Name (underline your surname):

Student number:

University of British Columbia

MATH 101 (Vantage): Final exam

Date: *April 21, 2017*

Time: *12:00 noon to 2:30 p.m.*

Number of pages: *16 (including cover page)*

Exam type: *Closed book*

Aids: *No calculators or other electronic aids*

Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- *Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;*
- *Speaking or communicating with other candidates;*
- *Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.*

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

For examiners' use only		
Question	Mark	Possible marks
1		12
2		22
3		8
4		5
5		5
6		6
7		10
8		2
Total		70

Note that your answers must be in “calculator-ready” form, but they do not have to be simplified.

In general, you may use any result proven in class or on assignments. You may use without proof the formulas

$$\begin{aligned} \sin^2(\theta) &= \tfrac{1}{2}\left(1-\cos(2\theta)\right), \\ \cos^2(\theta) &= \tfrac{1}{2}\left(1+\cos(2\theta)\right). \end{aligned}$$

This page may be used for rough work. It will not be marked.

1. Evaluate the following integrals.

   (a) **[3 marks]** $\int_1^4 \sqrt{t} \log(t) \, dt$.

   (b) **[3 marks]** $\int_1^\infty te^{-t^2} \, dt$.

(c) **[3 marks]** $\int_0^{\pi/4} \tan^4(t)\, dt.$

(d) **[3 marks]** $\int_0^1 \frac{t}{t^2 + 4t + 3}\, dt.$

2. (a) [**3 marks**] Find the sum of the series $\sum_{n\geq 0} \frac{n}{2^n}$.

(b) [**4 marks**] Find a power series representation for the function $f(x) = \frac{3x^2}{(x-1)^2}$, and state its interval of convergence.

(c) **[3 marks]** Find the interval of convergence of the power series $\sum_{n\geq 1} \frac{(x-3)^n 2^n}{n \cdot 7^{n+1}}$.

(d) **[3 marks]** The degree 5 Maclaurin approximation of $\sin(x)$ is $P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$. Find a reasonable upper bound on the difference between $\sin(2)$ and $P(2)$. You must justify your answer.

(e) **[3 marks]** Calculate $\lim_{x \to 0} \dfrac{\cos(x) - 1}{x \sin(x)}$.

(f) **[3 marks]** Solve the differential equation $\frac{dy}{dx} = 4 - 2y$, given $y(0) = 3$.

(g) **[3 marks]** Make a large sketch of the direction field and a few sample trajectories for the differential equation $\frac{dy}{dx} = x + y$.

3. For each of the following statements, determine if it is true. If it is true, provide a justification. If it is false, provide a counterexample.

   (a) **[2 marks]** *If* $f(x)$ *is defined on* $[0,1]$, *then* $\int_0^1 f(t)\,dt$ *exists.*

   (b) **[2 marks]** $\int_0^r \cos(r-t)\,dt = \int_{-r}^0 \cos(t)\,dt.$

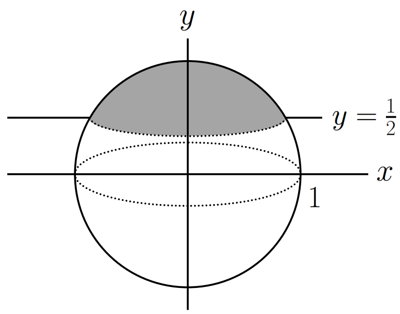
(c) **[2 marks]** *It takes more work to lift a* $75\,\mathrm{m}$ *rope to the top of a* $75\,\mathrm{m}$ *tall building than it does to lift the same rope to the top of a* $50\,\mathrm{m}$ *tall building. (In both cases, assume that the rope starts off hanging from the top of the building.)*

(d) **[2 marks]** *If* $\sum_{n\geq 0} a_n x^n$ *has radius of convergence* R*, then* $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = R$*.*

4. [**5 marks**] Find a function $f(x)$ and a real number a satisfying the equation

$$8 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}.$$

5. [**5 marks**] Calculate the volume of the spherical cap shown below. The sphere is of radius 1, and the cap is the portion of the sphere above $y = \frac{1}{2}$.



6. Recall that $f(x)$ is *odd* if $f(-x) = -f(x)$ for all x, and that $g(x)$ is *even* if $g(-x) = g(x)$ for all x. Let $f(x)$ be an odd, infinitely differentiable function. In this question we prove that the Maclaurin series for $f(x)$ has only odd powers of x.

(a) [**1 mark**] Prove that the derivative of a differentiable odd function is even.

(b) [**1 mark**] Prove that the derivative of a differentiable even function is odd.

(c) [**2 marks**] Explain why $f^{(2n)}(0) = 0$ for all nonnegative integers n.

(d) [**2 marks**] Let $f(x)$ have Maclaurin series $\sum_{n \geq 0} a_n x^n$. Explain why $a_{2n} = 0$ for all nonnegative integers n.

7. In this question we deduce the *binomial formula*

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n \quad \text{for } |x| < 1,$$

where α is a constant and

$$\binom{\alpha}{n} = \begin{cases} \dfrac{\alpha \cdot (\alpha-1)\cdots(\alpha-(n-1))}{n!} & \text{if } n = 1, 2, 3, \ldots \\ 1 & \text{if } n = 0 \end{cases} .$$

Let $f(x) = \sum_{n\geq 0} \binom{\alpha}{n} x^n$.

(a) **[2 marks]** Prove that $f(x)$ converges for $|x| < 1$.

(b) **[3 marks]** Prove that $(1+x)f'(x) = \alpha f(x)$ for $|x| < 1$.

(c) **[3 marks]** Prove that the differential equation

$$(1+x)\frac{dy}{dx} = \alpha y$$

has solutions of the form $y = C(1+x)^\alpha$ for some constant C.

(d) **[2 marks]** From part (b) and part (c), we may conclude that, when $|x| < 1$,

$$C(1+x)^\alpha = f(x) = \sum_{n\geq 0} \binom{\alpha}{n} x^n$$

for some constant C. Conclude the proof of the binomial formula by explaining why $C = 1$.

8. **[2 marks]** Answer *one* of the following two questions.

(a) State the Riemann Hypothesis.

(b) Write down, but do not evaluate, a formula describing the arc length of the parametrized curve $\vec{r}(t) = (e^t \cos(t), e^t \sin(t))$ from $t = 0$ to $t = 4\pi$.