MATH 101 V01 – ASSIGNMENT 2

Solutions

1. (a) Let $f(x)$ be continuous on an open interval that contains $x = 1$. Find an antiderivative $G(x)$ with $G(1) = 1$.

   (b) Solve the *integral equation*

$$f(x) = 1 + 2\int_x^3 f(t)\,dt$$

   for $f(x)$; that is, find an expression for $f(x)$ in terms of elementary functions. After you have found $f(x)$, verify that it is indeed a solution, by substituting your expression for $f(x)$ into the integral equation and evaluating the integral.

Solution:

(a) The function

$$F(x) = \int_1^x f(t)\,dt$$

is an antiderivative of $f(x)$: by the Fundamental Theorem of Calculus, $F'(x) = f(x)$ for x belonging to the open interval stated in the question.

If $G(x)$ is an antiderivative of $f(x)$, then $G'(x) = f(x)$ by definition. Therefore

$$G'(x) = F'(x),$$

and so

$$G(x) = F(x) + c,$$

for some constant c. Choose the constant c by evaluating G at $x = 1$:

$$G(1) = F(1) + c = \int_1^1 f(t)\,dt + c = 0 + c = c.$$

Since this is required to be 1, we take $c = 1$ and then

$$G(x) = \int_1^x f(t)\,dt + 1.$$

(b) Use the property of integrals

$$\int_r^l f(t)\,dt = -\int_l^r f(t)\,dt$$

to write the integral equation as

$$f(x) = 1 - 2\int_3^x f(t)\,dt,$$

then differentiate to obtain

$$\begin{aligned}
f'(x) &= -2\,f(x),\\
\frac{1}{f(x)}\,f'(x) &= -2,\\
(\log|f(x)|)' &= -2,\\
\log|f(x)| &= -2x + k,\\
|f(x)| &= e^{-2x+k} = e^k\,e{-}2x,\\
f(x) &= \pm e^k\,e^{-2x},
\end{aligned}$$

for some constant k. But from the integral equation, for $x = 3$ we must have

$$f(3) = 1,$$

therefore

$$1 = \pm e^k\, e^{-6},$$

so we should take the + sign and $k = 6$, so

$$f(x) = e^6\, e^{-2x} = e^{6-2x}.$$

Verify that $f(x) = e^{6-2x}$ is a solution of the integral equation:

$$\begin{aligned} 1 + 2\int_x^3 f(t)\,dt &= 1 + 2\int_x^3 e^{6-2t}\,dt \\ &= 1 + 2\left(-\tfrac{1}{2}\,e^{6-2t}\right)\Big|_{t=x}^{t=3} \\ &= 1 + 2\left[\left(-\tfrac{1}{2}\,e^{0}\right) - \left(-\tfrac{1}{2}\,e^{6-2x}\right)\right] \\ &= 1 + 2\left[-\tfrac{1}{2} + \tfrac{1}{2}\,e^{6-2x}\right] \\ &= e^{6-2x} \\ &= f(x), \end{aligned}$$

as is required to be a solution.

2. There is a line $y = mx$ through the origin that divides the finite region, bounded by the curve $y = x - x^2$ and the x-axis $y = 0$, into two regions with equal area. Find the slope m of the line.

Solution:
The finite region, referred to in the question, is

$$\{\,(x, y) :\ 0 \le y \le x - x^2,\ 0 \le x \le 1\,\},$$

and its area is

$$\begin{aligned} A &= \int_0^1 (x - x^2)\,dx \\ &= \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1 \\ &= \tfrac{1}{6}. \end{aligned}$$

For the curve (line) $y = mx$ to divide this region into two regions with equal areas, we need

$$m > 0.$$

Find the intersections of $y = x - x^2$ and $y = mx$:

$$\begin{aligned} x - x^2 &= mx, \\ x^2 + (m-1)x &= 0, \\ x(x + m - 1) &= 0, \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = 1 - m,$$

with

$$m < 1.$$

The area of the upper region is easier to find:

$$\begin{aligned} A_{upper} &= \int_0^{1-m} [(x - x^2) - mx]\,dx \\ &= \int_0^{m} [(1-m)x - x^2]\,dx \\ &= \left[\tfrac{1}{2}(1-m)x^2 - \tfrac{1}{3}x^3\right]\Big|_0^{1-m} \\ &= \tfrac{1}{6}(1-m)^3 \end{aligned}$$

Now we should choose $m \in (0,1)$ so that

$$\begin{aligned} A_{upper} &= \tfrac{1}{2}\,A \\ \tfrac{1}{6}(1-m)^3 &= \tfrac{1}{12} \\ (1-m)^3 &= \tfrac{1}{2} \\ 1-m &= \tfrac{1}{2^{1/3}} \\ m &= 1-\tfrac{1}{2^{1/3}}. \end{aligned}$$

The slope of the line is $m = 1 - \frac{1}{2^{1/3}} \approx 0.206 \in (0,1)$.