MATH 101 V01 – ASSIGNMENT 4
Solutions

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1. (a) Make the substitution $u = \sqrt{x+3}$ and then find the antiderivative (or indefinite integral)

$$\int \frac{1}{x + 2\sqrt{x+3}}\, dx.$$

(b) Make the substitution $u = \sqrt{x-10}$ and then find the antiderivative (or indefinite integral)

$$\int \frac{1}{x + 2\sqrt{x-10}}\, dx.$$

Solution:
(a) Making the substitution $u = \sqrt{x+3}$, we calculate $du = dx/(2\sqrt{x+3})$, or

$$x = u^2 - 3, \quad dx = 2u\, du.$$

Then the antiderivative becomes

$$\begin{aligned}\int \frac{1}{x + 2\sqrt{x+3}}\, dx &= \int \frac{1}{(u^2-3)+2u}\; 2u\; du \\ &= 2\int \frac{u}{u^2+2u-3}\, du,\end{aligned}$$

and now we must find the antiderivative of a proper rational function. (This type of substituion is called a *rationalizing* substitution, because it results in the antiderivative of a rational function.)
Notice the denominator is a reducible quadratic polynomial ($2^2 - 4(-3) > 0$) so we know it factors (over the real numbers) into a product of linear factors. Factoring the denominator and making a partial fraction decomposition, we get

$$\frac{u}{u^2+2u-3} = \frac{u}{(u+3)(u-1)},$$

and making a partial fraction decomposition, we have

$$\frac{u}{(u+3)(u-1)} = \frac{A}{u+3} + \frac{B}{u-1},$$

for some constants A and B, which we now find. Putting the right-hand side of the previous equation over the common denominator we get

$$\frac{u}{(u+3)(u-1)} = \frac{A(u-1)+B(u+3)}{(u+3)(u-1)}$$

and comparing the numerators in the above equation, we must find A, B to satisfy

$$u = A(u-1) + B(u+3), \tag{1}$$

for all u. Putting $u = 1$ we get $1 = 4B$, and putting $u = -3$ we get $-3 = -4A$, therefore

$$A = \tfrac{3}{4}, \quad B = \tfrac{1}{4}.$$

(Alternatively, we could expand (1) and collect coefficients of powers of u, obtaining the linear system of two equations in two unknowns A and B,

$$\begin{aligned}-A + 3B &= 0, \\ A + B &= 1,\end{aligned}$$

which can be solved to obtain the same values of A and B.)

Now returning to the antiderivative problem, we have

$$\begin{aligned} 2\int \frac{u}{u^2+2u-3}\,du &= \tfrac{3}{2}\int \frac{1}{u+3}\,du + \tfrac{1}{2}\int \frac{1}{u-1}\,du \\ &= \tfrac{3}{2}\log|u+3| + \tfrac{1}{2}\log|u-1| + C, \end{aligned}$$

where C is an arbitrary constant. Finally, expressing the antiderivative in terms of the original variable x we have

$$\int \frac{1}{x+2\sqrt{x+3}}\,dx = \tfrac{3}{2}\log(\sqrt{x+3}+3) + \tfrac{1}{2}\log|\sqrt{x+3}-1| + C,$$

for $x > -3$.

(b) Making the rationalizing substitution $u = \sqrt{x-10}$, we calculate $du = dx/(2\sqrt{x-10})$, or

$$x = u^2 + 10, \quad dx = 2u\,du.$$

Then the antiderivative becomes

$$\begin{aligned} \int \frac{1}{x+2\sqrt{x-10}}\,dx &= \int \frac{1}{(u^2+10)+2u}\,2u\,du \\ &= 2\int \frac{u}{u^2+2u+10}\,du, \end{aligned}$$

and now we must again find the antiderivative of a proper rational function.

