MATH 101 V01 – ASSIGNMENT 6

Solutions

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1. Use the Integral Test to determine if the series is convergent or divergent.

   (a) $1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \dots$

   (b) $\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^3}$

   (c) $\sum_{n=2}^{\infty} \frac{\log(n^2)}{n}$

Solution:

(a)

$$1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}.$$

The function $f(x) = \frac{1}{\sqrt{2x-1}} = (2x-1)^{-1/2}$ is continuous for $x > \frac{1}{2}$, positive for $x > \frac{1}{2}$, and

$$f'(x) = -\frac{1}{(2x-1)^{3/2}} < 0,$$

so $f(x)$ is decreasing for $x > \frac{1}{2}$. Therefore f is continuous, positive and decreasing on $[1, \infty)$. The improper integral

$$\begin{aligned}
\int_1^{\infty} \frac{1}{\sqrt{2x-1}}\,dx &= \lim_{t\to\infty} \int_1^t (2x-1)^{-1/2}\,dx \quad \text{(substitution: } u = 2x-1,\ du = 2\,dx) \\
&= \lim_{t\to\infty} \frac{1}{2} \int_1^{2t-1} u^{-1/2}\,du \\
&= \lim_{t\to\infty} u^{1/2}\Big|_1^{2t-1} \\
&= \lim_{t\to\infty} (\sqrt{2t-1} - 1) = \infty
\end{aligned}$$

is divergent. By the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ also is divergent.

(b) The function $f(x) = \frac{1}{x(\log(x))^3}$ is continuous for $x > 0$, positive for $x > 1$, and

$$f'(x) = -\frac{3 + \ln x}{x^2(\log(x))^4} < 0 \text{ for } x > e^{-3},$$

so $f(x)$ is decreasing for $x > e^{-3} \approx 0.05$. Therefore f is continuous, positive and decreasing on $[2, \infty)$. The improper integral

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\log(x))^3}\,dx &= \lim_{t\to\infty} \int_2^t (\log(x))^{-3}\,\frac{1}{x}\,dx \quad \text{(substitution: } u = \log(x),\ du = \tfrac{1}{x}\,dx) \\
&= \lim_{t\to\infty} \int_{\log(2)}^{\log(t)} u^{-3}\,du \\
&= \lim_{t\to\infty} \left(-\frac{1}{2}u^{-2}\right)\Big|_{\log(2)}^{\log(t)} \\
&= \lim_{t\to\infty} \left(-\frac{1}{2(\log(t))^2} + \frac{1}{2(\log(2))^2}\right) = \frac{1}{2(\log(2))^2}
\end{aligned}$$

is convergent. By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^3}$ also is convergent.

(c) The function $f(x) = \frac{\log(x^2)}{x} = \frac{2\log(x)}{x}$ is continuous for $x > 0$, positive for $x > 1$, and

$$f'(x) = \frac{2(1 - \log(x))}{x^2} < 0 \text{ for } x > e,$$

so $f(x)$ is decreasing for $x \geq e \approx 2.7$. Therefore f is continuous, positive and decreasing on $[3, \infty)$ (it is decreasing on $[e, \infty)$).
The improper integral

$$\begin{aligned}
\int_3^{\infty} \frac{\log(x^2)}{x}\, dx &= \lim_{t\to\infty} \int_3^t 2(\log(x))\, \frac{1}{x}\, dx \quad \text{(substitution: } u = \log(x),\ du = \tfrac{1}{x}\, dx) \\
&= \lim_{t\to\infty} \int_{\log(3)}^{\log(t)} 2u\, du \\
&= \lim_{t\to\infty} u^2\Big]_{\log(3)}^{\log(t)} \\
&= \lim_{t\to\infty} (\log(t))^2 - (\log(3))^2 = \infty
\end{aligned}$$

is divergent. By the Integral Test, $\sum_{n=3}^{\infty} \frac{\log(n^2)}{n}$ also is divergent, and so is

$$\sum_{n=2}^{\infty} \frac{\log(n^2)}{n} = \frac{\log(2^2)}{2} + \sum_{n=3}^{\infty} \frac{\log(n^2)}{n}.$$

