MATH 101 V01 – ASSIGNMENT 1

Solutions

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1. Consider $\displaystyle\int_0^2 (1-2t)\,dt$.

   (a) Calculate the Riemann sum for this integral using left endpoints and 4 subintervals.

   (b) Calculate the Riemann sum for this integral using right endpoints and 4 subintervals.

   (c) Explain why, if any function f is continuous on $[l,r]$, then $\int_l^r f(t)\,dt$ may be calculated by the definition of the integral as the limit of Riemann sums using *any* choice of sample points t_i^* in each subinterval $[t_{i-1},t_i]$ of the partition.

   (d) Find the value of the integral, using the definition of the integral as the limit of Riemann sums using right endpoints $t_i^* = t_i$ in every subinterval.

   (e) Interpret your result for part (d) in terms of the areas between the curve and the horizontal axis, between $t=0$ and $t=2$.

Solution:

The function is $f(t) = 1-2t$, $t \in [0,2]$, the interval width is $\Delta t = (r-l)/n = \frac{2}{n}$, the partition points are $t_i = l + i\Delta t = \frac{2i}{n}$, $i = 0,1,\ldots,n$.

(a) With $n=4$, $\Delta t = \frac{1}{2}$ and the partition points are $t_0 = 0$, $t_1 = \frac{1}{2}$, $t_2 = 1$, $t_3 = \frac{3}{2}$, $t_4 = 2$. The left endpoint of each subinterval $[t_{i-1},t_i]$ is $t_i^* = t_{i-1}$. The Riemann sum is

$$\begin{aligned}\sum_{i=1}^{n} f(t_i^*)\,\Delta t &= \sum_{i=1}^{4} f(t_{i-1})\,\tfrac{1}{2} \\ &= f(0)\,\tfrac{1}{2} + f(1/2)\,\tfrac{1}{2} + f(1)\,\tfrac{1}{2} + f(3/2)\,\tfrac{1}{2} \\ &= (1)\,\tfrac{1}{2} + (0)\,\tfrac{1}{2} + (-1)\,\tfrac{1}{2} + (-2)\,\tfrac{1}{2} \\ &= -1\end{aligned}$$

(b) The right endpoint of each subinterval $[t_{i-1},t_i]$ is $t_i^* = t_i$. The Riemann sum is

$$\begin{aligned}\sum_{i=1}^{n} f(t_i^*)\,\Delta t &= \sum_{i=1}^{4} f(t_i)\,\tfrac{1}{2} \\ &= f(1/2)\,\tfrac{1}{2} + f(1)\,\tfrac{1}{2} + f(3/2)\,\tfrac{1}{2} + f(2)\,\tfrac{1}{2} \\ &= (0)\,\tfrac{1}{2} + (-1)\,\tfrac{1}{2} + (-2)\,\tfrac{1}{2} + (-3)\,\tfrac{1}{2} \\ &= -3\end{aligned}$$

(c) If a function f is continuous on $[l,r]$, then it is integrable on $[l,r]$ and therefore the limit of the Riemann sums $\lim_{n\to\infty}\sum_{i=1}^n f(t_i^*)\,\Delta t = \int_l^r f(t)\,dt$ has the same value for any choice of sample points t_i^* in each subinterval $[t_{i-1},t_i]$.

(d) Since $f(t) = 1-2t$, $t\in[0,2]$, is continuous, by part (c) the value of the integral is the same as the limit of the Riemann sums using as sample points the right endpoints $t_i^* = t_i = \frac{2i}{n}$ of each subinterval $[t_{i-1},t_i]$. For each positive integer n, the Riemann sum is

