MATH 101 V01 – ASSIGNMENT 7

Solutions

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1. (a) Use linear approximation to estimate $\log(0.98)$.
   (b) Find the degree 2 Taylor polynomial $T_2(x)$, of the function $f(x) = x^{5/2}$, about $x = 4$.
   (c) Find the degree 3 Taylor polynomial $T_3(x)$, of the function $f(x) = \sqrt{x}$, about $x = 4$.
   (d) Find the degree 5 Taylor polynomial $T_5(x)$, of the function $f(x) = \cos(x)$, about $x = \frac{\pi}{3}$.
   (e) Find the degree 8 Taylor polynomial $T_8(x)$, of the function $f(x) = \cos(x)$, about $x = 0$ (a Taylor polynomial about $x = 0$ is called a **Maclaurin polynomial**).

   Solution: For all parts, we use the formula for the Taylor polynomial $T_n(x)$.

   (a) Let $f(x) = \log(x)$, $c = 1$, $n = 1$. Then

$$f(x) = \log(x), \quad f'(x) = \frac{1}{x},$$

   and evaluating at $x = c = 1$, we get

$$f(1) = 0, \quad f'(1) = 1,$$

   and the linear approximation of $\log(x)$ about $x = 1$ is

$$L(x) = T_1(x) = f(1) + f'(1)(x-1) = 0 + (x-1) = x - 1.$$

   Then $L(0.98) = 0.98 - 1 = -0.02$, So we estimate

$$\log(0.98) \approx -0.02.$$

   (The actual value of $\log(0.98)$ is -0.020203 accurate to six decimal places.)

   (b) Let $f(x) = x^{5/2}$, $c = 4$, $n = 2$. Then

$$f(x) = x^{5/2}, \quad f'(x) = \tfrac{5}{2}x^{3/2}, \quad f''(x) = \tfrac{15}{4}x^{1/2},$$

   and evaluating at $x = c = 4$, we get

$$f(4) = 4^{5/2} = (\sqrt{4})^5 = 32, \quad f'(4) = \tfrac{5}{2}4^{3/2} = \tfrac{5}{2}(\sqrt{4})^3 = 20, \quad f''(x) = \tfrac{15}{4}4^{1/2} = \tfrac{15}{2},$$

   and the degree 2 Taylor polynomial of $f(x) = x^{5/2}$ about $x = 4$ is

$$T_2(x) = 32 + 20(x-4) + \tfrac{15}{4}(x-4)^2.$$

   (c) Let $f(x) = \sqrt{x}$, $c = 4$, $n = 3$. Then

$$f(x) = x^{1/2}, \quad f'(x) = \tfrac{1}{2}x^{-1/2}, \quad f''(x) = -\tfrac{1}{4}x^{-3/2}, \quad f^{(3)}(x) = \tfrac{3}{8}x^{-5/2},$$

   and evaluating at $x = c = 4$, we get

$$f(4) = 2, \quad f'(4) = \tfrac{1}{4}, \quad f''(4) = -\tfrac{1}{32}, \quad f^{(3)}(4) = \tfrac{3}{256},$$

   and the degree 3 Taylor polynomial of $f(x) = \sqrt{x}$ about $x = 4$ is

$$T_3(x) = 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2 + \tfrac{1}{512}(x-4)^3.$$

(d) Let $f(x) = \cos(x)$, $c = \frac{\pi}{3}$, $n = 5$. Then

$f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f^{(3)}(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, $f^{(5)}(x) = -\sin(x)$,

and evaluating at $x = c = \frac{\pi}{3}$, we get

$$f\left(\tfrac{\pi}{3}\right) = \cos\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2},\ f'\left(\tfrac{\pi}{3}\right) = -\tfrac{\sqrt{3}}{2},\ f''\left(\tfrac{\pi}{3}\right) = -\tfrac{1}{2},\ f^{(3)}\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2},\ f^{(4)}\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2},\ f^{(5)}\left(\tfrac{\pi}{3}\right) = -\tfrac{\sqrt{3}}{2}.$$

Then the degree 5 Taylor polynomial of $f(x) = \cos(x)$ about $x = \frac{\pi}{3}$ is

$$T_5(x) = \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{3}\right) - \tfrac{1}{4}\left(x - \tfrac{\pi}{3}\right)^2 + \tfrac{\sqrt{3}}{12}\left(x - \tfrac{\pi}{3}\right)^3 + \tfrac{1}{48}\left(x - \tfrac{\pi}{3}\right)^4 - \tfrac{\sqrt{3}}{240}\left(x - \tfrac{\pi}{3}\right)^5.$$

(e) Let $f(x) = \cos(x)$, $c = 0$, $n = 8$. Then

$f^{(0)}(x) = \cos(x)$, $\quad f^{(1)}(x) = -\sin(x)$, $\quad f^{(2)}(x) = -\cos(x)$, $\quad f^{(3)}(x) = \sin(x)$, $\quad f^{(4)}(x) = \cos(x)$, $\quad$ etc.

