

Name (print):

Student number:



University of British Columbia
MATH 101 (Vantage): Final exam

Date: *April 18, 2016*

Time: *3:30 p.m. to 6:00 p.m.*

Number of pages: *16 (including cover page)*

Exam type: *Closed book*

Aids: *No calculators or other electronic aids*

Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- *Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;*
- *Speaking or communicating with other candidates;*
- *Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.*

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

For examiners' use only		
Question	Mark	Possible marks
1		15
2		12
3		22
4		6
5		7
6		7
7		4
8		7
Total		80

In your answers, you must simplify standard trigonometric values (*e.g.* $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$) and exponential values (*e.g.* $\log(e^3) = 3$). Otherwise, it is sufficient for your answers to be in “calculator-ready” form.

In general, you may use any result proven in class or on assignments. You may find the following identities helpful:

$$\begin{aligned}\sin^2(\theta) + \cos^2(\theta) &= 1 \\ \tan^2(\theta) + 1 &= \sec^2(\theta).\end{aligned}$$

This page may be used for rough work. It will not be marked.

1. (a) [**3 marks**] Let $f(t)$ be continuous. Prove that $\int_0^r f(t) dt = \int_0^r f(r-t) dt$ for all r .

(b) [**3 marks**] Determine whether the series $\sum_{n \geq 2} \frac{1}{n \log^2(n)}$ converges.

(c) **[3 marks]** Find the interval of convergence of $\sum_{n \geq 0} \frac{2^n}{n^2} (x - 3)^n$.

(d) **[3 marks]** Suppose the Maclaurin series $\sum_{n \geq 0} a_n x^n$ solves the differential equation

$$\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$$

with initial condition $y(0) = -1$. Find a_2 .

(e) [**3 marks**] Solve the differential equation $\frac{dy}{dx} + 3x^2y = 3x^2$.

2. For each of the following statements, determine if it is true. If it is true, provide a justification. If it is false, provide a counterexample.

(a) **[2 marks]** *If $|f(t)|$ is integrable, then $f(t)$ is integrable.*

(b) **[2 marks]** *If $f(x)$ is continuous on $[l, r]$, then $\frac{d}{dx} \int_l^r f(x) dx = f(x)$.*

(c) **[2 marks]** *All power series converge somewhere.*

(d) **[2 marks]** $\tan(0.1)$ is greater than the linear approximation of $\tan(0.1)$ about 0.

(e) **[2 marks]** If $\sum_{n \geq 0} a_n x^n$ converges for some nonzero x , then $\sum_{n \geq 0} a_n$ converges.

(f) **[2 marks]** $\sum_{n \geq 1} \frac{(-1)^{n+1} (\sqrt{5} - 1)^n}{n}$ converges to $\log(\sqrt{5})$.

3. Evaluate the following integrals.

(a) [**3 marks**] $\int_2^{\infty} \frac{t-1}{t^{5/2}} dt.$

(b) [**3 marks**] $\int_2^3 (t+2)(t-2)^{10} dt.$

(c) [4 marks] $\int_1^{\infty} \frac{\log(t)}{t^{1000}} dt.$

(d) [4 marks] $\int_0^{\pi} e^t \sin(t) dt.$

(e) [4 marks] $\int_0^{3/2} \frac{t^2}{\sqrt{9-t^2}} dt.$

(f) [4 marks] $\int_0^{\log(2)} \frac{1-e^t}{1+e^t} dt.$

4. Consider the differential equation

$$f''(x) = -f(x). \tag{1}$$

(a) [**4 marks**] Find all solutions to (1) of the form $\sum_{n \geq 0} a_n x^n$.

(b) [**2 marks**] Using your answer to part (a) or otherwise, write down four different solutions to (1) in closed form (that is, not in series form).

5. Let $f(x) = \frac{e^x - 1}{x}$.

(a) [**2 marks**] Prove that $f(x)$ has Maclaurin series $\sum_{n \geq 0} \frac{x^n}{(n+1)!}$.

(b) [**1 mark**] State the interval of convergence of the series in part (a).

(c) [**4 marks**] Determine what the series $\sum_{n \geq 1} \frac{n}{(n+1)!}$ converges to.

6. Let $f(x) = \int_0^x \sin(t^2) dt$.

(a) [**3 marks**] Find a power series representation for $f(x)$.

(b) [**1 mark**] State the interval of convergence of the series in part (a).

(c) [**3 marks**] Suppose we approximate $f(1)$ using the first three nonzero terms of the series in part (a). Find a good upper bound on the error for this approximation.

7. [4 marks] Consider a satellite moving rapidly overhead at a velocity v . According to special relativity, if a clock on the satellite reports the time T , a clock on the surface of the Earth, below, reports the time

$$T_E(v) = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c is the speed of light in a vacuum. GPS transmitters mounted on satellites are engineered to account for this difference between T_E and T using the approximation

$$T_E(v) \approx T \left(1 + \frac{v^2}{2c^2} \right).$$

Justify this approximation. (Hint: use an appropriate linear approximation on the function $\frac{T}{\sqrt{1-u}}$.)

8. *Newton's Law of Cooling* states that the rate at which an object changes temperature is proportional to the difference between its temperature and that of its surroundings. Equivalently, if $T(t)$ is the object's temperature at time t , and the environment has constant positive temperature E , then

$$\frac{dT}{dt} = -k(T - E) \quad (2)$$

where k is a positive constant.

- (a) [**3 marks**] Solve the differential equation (2) for $T(t)$.

- (b) [**1 mark**] Using your answer from part (a), find $\lim_{t \rightarrow \infty} T(t)$.

- (c) [**3 marks**] A cup of tea, initially at 100°C , is placed in a room kept at a constant temperature of 20°C . Suppose it takes 2 minutes for the cup of tea to cool down to 60°C . What is the temperature of the tea after 10 minutes?

9. [1 **bonus mark**] Draw a picture describing your love of mathematics.