2. Find a power series representation for the function and determine the interval of convergence.

(a) $f(x) = \frac{x^3}{4x^2+3}$
(b) $f(x) = \frac{x+2}{2x^2-x-1}$
(c) $f(x) = \ln(3 + x)$
(d) $f(x) = \arctan(3x)$
(e) $f(x) = \frac{2x}{(1+x^2)^2}$

Solution:

(a) Use some algebra to express the given function in terms of a geometric series:

$$\begin{aligned}
\frac{x^3}{4x^2+3} &= \frac{x^3}{3}\, \frac{1}{1 - \left(-\frac{4x^2}{3}\right)} \\
&= \tfrac{x^3}{3}\left[1 + \left(-\tfrac{4x^2}{3}\right) + \left(-\tfrac{4x^2}{3}\right)^2 + \left(-\tfrac{4x^2}{3}\right)^3 + \dots\right] \quad \text{if } \left|-\tfrac{4x^2}{3}\right| < 1 \text{ i.e. if } |x| < \tfrac{\sqrt{3}}{2} \\
&= \tfrac{1}{3}x^3 - \tfrac{4}{3^2}x^5 + \tfrac{4^2}{3^3}x^7 - \tfrac{4^3}{3^4}x^9 + \dots \\
&= \sum_{n=0}^{\infty} (-1)^n \tfrac{4^n}{3^{n+1}} x^{2n+3} \quad \text{if } |x| < \tfrac{\sqrt{3}}{2}
\end{aligned}$$

The interval of convergence is $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ (geometric series are always divergent at the endpoints of their interval of convergence), or

$$-\tfrac{\sqrt{3}}{2} < x < \tfrac{\sqrt{3}}{2}.$$

(b) First we use a partial fraction decomposition. Factor the denominator as $2x^2-x-1=(2x+1)(x-1)$

$$\frac{x+2}{(2x+1)(x-1)}=\frac{A}{2x+1}+\frac{B}{x-1}$$

gives $x+2=A(x-1)+B(2x+1)$, then $A=-1$, $B=1$.

$$\begin{aligned}
\frac{x+2}{2x^2-x-1} &= \frac{-1}{2x+1}+\frac{1}{x-1}\\
&= -\frac{1}{1-(-2x)}-\frac{1}{1-x}\\
&= -\left[1+(-2x)+(-2x)^2+(-2x)^3+\dots\right]-[1+x+x^2+x^3+\dots]\\
&\qquad \text{if } |-2x|<1 \text{ and } |x|<1, \text{ i.e. if } |x|<\tfrac{1}{2}\\
&= -1+2x-2^2x^2+2^3x^3-\dots-1-x-x^2-x^3-\dots\\
&= -2+(2-1)x+(-2^2-1)x^2+(2^3-1)x^3+\dots\\
&= \sum_{n=0}^{\infty}[(-1)^{n+1}2^n-1]x^n \quad \text{if } |x|<\tfrac{1}{2}
\end{aligned}$$

The interval of convergence is $\left(-\frac{1}{2},\frac{1}{2}\right)$ (geometric series are always divergent at the endpoints of their interval of convergence), or

$$-\tfrac{1}{2}<x<\tfrac{1}{2}.$$

(Alternatively, since the centre of the series was not specified, one could complete the square

$$2x^2-x-1=2(x^2-\tfrac{1}{2}x-\tfrac{1}{2})=2\left[\left(x-\tfrac{1}{4}\right)^2-\tfrac{9}{16}\right]=-\tfrac{9}{8}\left[1-\left(\tfrac{4x-1}{3}\right)^2\right],$$

write

$$\frac{x+2}{2x^2-x-1}=\frac{x-\frac{1}{4}+\frac{9}{4}}{2x^2-x-1}=-\tfrac{8}{9}\,\left[\tfrac{9}{4}+\tfrac{3}{4}\left(\tfrac{4x-1}{3}\right)\right]\,\frac{1}{1-\left(\frac{4x-1}{3}\right)^2},$$

and expand the term on the right in a geometric series that converges for $\left|\left(\frac{4x-1}{3}\right)^2\right|<1$ or

$$-\tfrac{1}{2}<x<1$$

with centre $\frac{1}{4}$, radius of convergence $\frac{3}{4}$).

