MATH 101 V01 – ASSIGNMENT 5

Solutions

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1. (a) Determine all values of the real number p such that the following integral converges:

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

If the integral converges, find its value.

(b) Determine all values of the real number q such that the following integral converges:

$$\int_0^1 \frac{1}{x^q}\,dx.$$

If the integral converges, find its value.

Solution:

(a) Here p is a constant. Because the formula for the antiderivative is different, depending on whether $p \neq 1$ or $p = 1$, we consider the two cases separately.

First, suppose $p \neq 1$ (so either $p < 1$ or $p > 1$). Then

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{r\to\infty}\int_1^r \frac{1}{x^p}\,dx \\
&= \lim_{r\to\infty}\int_1^r x^{-p}\,dx \\
&= \lim_{r\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right|_1^r \\
&= \lim_{r\to\infty}\frac{1}{1-p}\left[r^{1-p}-1\right] \quad (p \neq 1).
\end{aligned}$$

Now if $p < 1$, then $1 - p > 0$ (i.e. $1 - p$ is a positive power) so that the limit

$$\lim_{r\to\infty} r^{1-p} = \infty \quad (p < 1)$$

diverges and the improper integral diverges.

On the other hand if $p > 1$, then $p - 1$ is a positive power, $\lim_{r\to\infty} r^{p-1} = \infty$, and

$$\lim_{r\to\infty} r^{1-p} = \lim_{r\to\infty}\frac{1}{r^{p-1}} = 0 \quad (p > 1)$$

so this limit converges and the imporoper integral converges, to

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{r\to\infty}\frac{1}{1-p}\left[\frac{1}{r^{p-1}}-1\right] = \frac{1}{1-p}[0-1] = \frac{1}{p-1} \quad (p > 1).$$

Next, we consider the remaining case $p = 1$.

$$\begin{aligned}
\int_1^\infty \frac{1}{x}\,dx &= \lim_{r\to\infty}\int_1^r \frac{1}{x}\,dx \\
&= \lim_{r\to\infty}\log(|x|)\big|_1^r \\
&= \lim_{r\to\infty}[\log(r)-\log(1)] = \lim_{r\to\infty}\log(r) \quad (p = 1).
\end{aligned}$$

But $\lim_{r\to\infty} \log(r)$ diverges (to infinity), so the improper integral diverges.

Summarizing, all values of the real number p such that the integral converges are those for which $p > 1$, and in this case the value of the integral is

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad (p > 1).$$

For $p \le 1$ the integral diverges.

(b) Again, q is a constant, and we consider separately the cases $q \neq 1$ and $q = 1$.

First, suppose $q \neq 1$ (so either $q < 1$ or $q > 1$). There is a possible discontinuity (depending on the value of q; if $q \le 0$ there is no discontinuity) at the left endpoint $x = 0$, so we compute

$$\begin{aligned}
\int_0^1 \frac{1}{x^q}\,dx &= \lim_{\ell\to 0+} \int_\ell^1 \frac{1}{x^q}\,dx \\
&= \lim_{\ell\to 0+} \int_\ell^1 x^{-q}\,dx \\
&= \lim_{\ell\to 0+} \left.\frac{x^{-q+1}}{-q+1}\right|_\ell^1 \\
&= \lim_{\ell\to 0+} \frac{1}{1-q}\left[1 - \ell^{1-q}\right] \quad (q \neq 1).
\end{aligned}$$

Now if $q > 1$ then $1 - q$ is a negative power and the limit

$$\lim_{\ell\to 0+} \ell^{1-q} = \lim_{\ell\to 0+} \frac{1}{\ell^{q-1}} = \infty \quad (q > 1)$$

diverges and the improper integral diverges.
On the other hand if $q < 1$ then $1 - q$ is a positive power and the limit

$$\lim_{\ell\to 0+} \ell^{1-q} = 0 \quad (q < 1)$$

converges, and the improper integral converges

$$\int_0^1 \frac{1}{x^q}\,dx = \lim_{\ell\to 0+} \frac{1}{1-q}\left[1 - \ell^{1-q}\right] = \frac{1}{1-q} \quad (q < 1).$$

Now the remaining case is $q = 1$.

$$\begin{aligned}
\int_0^1 \frac{1}{x}\,dx &= \lim_{\ell\to 0+} \int_\ell^1 \frac{1}{x}\,dx \\
&= \lim_{\ell\to 0+} \log(|x|)\big|_\ell^1 \\
&= \lim_{\ell\to 0+} [\log(1) - \log(\ell)] = -\lim_{\ell\to 0+} \log(\ell) \quad (q = 1).
\end{aligned}$$

But $\lim_{\ell\to 0+} \log(\ell)$ diverges (to negative infinity), so the improper integral diverges.