(a pattern should be apparent), and evaluating at $x = c = 0$, we get

$$f^{(0)}(0) = 1, \quad f^{(1)}(0) = 0, \quad f^{(2)}(0) = -1, \quad f^{(3)}(0) = 0, \quad f^{(4)}(0) = 1, \quad \text{etc.}$$

and the degree 8 Maclaurin polynomial of $f(x) = \cos(x)$ is

$$T_8(x) = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \tfrac{1}{8!}x^8.$$

2. (a) Find an upper bound on the absolute value of the error made if linear approximation about $x = 4$ is used to estimate $(3.9)^{5/2}$, and determine (without calculating the "exact" value numerically) whether this approximation is greater than, or less than, the exact value $(3.9)^{5/2}$.
   (b) Find an upper bound on the absolute value of the error made if the degree 2 Taylor polynomial about $x = 4$ is used to estimate $\sqrt{4.2}$, and determine (without calculating the "exact" value numerically) whether this approximation is greater than, or less than, the exact value $\sqrt{4.2}$.
   (c) Determine what degree n of Taylor polynomial $T_n(x)$, of the function $f(x) = \cos(x)$, about $x = \frac{\pi}{3}$ is needed to guarantee that the Taylor polynomial approximation of $\cos(69°)$ is accurate within 5×10^{-6} (i.e. the error is guaranteed to have an absolute value no larger than 5×10^{-6}).
   (d) Determine what degree n of Maclaurin polynomial $T_n(x)$, of the function $f(x) = \log(1+x)$, is needed to guarantee that the Maclaurin polynomial approximation of $\log(1.4)$ is accurate within 10^{-3}.

Solution: For all parts, we use Taylor's Theorem (with Lagrange remainder), in particular the formula for the error,

$$E_n(x) = \tfrac{1}{(n+1)!} f^{(n+1)}(s)\,(x - c)^{n+1},$$

where s is some number between c and x.

(a) Let $f(x) = x^{5/2}$, $c = 4$, $n = 1$. Taylor's Theorem says that

$$(3.9)^{5/2} = f(3.9) = T_1(3.9) + E_1(3.9),$$

where the error (or remainder) is (see the solution of 1(b), $f''(x) = \frac{15}{4}x^{1/2}$)

$$E_1(3.9) = \tfrac{1}{2!} f''(s)\,(3.9 - 4)^2 = \tfrac{15}{8}\sqrt{s}\,(0.01),$$

for some number s between 3.9 and 4. Taking absolute values, we get the same right hand side

$$|E_1(3.9)| = \tfrac{15}{8}\sqrt{s}\,(0.01).$$

Since $\sqrt{s}$ is positive and increasing (one could assume this is "well known", or one could show that the derivative of $\sqrt{s}$ with respect to s is positive for $3.9 \le x \le 4$), as s increases from 3.9 to 4, the largest possible value of $|E_1(3.9)|$ would be at the right endpoint of the interval $[3.9, 4]$ known to contain s, if $s = 4$. Then we get an upper bound for the absolute value of the error

$$|E_1(3.9)| \le \tfrac{15}{8}\sqrt{4}\,(0.01) = \tfrac{15}{4}(0.01) = 0.0375.$$

Because the error is $E_1(3.9)$ is positive, we have from Taylor's Theorem

$$(3.9)^{5/2} = f(3.9) = T_1(3.9) + E_1(3.9) > T_1(3.9),$$

so

the linear approximation is less than the exact value $(3.9)^{5/2}$.

(The answers can be checked with a calculator: the linear approximation is $T_1(3.9) = 30$, the "exact" value is $(3.9)^{5/2} = 30.03734327$, the approximation is indeed less than the exact value and the error is 0.03734327, which is less than 0.0375 as predicted.)

(b) Let $f(x) = \sqrt{x}$. Taylor's Theorem says that

$$\sqrt{4.2} = f(4.2) = T_2(4.2) + E_2(4.2),$$

where the error is (see 1(c), $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$)

$$E_2(4.2) = \tfrac{1}{3!}f^{(3)}(s)(4.2-4)^3 = \tfrac{1}{16}\,s^{-5/2}\,0.008,$$

for some number s between 4.2 and 4. Since the expression is already positive, we get

$$|E_2(4.2)| = \tfrac{1}{16}\tfrac{0.008}{(\sqrt{s})^5}.$$

Since $s^{5/2} = (\sqrt{s})^5$ is positive and increasing, as s increases from 4 to 4.2, its reciprocal $1/(\sqrt{s})^5$ is positive and decreasing, as s increases from 4 to 4.2 (one could check that the derivative is negative on the interval), and the largest possible value for $|E_2(4.2)|$ would occur at the left endpoint of the interval known to contain s, if $s = 4$, so we get an upper bound for the absolute value of the error

$$|E_2(4.2)| \le \tfrac{1}{16}\tfrac{0.008}{(\sqrt{4})^5} = \tfrac{1}{64000} = 0.000015625.$$

