred solid torus** is a torus with a foliation by circles that is finitely covered by a trivial fibred solid torus (by this is meant that the covering map restricted to a leaf of the trivial fibred solid torus covers a leaf of the fibred solid torus).

Proposition 1. Let $T_{p,q}$ be the solid torus with foliation obtained by taking $D^2 \times [0,1]$ (with the product foliation by intervals) and gluing $D^2 \times \{0\}$ to $D^2 \times \{1\}$ with a p/q twist. To be completely unambiguous, let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ and suppose that p, q are coprime integers. Then the gluing map is

$$re^{i2\pi t} \mapsto re^{i2\pi(t+\frac{p}{q})}.$$

$T_{p,q}$ is a fibred solid torus.

Proof. It is easier to describe the group of deck transformations on $D^2 \times S^1$: let $\psi : D^2 \times S^2 \to D^2 \times S^2$ rotate the disc factor of by q/p of a turn and the circle factor by $1/p$ of a turn. Explicitly,

$$\psi(z, e^{i2\pi t}) = (ze^{i2\pi\frac{q}{p}}, e^{i2\pi(t+\frac{1}{p})}).$$

Then $\psi^p = id$ and the quotient map $D^2 \times S^1 \to (D^2 \times S^1)/\langle\psi\rangle$ is a covering map of degree p. The quotient is $T_{p,q}$. ☐

Remark. We can now see why using S^1-bundles would be wrong. $T_{p,q}$ has two different types of fibre: the central fibre $\{0\} \times S^1$ is p-covered by the central fibre of the trivial fibred solid torus and represents the generator of $\pi_1(T_{p,q})$, whereas the other fibres of $T_{p,q}$ are homeomorphic images of the non-central fibres of the trivial fibred solid torus and each of them represents p times the generator of $\pi_1(T_{p,q})$. No neighbourhood U of $\{0\} \times S^1$ is a trivial S^1-bundle.

Fact: Every standard fibred solid torus is isomorphic to $T_{p,q}$, for some p, q. WLOG, assume that p, q are coprime. The p is unique, but the q is only unique in the interval $1 \leq q < p$.

Definition 2. A **Seifert fibre space** is a 3-manifold which decomposes as a disjoint union of circles such that each circle has a regular neighbourhood which is isomorphic (in a fibre-preserving way) to either a trivial fibred solid torus or a standard fibred solid torus. If a fibre has a neighbourhood isomorphic to $T_{p,q}$, then the fibre is called **singular** of **order** p and (p, q) are called the **orbit invariants** of the singular fibre. Otherwise the fibre is called **regular**.

Consider now the quotient space X obtained by identifying every fibre of a Seifert fibre space M to a point. Let $\eta : M \to X$ denote the quotient map. So each $x \in X$ represents a fibre of M. If x represents a non-singular fibre, then a neighbourhood of x is homeomorphic to $\mathbb{R}^2$. If x represents a singular fibre of order p, then a small disc neighbourhood of x is p-covered by η restricted to some embedded disc $D^2 \hookrightarrow M$ which is transverse to the fibre represented by x. Such singular points are the only feature of orbifolds which distinguishes them from usual surfaces, for the orbifolds which arise in this paper.

3 Elementary Orbifold Facts

This section is mostly a restatement of section §2 in [5]. The reader should refer to it for more details.

Example 1. We restate the example of an orbifold mentioned above. Consider $\mathbb{Z}/n\mathbb{Z}$ acting on D^2 by $z \mapsto e^{i2\pi/n}z$ and the quotient of D^2 by this action. A fundamental domain for the action is a wedge of the disc of angle $2\pi/n$. The quotient of D^2 by this action is an orbifold. The image of 0 under the quotient map is called a **cone point**. This is because the quotient is obtained from the fundamental domain by gluing the sides, yielding a cone. Of course, topologically, the orbifold is just another copy of D^2. We should however consider it as more than a topological space if we want to encode some information about the fibres of a Seifert fibre space.

Definition 3. An orbifold is a topological manifold such that every point has a neighbourhood which is homeomorphic to a quotient of $\mathbb{R}^2$ by a finite subgroup of $\text{Isom}(\mathbb{R}^2)$. A point whose every neighbourhood is a quotient by a nontrivial action will be called **singular**.