$$\begin{aligned}\sum_{i=1}^{n} f(t_i^*)\,\Delta t &= \sum_{i=1}^{n} f(t_i)\,\tfrac{2}{n} = \textstyle\sum_{i=1}^n f\left(\frac{2i}{n}\right)\,\frac{2}{n} = \sum_{i=1}^n \left(1-\frac{4i}{n}\right)\,\frac{2}{n} \\ &= \tfrac{2}{n}\textstyle\sum_{i=1}^n 1 - \frac{8}{n^2}\sum_{i=1}^n i \\ &= \tfrac{2}{n}\,n - \tfrac{8}{n^2}\,\tfrac{n(n+1)}{2} \\ &= 2 - 4\left(\tfrac{n+1}{n}\right).\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}\int_0^2 (1-2t)\,dt &= \lim_{n\to\infty} \sum_{i=1}^n f(t_i^*)\,\Delta t \\ &= \lim_{n\to\infty} \left[2 - 4\left(\tfrac{n+1}{n}\right)\right] \\ &= 2-4 \\ &= -2.\end{aligned}$$

(e) Between $t=0$ and $t=2$, the area above the t-axis and below the curve $y=f(t)=1-2t$ is $A_+ = \frac{1}{4}$ (area of a triangle), and the area below the t-axis and above the curve $y=f(t)=1-2t$ is $A_- = \frac{9}{4}$ (area of a larger triangle). The value of the integral is the **net area**

$$\int_0^2 (1-2t)\,dt = A_+ - A_- = \tfrac{1}{4} - \tfrac{9}{4},$$

or the difference of areas.

2. Prove both the following statements, using the definition of the integral.

(a) If f and g are integrable on $[l,r]$, then $f+g$ is integrable on $[l,r]$, and

$$\int_l^r [f(t)+g(t)]\,dt = \int_l^r f(t)\,dt + \int_l^r g(t)\,dt,$$

(b) If f is integrable on $[l,r]$, then for any constant c, the function cf is integrable on $[l,r]$, and

$$\int_l^r cf(t)\,dt = c\int_l^r f(t)\,dt.$$

Solution:

(a) Let n be a positive integer, $\Delta t = \frac{r-l}{n}$, $t_i = l + i\Delta t = l + \frac{i(r-l)}{n}$ for $i=0,1,\ldots,n$. Let $t_i^* \in [t_{i-1}, t_i]$ be any choice of sample point in the ith subinterval, $i=1,\ldots,n$. Then

$$\begin{aligned}\sum_{i=1}^n [f(t_i^*)+g(t_i^*)]\,\Delta t &= [f(t_1^*)+g(t_1^*)]\,\Delta t + [f(t_2^*)+g(t_2^*)]\,\Delta t + \cdots + [f(t_n^*)+g(t_n^*)]\,\Delta t \\ &= f(t_1^*)\,\Delta t + g(t_1^*)\,\Delta t + f(t_2^*)\,\Delta t + g(t_2^*)\,\Delta t + \cdots + f(t_n^*)\,\Delta t + g(t_n^*)\,\Delta t \\ &= f(t_1^*)\,\Delta t + f(t_2^*)\,\Delta t + \cdots + f(t_n^*)\,\Delta t + g(t_1^*)\,\Delta t + g(t_2^*)\,\Delta t + \cdots + g(t_n^*)\,\Delta t \\ &= \sum_{i=1} f(t_i^*)\,\Delta t + \sum_{i=1} g(t_i^*)\,\Delta t,\end{aligned}$$

and taking the limit as $n \to \infty$ we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^n [f(t_i^*)+g(t_i^*)]\,\Delta t &= \lim_{n\to\infty} \sum_{i=1}^n f(t_i^*)\,\Delta t + \lim_{n\to\infty} \sum_{i=1}^n g(t_i^*)\,\Delta t \\ &= \int_l^r f(t)\,dt + \int_l^r g(t)\,dt,\end{aligned}$$

since f and g are both integrable on $[l,r]$. Therefore the limit exists and has the value $\int_l^r f(t)\,dt + \int_l^r g(t)\,dt$, for any choice of sample points $t_i^* \in [t_{i-1}, t_i]$, so $f+g$ is integrable on $[l,r]$ and we can write

$$\lim_{n\to\infty} \sum_{i=1}^n [f(t_i^*)+g(t_i^*)]\,\Delta t = \int_l^r [f(t)+g(t)]\,dt = \int_l^r f(t)\,dt + \int_l^r g(t)\,dt.$$