(c) Note that $f'(x)=\frac{1}{3+x}$, so we have

$$\begin{aligned}
\log(3+x) &= \textstyle\int \frac{1}{3+x}\,dx\\
&= \int \frac{1}{3}\,\frac{1}{1-(-\frac{x}{3})}\,dx\\
&= \tfrac{1}{3}\int \left[1+\left(-\tfrac{x}{3}\right)+\left(-\tfrac{x}{3}\right)^2+\left(-\tfrac{x}{3}\right)^3+\dots\right]dx \quad \text{if } \left|-\tfrac{x}{3}\right|<1 \text{ i.e. if } |x|<3\\
&= \tfrac{1}{3}\int \left[1-\tfrac{1}{3}x+\tfrac{1}{3^2}x^2-\tfrac{1}{3^3}x^3+\dots\right]dx\\
&= \tfrac{1}{3}\left[x-\tfrac{1}{2\cdot 3}x^2+\tfrac{1}{3\cdot 3^2}x^3-\tfrac{1}{4\cdot 3^3}x^4+\dots+c\right] \quad \text{substitute } x=0 \text{ to get } c=3\log(3), \text{ or } \tfrac{1}{3}\,c=\log(3)\\
&= \log(3)+\tfrac{1}{3}x-\tfrac{1}{2\cdot 3^2}x^2+\tfrac{1}{3\cdot 3^3}x^3-\tfrac{1}{4\cdot 3^4}x^4+\dots\\
&= \log(3)+\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{n\cdot 3^n}x^n \quad \text{if } |x|<3
\end{aligned}$$

The radius of convergence is $R = 3$, the same as for $1/(1-(-x/3))$, but since we have integrated or differentiated, convergence at the endpoints of the interval of convergence must be investigated separately.
$x = 3$ gives $\log(3) + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$ which is convergent (alternating series test).
$x = -3$ gives $\log(3) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \dots$ which is divergent (harmonic series).
The interval of convergence is $(-3, 3]$, or

$$-3 < x \leq 3.$$

(d) From the February 26 lecture we have

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

with radius of convergence 1, and interval of convergence

$$-1 \leq x \leq 1.$$

Now replace x with $3x$:

$$\begin{aligned} \tan^{-1}(3x) &= 3x - \frac{(3x)^3}{3} + \frac{(3x)^5}{5} - \frac{(3x)^7}{7} + \dots \\ &= 3x - \tfrac{3^3}{3}x^3 + \tfrac{3^5}{5}x^5 - \tfrac{3^7}{7}x^7 + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \tfrac{3^{2n+1}}{2n+1} x^{2n+1} \end{aligned}$$

which is convergent for $-1 \leq 3x \leq 1$, i.e. in the interval of convergence $\left[-\frac{1}{3}, \frac{1}{3}\right]$, or

$$-\tfrac{1}{3} \leq x \leq \tfrac{1}{3}.$$

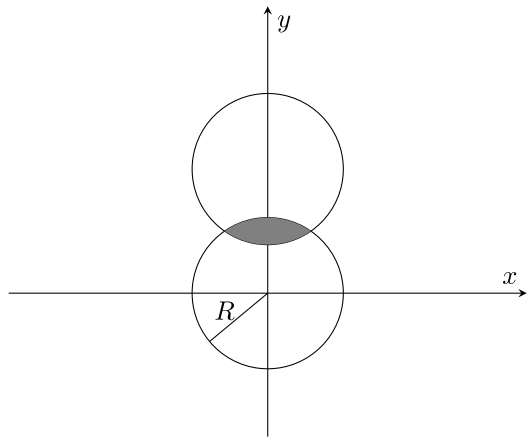
(e) We observe that the given function is the derivative of a function that can be expressed in terms of the geometric series:

$$\begin{aligned} \frac{2x}{(1+x^2)^2} &= -\tfrac{d}{dx}\left[\frac{1}{1+x^2}\right] \\ &= -\tfrac{d}{dx}\left[\frac{1}{1-(-x^2)}\right] \\ &= -\tfrac{d}{dx}\left[1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots\right] \quad \text{if } |-x^2| < 1\text{, i.e. if } |x| < 1 \\ &= -\tfrac{d}{dx}\left[1 - x^2 + x^4 - x^6 + \dots\right] \\ &= -\left[-2x + 4x^3 - 6x^5 + \dots\right] \\ &= 2x - 4x^3 + 6x^5 - \dots \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} 2n x^{2n-1} \end{aligned}$$