Remark. (1) In general orbifolds are defined in all dimensions, but we only need 2 dimensional orbifolds in this paper. (2) Since the action in the definition can be trivial, smooth surfaces are orbifolds.

Since in this paper we are only considering orientable Seifert manifolds, our orbifolds always arise as topological surfaces with isolated cone points, i.e. points with neighbourhoods homeomorphic to $\mathbb{R}^2/\mathbb{Z}_n$, as in the above example. There are 2 other kinds of singular points on a 2-dimensional orbifold, arising from acting on $\mathbb{R}^2$ by $\mathbb{Z}_2$ generated by a reflection or by the dihedral group generated by reflections across two lines that intersect at a point. We need only mention them again once.

Definition 4. Let Σ be a surface and $\alpha_i \in \mathbb{N}$. $\Sigma(\alpha_1, \ldots, \alpha_n)$ will denote an orbifold with n cone points of order $\alpha_1, \ldots, \alpha_n$ and which is homeomorphic to Σ.

We would like to define covering spaces and a useful notion of fundamental group for orbifolds. For instance the disc with one cone point in the example above is n-covered by a disc, so its fundamental group should be $\mathbb{Z}/n\mathbb{Z}$.

3.1 Orbifold Cover and Fundamental Group

We define orbifold covers so that $D^2(n)$ covers $D^2(m)$ iff $n|m$.

Definition 5. We say that a map $f : X \to Y$ between orbifolds is an **orbifold covering** if every $y \in Y$ has a neighbourhood U such that, if $U \simeq \mathbb{R}^2/G$, then there is a subgroup $H < G$ and every $x \in f^{-1}(y)$ has a neighbourhood V which is homeomorphic to $\mathbb{R}^n/H$.

Definition 6. An orbifold is called **good** if it is covered by a smooth surface. Otherwise it is **bad**.

For example $S^2(p)$ and $S^2(p,q)$ for $p \neq q$ are bad orbifolds. It turns out that there aren't many other closed bad orbifolds:

Proposition 2. There are only 4 bad orbifolds: $S^2(p)$, $S^2(p,q)$ with $p \neq q$ and D^2 with one or two "corner reflectors".

Proof. We do not need to worry about the last 2 bad orbifolds. See Theorem 13.3.6 in [6] for a proof. □

Here is how to define the fundamental group consistently with the desired result $\pi_1 D^2(n) = \mathbb{Z}_n$. It is a fact that every orbifold X has a universal orbifold cover $\widetilde{X}$, i.e. that $\widetilde{X}$ is connected and $\widetilde{X} \to X$ is an orbifold cover which factors through any other orbifold cover of X. Furthermore, if X is good, then the universal orbifold cover of X coincides with the universal cover of a surface covering X and the universal orbifold cover is automatically a regular cover. See Proposition 13.2.4 in [6] for proofs of these facts.

Definition 7. Let X be an orbifold. $\pi_1 X$ is defined to be the group of deck transformations acting on the universal orbifold cover of X.

In [5] the universal cover of the orbifolds with cone point singularities is constructed by hand, by removing the singularities, taking the universal cover of the resulting non-compact manifold and gluing back appropriate disc orbifolds. At the same time, this construction allows one to easily obtain a presentation for the orbifold fundamental group:

Let X be the orientable surface of genus g with n cone points of order α_i and let Y be the non-orientable surface of genus g with n cone points of order α'_i. We have the following presentations for their orbifold fundamental groups:

$$\pi_1 X = \langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_n | x_i^{\alpha_i}, \Pi_i [a_i, b_i] x_1 \cdots x_n \rangle$$
$$\pi_1 Y = \langle a_1, \ldots, a_g, x_1, \ldots, x_n | x_i^{\alpha'_i}, \Pi_i a_i x_1 \cdots x_n \rangle,$$

where the relators indicate which words are trivial and $\Pi_i a_i$ denotes the product $a_1 a_2 \cdots a_g$.