(b) Let n be a positive integer, $\Delta t = \frac{r-l}{n}$, $t_i = l + i\Delta t = l + \frac{i(r-l)}{n}$ for $i = 0, 1, \ldots, n$. Let $t_i^* \in [t_{i-1}, t_i]$ be any choice of sample point in the ith subinterval, $i = 1, \ldots, n$, and let c be a constant. Then

$$\begin{aligned}\sum_{i=1}^{n} [c\, f(t_i^*)]\, \Delta t &= [c\, f(t_1^*)]\, \Delta t + [c\, f(t_2^*)]\, \Delta t + \cdots + [c\, f(t_n^*)]\, \Delta t \\ &= c[f(t_1^*)\, \Delta t + f(t_2^*)\, \Delta t + \cdots + f(t_n^*)\, \Delta t] \\ &= c \sum_{i=1}^{n} f(t_i^*)\, \Delta t\end{aligned}$$

and taking the limit as $n \to \infty$ we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} [c\, f(t_i^*)]\, \Delta t &= c \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\, \Delta t \\ &= c \int_l^r f(t)\, dt,\end{aligned}$$

since f is integrable on $[l, r]$. Therefore the limit exists and has the value $c \int_l^r f(t)\, dt$, for any choice of sample points $t_i^* \in [t_{i-1}, t_i]$, so cf is integrable on $[l, r]$ and we can write

$$\lim_{n\to\infty} \sum_{i=1}^{n} [c\, f(t_i^*)]\, \Delta t = \int_l^r [c\, f(t)]\, dt = c \int_l^r f(t)\, dt.$$

3. Let

$$f(t) = \begin{cases} 1 & \text{if } 1 \le t \le \sqrt{2}, \\ 2 & \text{if } \sqrt{2} < t \le 2, \end{cases}$$

and note that this function is not continuous on $[1, 2]$.

(a) Prove that f is integrable on $[1, 2]$, and calculate $\int_1^2 f(t)\, dt$ using the definition of the integral: let n be a positive integer and use a regular partition $1 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 2$ of $[1, 2]$ into n subintervals of equal width $\Delta t = (2-1)/n = 1/n$, choose sample points $t_i^* \in [t_{i-1}, t_i]$ in each subinterval $i = 1, \ldots, n$, and prove that the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\, \Delta t$ exists and is equal for all choices of sample points.

(b) Explain in a sentence or two how some other function f on $[1, 2]$ (not the function in part (a)) could be continuous everywhere except at a single point $\sqrt{2}$ in $[1, 2]$, and the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\, \Delta t$ does not exist, i.e. f is not integrable on $[1, 2]$.

Solution:

(a) Let n be a positive integer, let $\Delta t = \frac{r-l}{n} = \frac{1}{n}$ be the distance between partition points

$$t_i = l + i\Delta t = l + \frac{i(r-l)}{n} = 1 + \frac{i}{n},$$

for $i = 0, 1, \ldots, n$, and form subintervals between consecutive partition points. Let $t_i^* \in [t_{i-1}, t_i]$ be any choice of sample point in the ith subinterval, $i = 1, \ldots, n$.

For every n, the partition points t_i (endpoints of the subintervals) are rational numbers. The point of discontinuity $t = \sqrt{2}$ is irrational, so it always lies in the interior of a unique, critical subinterval. So for every n, there exists a unique positive integer i_n such that

$$t_{i_n - 1} = 1 + \frac{i_n - 1}{n} < \sqrt{2} < 1 + \frac{i_n}{n} = t_{i_n}.$$

For example,

$$n=1: \; t_0=1,\, t_1=2;\; t_0<\sqrt{2}<t_1,\; i_n=i_1=1,$$
$$n=2: \; t_0=1,\, t_1=1.5,\, t_2=2;\; t_0<\sqrt{2}<t_1,\; i_n=i_2=1,$$
$$n=3: \; t_0=1,\, t_1=1\tfrac{1}{3},\, t_2=1\tfrac{2}{3},\, t_3=2;\; t_1<\sqrt{2}<t_2,\; i_n=i_3=2,$$
$$n=4: \; t_0=1,\, t_1=1.25,\, t_2=1.5,\, t_3=1.75,\, t_4=2;\; t_1<\sqrt{2}<t_2,\; i_n=i_4=2,$$
$$n=5: \; t_0=1,\, t_1=1.2,\, t_2=1.4,\, t_3=1.6,\, t_4=1.8,\, t_5=2;\; t_2<\sqrt{2}<t_3,\; i_n=i_5=3.$$