But this time, we notice that the denominator is an irreducible quadratic polynomial ($2^2 - 4\cdot 10 < 0$) so it cannot be factored (over the real numbers) any further, it is *already* decomposed into partial fractions. To deal with this, we first recognize that the derivative of the denominator $2u+2$ is a constant multiple of $u+1$, so we arrange for $u+1$ to appear in the numerator, and split the expression into two parts:

$$\begin{aligned} \frac{u}{u^2+2u+10} &= \frac{u+1-1}{u^2+2u+10} \\ &= \frac{u+1}{u^2+2u+10} - \frac{1}{u^2+2u+10}. \end{aligned}$$

Now the antiderivative of the first term on the right hand side of the equation can be found by "guess and check", or at least with a simple substitution. In the second term on the right hand side we complete the square in the denominator

$$\frac{1}{u^2+2u+10} = \frac{1}{(u+1)^2+9},$$

so we have

$$\frac{u}{u^2+2u+10} = \frac{u+1}{u^2+2u+10} - \frac{1}{(u+1)^2+3^2}.$$

It is handy to remember that $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$ for any constant $a > 0$, which was derived in class. The two antiderivatives we need are

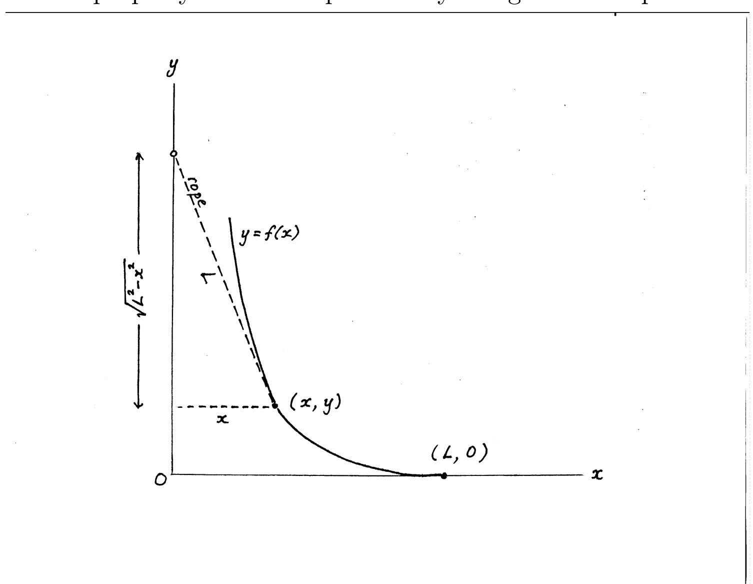
$$\int \frac{u+1}{u^2+2u+10}\,du = \tfrac{1}{2}\log(u^2+2u+10) + C, \qquad \int \frac{1}{(u+1)^2+3^2}\,du = \tfrac{1}{3}\arctan\left(\tfrac{u+1}{3}\right) + C.$$

Finally, assembling all the pieces and expressing the antiderivative in terms of the original variable x, we have

$$\int \frac{1}{x+2\sqrt{x-10}}\,dx = \log(x+2\sqrt{x-10}) - \tfrac{2}{3}\arctan\left(\tfrac{\sqrt{x-10}+1}{3}\right) + C,$$

for $x > 10$.

2. Suppose the open first quadrant of the xy-plane (i.e. $x > 0$, $y > 0$) represents water, viewed from above. The rest of the xy-plane represents solid land. You are initially at the origin, holding the end of a rope of length $L > 0$ which is tied to a boat at the point $(L, 0)$ in the xy-plane. Then you walk along the positive y-axis, keeping the rope straight and taut (assume you are strong enough to pull the boat, the rope doesn't stretch, and you don't fall into the water trying to balance on the y-axis). The boat follows a path $y = f(x)$, with the property that the rope is always tangent to the path of the boat (figure below).



(a) Show that

$$f'(x) = \frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}.$$

(b) Determine the function $y = f(x)$.