The radius of convergence is $R = 1$, the same as for $1/(1+x^2)$, but since we have integrated or differentiated, convergence at the endpoints of the interval of convergence must be investigated separately.
$x = 1$ gives $2 - 4 + 6 - \dots$ which is a divergent alternating series (Divergence Test).
$x = -1$ gives $-2 + 4 - 6 + \dots$ which which is also a divergent alternating series (Divergence Test).
The interval of convergence is $(-1, 1)$, or

$$-1 < x < 1.$$

3. Let D_1 be the closed disk (circle together with its interior region) of radius R centred at the origin and let D_2 be the closed disk of radius R centred at the point $(0, \sqrt{3}R)$. Determine the area of the region of the intersection (or overlap) of D_1 and D_2 (the shaded region in the figure below).



Solution:

The closed disk D_1 is $x^2+y^2 \le R^2$, its boundary is the circle $x^2+y^2 = R^2$, and the *top half* of this circle is

$$y = \sqrt{R^2 - x^2} \quad (-R \le x \le R).$$

The closed disk D_2 is $x^2 + (y - \sqrt{3}\,R)^2 \le R^2$, its boundary is the circle $x^2 + (y - \sqrt{3}\,R)^2 = R^2$, and the *bottom half* of this circle is

$$y = \sqrt{3}\,R - \sqrt{R^2 - x^2} \quad (-R \le x \le R).$$

We find the intersections of the two curves by solving

$$\begin{aligned} \sqrt{R^2 - x^2} &= \sqrt{3}\,R - \sqrt{R^2 - x^2} \\ 2\sqrt{R^2 - x^2} &= \sqrt{3}\,R \\ 4(R^2 - x^2) &= 3R^2 \\ R^2 &= 4x^2 \end{aligned}$$

so the intersection points are at

$$x = -\frac{R}{2}, \quad \frac{R}{2},$$

and the area of the region of the intersection of D_1 and D_2 is (noting that the integrand is an even function of x)

$$\begin{aligned} \int_{-R/2}^{R/2} \left[\sqrt{R^2 - x^2} - \left(\sqrt{3}\,R - \sqrt{R^2 - x^2}\right)\right] dx &= 2\int_0^{R/2} \left[\sqrt{R^2 - x^2} - \left(\sqrt{3}\,R - \sqrt{R^2 - x^2}\right)\right] dx \\ &= 4\int_0^{R/2} \sqrt{R^2 - x^2}\, dx - 2\sqrt{3}\,R \int_0^{R/2} dx \\ &= 4\int_0^{R/2} \sqrt{R^2 - x^2}\, dx - \sqrt{3}\,R^2. \end{aligned}$$

In the last integral make a trigonometric substitution

$$x = R\sin(\theta), \quad dx = R\cos(\theta)\, d\theta, \quad \sqrt{R^2 - x^2} = R\cos(\theta),$$

to get

$$\begin{aligned}
\int_0^{R/2} \sqrt{R^2 - x^2}\, dx &= \int_0^{\pi/6} R\cos(\theta)(R\cos(\theta)\, d\theta) \\
&= R^2 \int_0^{\pi/6} \cos^2(\theta)\, d\theta \\
&= R^2 \int_0^{\pi/6} \tfrac{1}{2}[1 + \cos(2\theta)]\, d\theta \\
&= R^2\ \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin(2\theta)\right]\Big|_0^{\pi/6} \\
&= R^2 \tfrac{1}{2}\left[\tfrac{\pi}{6} + \tfrac{1}{2}\sin\left(\tfrac{\pi}{3}\right)\right] \\
&= R^2 \tfrac{1}{2}\left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{4}\right)
\end{aligned}$$

and therefore the area is

$$\begin{aligned}
4\int_0^{R/2} \sqrt{R^2 - x^2}\, dx - \sqrt{3}\, R^2 &= R^2\left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}\right) - \sqrt{3}\, R^2 \\
&= \left(\tfrac{\pi}{3} - \tfrac{\sqrt{3}}{2}\right) R^2
\end{aligned}$$

(which is positive).