3.2 Geometrization

Just as surfaces inherit a unique geometry as a quotient of $S^2, \mathbb{R}^2$ or H^2, so do orbifolds. The following theorem states this precisely. We will use it to put geometric structures on Seifert fibre spaces without getting our hands too dirty. It will be a very useful black box for us in section 6.

Theorem 2. Every good orbifold is isomorphic as orbifold to a quotient of $S^2, \mathbb{R}^2$ or H^2 by some discrete group of isometries.

Proof. Theorem 13.3.6 in [6]. □

The above cover by surfaces of good orbifolds is finite:

Theorem 3. Every good compact orbifold without boundary is finitely covered by a surface.

This is Theorem 2.5 in [5], where Scott gives references for the proof of the group-theoretic reformulation of this theorem in terms of properties of subgroups of $PSL_2(\mathbb{R})$.

3.3 Orbifold Euler Characteristic

Definition 8. Let X be a good orbifold, so, by the above theorem, X is finitely covered by a surface $\widehat{X}$. Let n be the degree of the covering. The Euler characteristic of X is defined to be

$$\chi(X) = \frac{1}{n}\chi(\widehat{X}).$$

Proposition 3. (Riemann-Hurwitz) Let $X = \Sigma(\alpha_1, \ldots, \alpha_n)$ be a good orbifold. Then

$$\chi(X) = \chi(\Sigma) - \sum_{i=1}^{n} 1 - \frac{1}{\alpha_i}.$$

Proof. The proof is elementary and explained in [5]. The idea is to remove from Σ a small disc neighbourhood D_i of each cone point. Since $\chi(D^2) = 1$, $\chi(\Sigma) = \chi(\Sigma \setminus \cup_i D_i) + n$. Now X is finitely covered by a manifold $\widehat{X}$. Let d be the degree of the covering. The preimage of a disc containing a singularity of order α_i consists of d/α_i discs. Altogether this implies the formula:

$$\chi(X) = \frac{1}{d}\chi(\widehat{X}) = \left(\chi(\Sigma \setminus \cup_i D_i) + \frac{1}{d}\sum_{i=1}^{n}\frac{d}{\alpha_i}\right) = \chi(\Sigma) - \sum_{i=1}^{n} 1 - \frac{1}{\alpha_i}.$$

□

The Riemann-Hurwitz formula is also used to define the Euler characteristic of bad orbifolds. Here is a precision to Theorem 2:

Corollary 1. The Euler characteristics of a good orbifold and of a manifold covering the orbifold have the same sign.

Thus the geometrization of orbifolds is exactly like the geometrization of surfaces: if $\chi(X) < 0$ then X is finitely covered by H^2, etc.

4 More on Seifert fibre spaces

Many authors call Seifert fibre spaces Seifert fibrations, with good reason:

Theorem 4. Let M be a Seifert fibre space with base orbifold X. Then there is a short exact sequence

$$1 \to K \to \pi_1 M \to \pi_1 X \to 1,$$

where K is the cyclic subgroup of $\pi_1 M$ generated by a regular fibre. It is infinite unless M is covered by S^3. We call the quotient map $\eta : M \to X$ a **Seifert fibration**.

Definition 9. If a singular fibre has orbit invariants (p, q), then it has **Seifert invariants** (α, β), where $\alpha = p$ and β is such that $\beta q \equiv 1 \pmod p$ and $1 \le \beta < p$. The reason for these invariants is their use in describing the fundamental group.

Remark. Note that we are using normalized Seifert invariants, so that they are uniquely determined by the normalized orbit invariants of the fibres. For this reason, we need an additional invariant in order to characterize Seifert fibre spaces (it is not mentioned in [3], for example). For a Seifert fibration $\eta : M \to X$, the invariant is denoted $b(\eta)$ and it is an integer if M is oriented. In the case where X has no singular points, $b(\eta)$ is the obstruction to the existence of a section of the S^1-bundle and it is an element of $H^2(S; \mathbb{Z})$. b is defined for orbifolds as a generalization of the definition given for usual manifolds. For reasons explained in §3 of [5], we should in fact consider a slightly different obstruction for Seifert fibre spaces, which is the Euler number of M, denoted $e(\eta)$ or $e(M)$. Briefly, the point is to find an obstruction that is in a sense multiplicative with respect to finite covering spaces. If M has n cone points with Seifert invariants (α_i, β_i), the the Euler number of M is related to b by the formula

$$e = -b - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}.$$

However it is the invariant b that shows up in the fundamental groups of Seifert fibre spaces.