Solution:

(a) Let $(x_0, f(x_0))$ on the path be the point of attachment to the boat (we use x_0 so as not to confuse it with the variable x in the equation for the tangent line). The tangent line to the curve at that point has the equation

$$y = f(x_0) + f'(x_0)(x - x_0).$$

The y-intercept of this tangent line (where you are holding one end of the rope) is

$$y_1 = f(x_0) - f'(x_0)x_0.$$

The length of the rope is L, it forms the hyponenuse of a right triangle (see the figure), and its ends are $(0, y_1)$ and $(x_0, f(x_0))$, so we can write

$$\begin{aligned} L^2 &= x_0^2 + [f(x_0) - y_1]^2 \\ &= x_0^2 + [f'(x_0)x_0]^2 \\ &= x_0^2\{1 + [f'(x_0)]^2\}. \end{aligned}$$

Solving for $f'(x_0)$, we get

$$\begin{aligned} [f'(x_0)]^2 &= \frac{L^2}{x_0^2} - 1 \\ &= \frac{L^2 - x_0^2}{x_0^2}, \\ f'(x_0) &= \pm\frac{\sqrt{L^2 - x_0^2}}{x_0^2}, \end{aligned}$$

for any $0 < x_0 < L$. We take the $-$ sign because the slope of the tangent line is negative (see the figure). Alternatively, from the figure (and Pythagoras' Theorem), the difference between the y-values of your position $(0, y_1)$ and the point of attachment to the boat $(x_0, f(x_0))$ is $\sqrt{L^2 - x_0^2}$, the length of the vertical leg of the right triangle,

$$y_1 - f(x_0) = -f'(x_0)x_0 = \sqrt{L^2 - x_0^2}.$$

So for any point $(x, f(x))$ on the curve, we have

$$f'(x) = -\frac{\sqrt{L^2 - x^2}}{x^2} \qquad (0 < x < L).$$

(b) From part (a), $f(x)$ is an antiderivative of $-\sqrt{L^2 - x^2}/x^2$,

$$f(x) = \int \left(-\frac{\sqrt{L^2 - x^2}}{x^2} \right) dx.$$

To find this antiderivative, make a trigonometric substitution

$$x = L \sin\theta,$$

then

$$dx = L\cos\theta \, d\theta, \qquad \sqrt{L^2 - x^2} = L\cos\theta,$$

and

$$\begin{aligned} f(x) = \int \left(-\frac{\sqrt{L^2 - x^2}}{x^2} \right) dx &= -\int \frac{L\cos\theta}{L\sin\theta} L\cos\theta \, d\theta \\ &= -L \int \frac{\cos^2\theta}{\sin\theta} \, d\theta \\ &= -L \int \frac{1 - \sin^2\theta}{\sin\theta} \, d\theta \\ &= -L \int \left(\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta} \right) d\theta \\ &= -L \int (\csc\theta - \sin\theta) \, d\theta \\ &= -L(\log|\csc\theta - \cot\theta| + \cos\theta) + C \\ &= -L \log \left| \frac{L}{x} - \frac{\sqrt{L^2 - x^2}}{x} \right| - \sqrt{L^2 - x^2} + C, \end{aligned}$$

where C is some constant. Now we have to find the constant, which is possible because we know that when $x = L$, the boat starts with

$$y = f(L) = 0,$$

so using our formula for $f(x)$, we must have

$$f(L) = -L \log|1| - 0 + C = 0.$$

Therefore $C = 0$ and

$$f(x) = -L \log \left| \frac{L}{x} - \frac{\sqrt{L^2 - x^2}}{x} \right| - \sqrt{L^2 - x^2}.$$

Above, we have used the antiderivative of $\csc\theta$,

$$\int \csc\theta\, d\theta = \log|\csc\theta - \cot\theta| + C,$$

which can be found in most calculus textbooks, or derived by putting

$$\begin{aligned}\csc\theta &= \csc\theta\,\frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}\\ &= \frac{-\csc\theta\cot\theta + \csc^2\theta}{\csc\theta - \cot\theta}\end{aligned}$$

and recalling that

$$\frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta, \quad \frac{d}{d\theta}\cot\theta = -\csc^2\theta,$$

so that making the substitution

$$u = \csc\theta - \cot\theta, \quad du = (-\csc\theta\cot\theta + \csc^2\theta)\, d\theta$$

we get

$$\begin{aligned}\int \csc\theta\, d\theta &= \int \frac{-\csc\theta\cot\theta + \csc^2\theta}{\csc\theta - \cot\theta}\, d\theta\\ &= \int \frac{1}{u}\, du\\ &= \log|u| + C\\ &= \log|\csc\theta - \cot\theta| + C.\end{aligned}$$