MATH 101 V01 – ASSIGNMENT 3

Solutions

---

1. (a) Let $a > 0$. Use properties of the integral and integration by substitution to prove for any function f that is continuous on the closed interval $[-a, a]$:
   i. If f is an *even* function ($f(-x) = f(x)$ for all x), then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$
   ii. If f is an *odd* function ($f(-x) = -f(x)$ for all x), then
$$\int_{-a}^{a} f(x)\,dx = 0.$$

   (b) Let $L > 0$ be a constant and let n be a positive integer. Evaluate the two integrals
$$a_n = \frac{1}{L}\int_{-L}^{L} x^2 \cos\left(\frac{n\pi x}{L}\right)\,dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} x^2 \sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Solution:

(a) For both i. and ii. split the integral on $[-a, a]$ into a sum of an integral from $[-a, 0]$ and and integral from $[0, a]$, make the substitution $u = -x$, $du = -dx$ in the first integral, then use the definitions of even and odd functions:

i. For an even function f we have
$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{a}^{0} f(-u)\,(-du) + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

ii. For an odd function f we have
$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{a}^{0} f(-u)\,(-du) + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 0.\end{aligned}$$

(b) For a_n we note that $f(x) = x^2\cos(n\pi x/L)$ is an even function, so by part (a) we have, using integration by parts twice
(in the first integration by parts put $u = x^2$, $dv = \cos(n\pi x/L)\,dx$; $du = 2x\,dx$, $v = (L/n\pi)\sin(n\pi x/L)$, then in the second integration by parts put $u = x$, $dv = \sin(n\pi x/L)\,dx$; $du = dx$, $v = -(L/n\pi)\cos(n\pi x/L)$),

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_{0}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)\,dx \qquad \text{(first integration by parts)} \\
&= \frac{2}{L}\left\{\frac{L}{n\pi}x^2\sin\left(\frac{n\pi x}{L}\right)\bigg|_0^L - \frac{2L}{n\pi}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\,dx\right\} \qquad \text{(second integration by parts)} \\
&= \frac{2}{L}\left\{\frac{L}{n\pi}\,x^2\sin\left(\frac{n\pi x}{L}\right)\bigg|_0^L - \frac{2L}{n\pi}\left[-\frac{L}{n\pi}\,x\cos\left(\frac{n\pi x}{L}\right)\bigg|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx\right]\right\} \\
&= \frac{2}{L}\left\{\frac{L}{n\pi}\,x^2\sin\left(\frac{n\pi x}{L}\right)\bigg|_0^L - \frac{2L}{n\pi}\left[-\frac{L}{n\pi}\,x\cos\left(\frac{n\pi x}{L}\right)\bigg|_0^L + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\bigg|_0^L\right]\right\} \\
&= \frac{2}{n\pi}\,x^2\sin\left(\frac{n\pi x}{L}\right)\bigg|_0^L + \frac{4L}{n^2\pi^2}\,x\cos\left(\frac{n\pi x}{L}\right)\bigg|_0^L - \frac{4L^2}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)\bigg|_0^L \\
&= \frac{4L^2}{n^2\pi^2}\cos(n\pi) \qquad \text{(note that } \cos(n\pi) = (-1)^n\text{)}
\end{aligned}$$

while for b_n we note that $f(x) = x^2\sin(n\pi x/L)$ is an odd function, so by part (a) we have

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= 0
\end{aligned}$$

2. (a) Use integration by parts to derive the following *reduction formula*

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx,$$

where $n \geq 2$ is an integer.

(b) Use the reduction formula to evaluate

$$\int_0^{\pi/2} \cos^4 x\,dx.$$

Solution:

(a) Integrate by parts: put

$$u = \cos^{n-1}x\,dx,\ dv = \cos x\,dx; \quad du = -(n-1)\cos^{n-2}x\sin x\,dx,\ v = \sin x$$

to get

$$\begin{aligned}
\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx \\
&= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx \\
&= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx
\end{aligned}$$

Then rearranging terms we get

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

and dividing by n gives the reduction formula.

(b) Using the reduction formula with $n = 4$, and then with $n = 2$ we get

$$\int \cos^4 \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx,$$

$$\begin{aligned}
\int_0^{\pi/2} \cos^4 \, dx &= \frac{1}{4} \cos^3 x \sin x \Big|_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} \cos^2 x \, dx \\
&= \frac{1}{4} \cos^3 x \sin x \Big|_0^{\pi/2} + \frac{3}{4} \left(\frac{1}{2} \cos x \sin x \Big|_0^{\pi/2} + \frac{1}{2} \int_0^{\pi/2} \cos^0 x \, dx \right) \\
&= \frac{1}{4} \cos^3 x \sin x \Big|_0^{\pi/2} + \frac{3}{8} \cos x \sin x \Big|_0^{\pi/2} + \frac{3}{8} x \Big|_0^{\pi/2} \\
&= 0 + 0 + \frac{3\pi}{16}
\end{aligned}$$