Proposition 4. Closed Seifert fibre spaces are determined by their b invariant, their Seifert invariants and the underlying surface of the base orbifold. Therefore we write $M = M(g, b, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, with the convention that g is negative for non-orientable surfaces.

4.1 Fundamental Group

We can now give the presentation for the fundamental group of any orientable Seifert fibre space $M = M(g, b, (\alpha_1, \beta_1), \ldots, \alpha_n, \beta_n))$. Suppose first that the underlying surface of the base orbifold of M is orientable (equivalently, that $g \ge 0$, according to our conventions). Then

$$\pi_1 M = \langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_n, k | [k, a_i], [k, b_i], [k, x_i], x_i^{\alpha_i} k^{\beta_i}, \Pi_i [a_i, b_i] x_1 \cdots x_n = k^b \rangle \quad (1)$$

If $g < 0$, then

$$\pi_1 M = \langle a_1, \ldots, a_{-g}, x_1, \ldots, x_n | a_i^{-1} k a_i = k^{-1}, [k, x_i], x_i^{\alpha_i} k^{\beta_i}, \Pi_i a_i^2 x_1 \cdots x_n = k^b \rangle \quad (2)$$

Refer to [5] for the construction of $\pi_1 M$ in the illuminating special case of a Seifert bundle $\eta : M \to T^2(\alpha)$. For the general construction, see [4].

The following theorem says that most Seifert fibre spaces have a unique Seifert structure.

Theorem 5. If M is a closed 3-manifold which is homeomorphic to two non-isomorphic Seifert fibre spaces, then either M is covered by T^3, $S^2 \times S^1$ or S^3.

Proof. This is Theorem 3.8 in [5]. □

This implies that if M is a Seifert fibre space with $\chi < 0$ or $\chi = 0$ and $e \ne 0$, then its Seifert invariants are determined by its fundamental group. See sections 5.2 and 5.3 in [4] and Theorem 6 therein, where Seifert manifolds covered by T^3, $S^2 \times \mathbb{R}$ or S^3 are called "small". This is not the case for S^3, for example, which has infinitely many descriptions as a Seifert fibre space.

5 Why Seifert fibred manifolds are geometric

In this section, we wish to prove the direction (*M is Seifert $\Rightarrow$ M has the prescribed geometric structure*) of Theorem 1, which appears as Theorem 5.3 in [5]. In short, the core idea is to fully adopt the Kleinian paradigm, that is to define the geometry of a space by describing the group of isometries of the said space; so there is no discussion of Riemannian metrics here.

Throughout this section M is a Seifert fibre space and X is its base orbifold. Recall the short exact sequence

$$1 \to K \to \pi_1 M \to \pi_1 X \to 1,$$

where K is a cyclic group that is infinite unless M is covered by S^3.

In just a bit more detail, the strategy for proving that the Seifert fibre space M has a geometry modelled on $(\widetilde{M}, G)$ is to correctly embed $\pi_1 M$ into G, i.e. to specify how each generator of $\pi_1(M)$ acts as an isometry of $\widetilde{M}$ such that the action of $\pi_1(M)$ is free and discrete. This will then imply that the quotient space is a 3-manifold with geometry modelled on $(\widetilde{M}, G)$.

We split our discussion in the following cases, according to whether $\chi(X)$ is negative, zero or positive and whether $e(M)$ is zero or not.

5.1 $\chi(X) > 0,\ e(M) = 0$

Proposition 5. If $\eta : M \to X$ is a Seifert fibration with $\chi(X) > 0$, then X is one of $\mathbb{RP}^2, S^2$ (with no singular points) or $S^2(p, p)$.