In summary, all values of the real number q such that the integral converges are those for which $q < 1$, and in this case the value of the integral is

$$\int_0^1 \frac{1}{x^q}\,dx = \frac{1}{1-q} \quad (q < 1).$$

For $q \ge 1$ the integral diverges.

2. Let R be the bounded region between the two curves $y = \sqrt[4]{x}$ and $y = x$. Find the volume of the solid that is generated by rotating the region R about the vertical line $x = 1$:
   (a) Using slices.
   (b) Using cylindrical shells.

Solution:
First we note that $\sqrt[4]{x}$ is only defined for $x \geq 0$. Then we find the intersection(s) of the two curves $y = \sqrt[4]{x}$ and $y = x$, by setting

$$\sqrt[4]{x} = x \quad (x \geq 0),$$

which is equivalent to

$$\begin{aligned} \sqrt[4]{x} - x &= 0 \quad (x \geq 0) \\ \sqrt[4]{x}[1 - (\sqrt[4]{x})^3] &= 0 \quad (x \geq 0) \end{aligned}$$

therefore

$$\sqrt[4]{x} = 0 \quad \text{or} \quad (\sqrt[4]{x})^3 = 1 \quad (x \geq 0).$$

The only solution of the first equation is $x = 0$, and the only solution of the second equation is $x = 1$, so we have found all the intersections of the two curves, at $(x, y) = (0, 0)$ and at $(x, y) = (1, 1)$. Furthermore, for $x \geq 1$ the region between the two curves is unbounded, and the bounded region is

$$R = \{\, (x, y) : \quad x \leq y \leq \sqrt[4]{x}, \quad 0 \leq x \leq 1 \,\}.$$

(a) Using slices, perpendicular to the axis of rotation, the slices are horizontal with infinitesimal thickness dy, so the integrand and limits of integration should be expressed in terms of y, and the two curves are expressed as $x = y$ and $x = y^4$. The slices are washers, with outer (large) radius $1 - y^4$ (the positive distance between the x-value on the farther curve at height y, to the x-value 1 of the axis of rotation) and and inner (small) radius $1 - y$ (the positive distance between the nearer curve at height y, to the vertical axis of rotation). The volume is

$$\begin{aligned} V &= \int_{y=0}^{y=1} [\pi(1 - y^4)^2 - \pi(1 - y)^2]\, dy \\ &= \pi \int_0^1 [(1 - 2y^4 + y^8) - (1 - 2y + y^2)]\, dy \\ &= \pi \int_0^1 [2y - y^2 - 2y^4 + y^8]\, dy \\ &= \pi \left(y^2 - \tfrac{1}{3}y^3 - \tfrac{2}{5}y^5 + \tfrac{1}{9}y^9\right)\Big|_0^1 \\ &= \pi \left(1 - \tfrac{1}{3} - \tfrac{2}{5} + \tfrac{1}{9}\right) = \tfrac{17}{45}\pi. \end{aligned}$$

(b) Using cylindrical shells, parallel to the axis of rotation, the cylindrical shells are vertical with infinitesimal thickness dx, so the integrand and limits of integration should be expressed in terms of x. The radius of each cylindrical shell is $1 - x$ (the positive distance from typical position x to the axis of rotation $x = 1$) and the height of each cylindrical shell is the positive distance between the y-values of the two curves at position x, which is $\sqrt[4]{x} - x$. The volume is

$$\begin{aligned} V &= \int_{x=0}^{x=1} 2\pi(1 - x)(\sqrt[4]{x} - x)\, dx \\ &= 2\pi \int_0^1 (x^{1/4} - x^{5/4} - x + x^2)\, dx \\ &= 2\pi \left(\tfrac{4}{5}x^{5/4} - \tfrac{4}{9}x^{9/4} - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right)\Big|_0^1 \\ &= 2\pi \left(\tfrac{4}{5} - \tfrac{4}{9} - \tfrac{1}{2} + \tfrac{1}{3}\right) = \tfrac{17}{45}\pi. \end{aligned}$$

(Of course, the answers to parts (a) and (b) should agree.)