Furthermore, since the widths $\Delta t = \frac{1}{n}$ of the critical subintervals $[t_{i_n-1}, t_{i_n}]$ that contain $\sqrt{2}$ shrink to zero as $n\to\infty$, both the left endpoints t_{i_n-1} and the right endpoints t_{i_n} of the critical subinterval converge to $\sqrt{2}$,

$$\lim_{n\to\infty}\left(1+\frac{i_n-1}{n}\right)=\sqrt{2}=\lim_{n\to\infty}\left(1+\frac{i_n}{n}\right). \tag{1}$$

We split the Riemann sum into three parts, one part for the subintervals all to the left of $\sqrt{2}$ where we know that $f(t_i^*)=1$, one part (a single term) for the subinterval that contains $\sqrt{2}$ where we can't be sure what value $f(t_{i_n}^*)$ takes, and one part for the subintervals to the right of $\sqrt{2}$ where we know that $f(t_i^*)=2$:

$$\begin{aligned}
\sum_{i=1}^{n} f(t_i^*)\,\Delta t &= \sum_{i=1}^{i_n-1} f(t_i^*)\,\Delta t + f(t_{i_n}^*)\,\Delta t + \sum_{i=i_n+1}^{n} f(t_i^*)\,\Delta t\\
&= \sum_{i=1}^{i_n-1} (1)\,\Delta t + f(t_{i_n}^*)\,\Delta t + \sum_{i=i_n+1}^{n} (2)\,\Delta t\\
&= \Delta t \sum_{i=1}^{i_n-1} 1 + f(t_{i_n}^*)\,\Delta t + 2\Delta t \sum_{i=i_n+1}^{n} 1\\
&= \tfrac{1}{n}(i_n-1) + f(t_{i_n}^*)\,\tfrac{1}{n} + \tfrac{2}{n}(n-i_n)\\
&= \tfrac{i_n-1}{n} + f(t_{i_n}^*)\,\tfrac{1}{n} + 2\,\tfrac{n-i_n}{n}.
\end{aligned}$$

Now taking the limit as $n\to\infty$, we use (1) and the fact that $f(t_{i_n}^*)$ is either 1 or 2, so is bounded in any case and the limit of the middle term is zero:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\,\Delta t &= \lim_{n\to\infty}\left[\tfrac{i_n-1}{n} + f(t_{i_n}^*)\,\tfrac{1}{n} + 2\,\tfrac{n-i_n}{n}\right]\\
&= (\sqrt{2}-1)+0+2\,(2-\sqrt{2}),
\end{aligned}$$

which can be seen to be the area under the graph of $f(t)$, between $t=1$ and $t=2$.

This limit does not depend on particular choices of sample points t_i^*, so f is integrable on $[1,2]$, and we are justified in writing

$$\int_1^2 f(t)\,dt = \sqrt{2}-1+2(2-\sqrt{2}) = 3-\sqrt{2}.$$

The contribution to the Riemann sum from the single term $f(t_{i_n}^*)\,\frac{1}{n}$ near the jump discontinuity shrinks to zero as $n\to\infty$ and does not affect integrability or the value of the limit.

(b) If some other function f that is continous everywhere in $[1,2]$ except at $\sqrt{2}$ is unbounded near $\sqrt{2}$, then it might not be integrable. The function should "grow fast enough" near the point of discontinuity like the example on p. 55 of the textbook, for example

$$f(t)=\begin{cases}\frac{1}{t-\sqrt{2}} & \text{if } t\neq\sqrt{2}\\ 0 & \text{if } t=\sqrt{2}\end{cases}$$

is not integrable.