Proof. Let Σ denote the underlying surface of X, let n be the number of cone points of X and let α_i be the order oft the i^{th} cone point. Since $\alpha_i \geq 2$, we have the following inequality:

$$\chi(X) = \chi(\Sigma) - \sum_{i=1}^{n} 1 - \frac{1}{\alpha_i} \leq \chi(\Sigma) - \frac{n}{2}.$$

It follows that $\chi(X) > 0$ only if $\Sigma = S^2$ and $n \leq 3$ or $\Sigma = \mathbb{RP}^2$ and $n \leq 1$. Now a Seifert fibred manifold with a single cone point must have

$$e(M) = -b - \frac{\beta}{\alpha},$$

which cannot be an integer, let alone 0. The only options we are left with are for X to be S^2 with 0,2 or 3 cone points or $\mathbb{RP}^2$ with no cone points. By a similar argument we can eliminate the case of S^2 with 3 cone points: $\chi(X) > 0$ implies that the triple of multiplicities $(\alpha_1, \alpha_2, \alpha_3)$ of the 3 cone points is either $(2, 2, k)$, for some $k \geq 2$, or $(2, 3, k)$, for $k \in \{3, 4, 5\}$. For all choices of β_i satisfying $1 \leq \beta_i < \alpha_i$ and $(\alpha_i, \beta_i) = 1$, it is easy to see that

$$\sum_{i=1}^{3} \frac{\beta_i}{\alpha_i} \notin \mathbb{Z},$$

thus the Euler number e cannot vanish. For example,

$$\frac{1}{2} + \frac{\beta_2}{3} + \frac{\beta_3}{4} = \frac{6 + 4\beta_2 + 3\beta_3}{12}$$

and the denominator of the above fraction is odd because $\beta_3 \in \{1, 3\}$ so that $e(M(0, b, (2, 1), (3, \beta_2), (4, \beta_3)) \neq 0$. □

Thus the only possible closed and orientable Seifert fibre spaces M with $\chi > 0$ and $e = 0$ are the orientable S^1 bundle over $\mathbb{RP}^2$, $S^2 \times S^1$, and the Seifert fibre spaces $M(0, -1, (\alpha, \beta), (\alpha, \alpha - \beta))$. It turns out that the orientable S^1-bundle over $\mathbb{RP}^2$ is $\mathbb{RP}^3 \# \mathbb{RP}^3$, which can be obtained as the quotient of $S^2 \times S^1$ by the $\mathbb{Z}_2$ action $(p, e^{it}) \mapsto (-p, e^{-it})$, and the Seifert fibre spaces $M(0, -1, (\alpha, \beta), (\alpha, \alpha - \beta))$ are all $S^2 \times S^1$. This completes the picture for the case with $\chi > 0$ and $e = 0$.

5.2 $\chi(X) = 0,\ e(M) = 0$

By a similar argument to the one provided in the proof of Proposition 5, we have

Proposition 6. If $\eta : M \to X$ is a Seifert fibration with $\chi(X) = 0$, then X is one of the following: $\mathbb{RP}^2(p, p)$, $S^2(p, q, r)$ with $(p, q, r) \in \{(2, 3, 6), (2, 4, 4), (3, 3, 3)\}$, $S^2(2, 2, 2, 2)$, T^2 or K, the Klein bottle.

We can use the proposition above to list all the possible Seifert fibrations $\eta : M \to X$ with $\chi(X) = 0$. There are 7 distinct Seifert fibrations with orientable total space, one for each orbifold given in the above proposition. It turns out that 2 of them are isomorphic as smooth manifolds. They can each be given a flat ($\mathbb{R}^3$) structure by tiling Euclidean space. See section 13.1.6 in [3], specifically figure 12.2.

5.3 $\chi(X) < 0,\ e(M) = 0$

This is done in [5]. Let $\pi_1 M$ have a presentation as above, with generating set $\{a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_n, k\}$ and let $\pi_1 X$ be generated by $\{\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g, \overline{x}_1, \ldots, \overline{x}_n\}$, where the elements of $\pi_1 X$ are images of the corresponding elements of $\pi_1 M$ under the fibration $M \to X$. Since $\chi(X) < 0$, we know from the geometrization of orbifolds that X is a quotient of H^2 by $\pi_1 X$ acting by isometries. Define then an action of $\pi_1 M$ on $\mathrm{H}^2 \times \mathbb{R}$ as follows:

$$\begin{aligned}
a_i \cdot (x,t) &= (\overline{a}_i \cdot x, t) \\
b_i \cdot (x,t) &= (\overline{b}_i \cdot x, t) \\
x_i \cdot (x,t) &= (\overline{x}_i \cdot x, t - \beta_i/\alpha_i) \\
k \cdot (x,t) &= (x, t+1)
\end{aligned}$$

Proposition 7. The isometries defined by the generators of $\pi_1 M$ satisfy the relations that the generators themselves satisfy in the presentation for $\pi_1 M$.

Proof. Clearly k commutes with everything and $x_i^{\alpha_i} \cdot (x,t) = (\overline{x}_i^{\alpha_i} \cdot x, t - \beta_i) = k^{-\beta_i} \cdot (x,t)$. We only need to check that they also satisfy the long relation. This will follow from $e(M) = 0$, which is equivalent to $b = \sum_i \beta_i/\alpha_i$:

$$\prod_{i=1}^{g}[a_i, b_i] x_1 \cdots x_n \cdot (x,t) = \left(\prod_{i=1}^{g}[\overline{a}_i, \overline{b}_i] \overline{x}_1 \cdots \overline{x}_n \cdot x, t - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} \right) = (x, t+b) = k^b \cdot (x,t).$$

□

This proves that we have a homomorphism $\phi : \pi_1 M \to \mathrm{Isom}(\mathrm{H}^2 \times \mathbb{R}) = \mathrm{Isom}(\mathrm{H}^2) \times \mathrm{Isom}(\mathbb{R})$. The image of $\pi_1 M$ is a discrete subgroup of $\mathrm{Isom}(\mathrm{H}^2 \times \mathbb{R})$. Following [5], we check that the action on $\mathrm{H}^2 \times \mathbb{R}$ defined by its image is free: if $g \in \pi_1 M$ fixes a point of $\mathrm{H}^2 \times \mathbb{R}$, then the image $\overline{g}$ of g in $\pi_1 X$ must fix a point of H^2. Either $\overline{g}$ is identity or not. If $\overline{g}$ is identity, we can conclude that g itself must be identity. If $\overline{g}$ is non-trivial, then it must be conjugate to some power of some $\overline{x}_i$, as these are the only elements of $\pi_1 X$ which act on H^2 with fixed points. It follows then that g is conjugate to an element of the form $k^a x_i^b$. But the subgroup of $\pi_1 M$ generated by x_i and k acts freely on $\mathrm{H}^2 \times \mathbb{R}$, thus we can again conclude that g is identity. This shows that M is a quotient of $\mathrm{H}^2 \times \mathbb{R}$ by a free and discrete action of a subgroup of isometries, i.e. that it has $\mathrm{H}^2 \times \mathbb{R}$ geometry.

5.4 $\chi(X) > 0,\ e(M) \neq 0$

These manifolds should have spherical geometry. The method for finding out how to take a quotient of S^3 to obtain a specific space is on the one hand to list all possible Seifert manifolds with $\chi > 0$ and $e \neq 0$ and on the other hand to list all possible finite subgroups of $\mathrm{Isom}(S^3) = SO(4)$ which act freely on S^3. This is done in both [3] and [5], but perhaps the first reference is more systematically organized. See also chapter 6 in W. Jaco's "Lectures on Three-Manifold Topology".

One then compares the two lists to find that manifolds with $\chi > 0$ and $e \neq 0$ indeed arise as quotients of S^3.

5.5 $\chi(X) \leq 0,\ e(M) \neq 0$

This should be $\widetilde{SL_2}$ or Nil geometry. Recall that the isometry group of $\widetilde{SL_2}$ falls into the short exact sequence

$$1 \to \mathbb{R} \to \mathrm{Isom}^+(\widetilde{SL_2}) \to \mathrm{Isom}(\mathrm{H}^2) \to 1.$$

Similarly, we have

$$1 \to \mathbb{R} \to \text{Isom}^+(\text{Nil}) \to \text{Isom}(\mathbb{R}^2) \to 1.$$

The argument presented here works for both the case $\chi(X) < 0$ and the case $\chi(X) = 0$. In both cases the Seifert fibre structure is unique and determined by the fundamental group. There are infinitely many distinct Seifert fibre spaces with $\chi < 0$ and $e \neq 0$. Consider first the case where the base orbifold X is $S^2(\alpha_1, \ldots, \alpha_n)$ and the Seifert fibre space $M = M(0, b, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$. We have

$$\pi_1 X = \langle \overline{x}_1, \ldots, \overline{x}_n | \overline{x}_i^{\alpha_i}, \Pi_i \overline{x_i} \rangle.$$

And X is the quotient of H^2 by $\pi_1 X$ acting by isometries. Again there is a projection of $\pi_1 M$ onto $\pi_1 X$:

$$1 \to \mathbb{Z} \to \pi_1 M \to \pi_1 X \to 1$$

and we can find x_i which project to $\overline{x}_i$ such that

$$\pi_1 M = \langle x_1, \ldots, x_n, k | x_i^{\alpha_i} k^{\beta_i}, [x_i, k], \Pi_i x_i k^{-b} \rangle.$$

Let k be an isometry of $\widetilde{SL_2}$ which projects to identity on H^2. The relations $x_i^{\alpha_i} k^{\beta_i} = 1$ and the requirement that the isometry x_i project to $\overline{x}_i \in \text{Isom}(\text{H}^2)$ determine x_i. However this does not guarantee that the long relation $\Pi_i x_i = k^b$ holds. It is however clear that $\Pi_i x_i = k^\epsilon$ for some ϵ because $\Pi_i x_i$ projects to the identity in $\text{Isom}(\text{H}^2)$. To have the long relation hold, we should perform the following Tietze transformations: let $K = k^u$ for some $u \in \mathbb{R}$ and let $X_i = x_i k^{v_i}$, where v_i is chosen so that $X_i^{\alpha_i} K^{\beta_i} = 1$ and replace the generators x_i and k in the group presentation with X_i and K. This implies that $v_i = (1-u)\frac{\beta_i}{\alpha_i}$. By expanding the relation $\Pi_i X_i = K^b$ in terms of k, we see that it is equivalent to the equation

$$\epsilon + (1-u)\left(\frac{\beta_1}{\alpha_1} + \cdots + \frac{\beta_n}{\alpha_n}\right) = ub,$$

which has a solution for u iff the coefficient multiplying u is not 0. But this coefficient is precisely $e(M)$, so we can solve for u and have $\pi_1 M$ act on $\widetilde{SL_2}$ by isometries.

Remark. This procedure applies to general Seifert fibre spaces $M \to X$ with $\chi(X) < 0$ and $e(M) \neq 0$: if the underlying manifold of the base orbifold has genus g, then a presentation $\pi_1 M$ is obtained from the one above by adding generators $a_1, b_1, \ldots, a_g, b_g$ (in the orientable case) which commute with k and by replacing the long relation $\Pi_i x_i = k^b$ with the relation $\Pi_i [a_i, b_i] x_1, \ldots, x_= k^b$. We may then apply the same Tietze transformations to find the correct isometry K, i.e. there is no need to replace the generators a_i, b_i.

References

1. A.T. Fomenko and S.V. Matveev, "Algorithmic and Computer Methods for Three-Manifolds", Springer, (1991)

2. A. Hatcher, "Notes on Basic 3-Manifold Topology" (2007)

3. B. Martelli, "An Introduction to Geometric Topology", (2016)

4. P. Orlik, "Seifert Manifolds", Springer Lecture Notes in Mathematics, **291** (1972)

5. P. Scott, "The Geometries of 3-Manifolds", Bull. London Math. Soc. **15**, pp.401-487 (1983)

6. W.P Thurston, "The Geometry and Topology of Three-Manifolds", Lecture notes (1980)

7. W.P. Thurston, "Three-Dimensional Geometry and Topology", Princeton Mathematical Series **35**, (1997)