The error is positive ($E_2(4.2) > 0$) and therefore

$$T_2(4.2) < \sqrt{4.2}.$$

(Checking with a calculator we get $T_2(4.2) = 2 + \frac{1}{4}(4.2-4) - \frac{1}{64}(4.2-4)^2 = 2.049375$, $f(4.2) = \sqrt{4.2} = 2.049390153$ accurate to 10 significant digits, therefore $E_2(4.2) = f(4.2) - T_2(4.2) = 0.000015153$, which is positive, as predicted, and its absolute value is not larger than the upper bound 0.000015625, as predicted).

(c) Note that $69°$ is $\frac{23\pi}{60}$ radians. We are required to find a positive integer n such that the absolute value of the error satisfies

$$|E_n\left(\tfrac{23\pi}{60}\right)| < 5 \times 10^{-6}.$$

Taylor's Theorem gives

$$|E_n\left(\tfrac{23\pi}{60}\right)| = \tfrac{1}{(n+1)!}|f^{(n+1)}(s)|\left|\tfrac{23\pi}{60} - \tfrac{\pi}{3}\right|^{n+1}$$

where s is some number between $\frac{23\pi}{60}$ and $\frac{\pi}{3}$. The derivatives of $f(x) = \cos(x)$ are $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f^{(3)}(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, etc., so the absolute value $|f^{(n+1)}(s)|$ is either

$|\sin(s)|$ or $|\cos(s)|$. Both of these are always less than or equal to 1 in absolute value, so it is always true that

$$|f^{(n+1)}(s)| \le 1$$

for all n, without using any specific information about s. Using this, we get an upper bound for the absolute value of the error for the Taylor polynomial $T_n(x)$:

$$|E_n\left(\tfrac{23\pi}{60}\right)| \le \tfrac{1}{(n+1)!}\left|\tfrac{\pi}{20}\right|^{n+1}.$$

Trying different values of n, we get

$$|E_1\left(\tfrac{23\pi}{60}\right)| \le \frac{1}{2!}\frac{\pi^2}{20^2} \le 0.013, \quad |E_2\left(\tfrac{23\pi}{60}\right))| \le \frac{1}{3!}\frac{\pi^3}{20^3} \le 0.00065,$$

$$|E_3\left(\tfrac{23\pi}{60}\right)| \le \frac{1}{4!}\frac{\pi^4}{20^4} \le 0.000026, \quad |E_4\left(\tfrac{23\pi}{60}\right)| \le \frac{1}{5!}\frac{\pi^5}{20^5} \le 0.0000008 = 8\times 10^{-7},$$

and we can stop at

$$n = 4$$

since $8\times 10^{-7} < 5\times 10^{-6}$. Therefore the 4th-degree Taylor polynomial $T_4(x)$, of $\cos(x)$ centred at $x = \frac{\pi}{3}$, is guaranteed to be within 5×10^{-6} of the exact value of $\cos(69°)$.
(You were not required to calculate $T_4\left(\frac{23\pi}{60}\right)$, but it is approximately 0.3583686496, while $\cos\left(\frac{23\pi}{60}\right) =$ 0.3583679494, so the error is approximately -7×10^{-7}, the absolute value is indeed not larger than 5×10^{-6}. It *might* be true that a lower value of n would be accurate enough, but we can't guarantee it unless we do more work, by finding more accurate upper bounds for $|f^{(n+1)}(s)|$.)

(d) "Maclaurin" means "Taylor centred at $c = 0$". Here we use

$$x = 0.4$$

since we want $\log(1+x) = \log(1.4)$. We need to find a positive integer n such that the absolute value of the error satisfies

$$|E_n(0.4)| < 0.001.$$

Taylor's Theorem gives

$$E_n(0.4) = \tfrac{1}{(n+1)!}\, f^{(n+1)}(s)\,(0.4)^{n+1}$$

where s is some number between 0 and 0.4. The derivatives of $f(x) = \log(1+x)$ are

$$f'(x) = (1+x)^{-1},\ f''(x) = (-1)(1+x)^{-2},\ f^{(3)}(x) = (-1)(-2)(1+x)^{-3},$$

$$f^{(4)}(x) = (-1)(-2)(-3)(1+x)^{-4},$$

etc., so

$$|f^{(n+1)}(s)| = (1)(2)(3)\cdots(n)(1+s)^{-n-1} = \frac{n!}{(1+s)^{n+1}}$$

and

$$|E_n(0.4)| = \tfrac{1}{(n+1)!}\,|f^{(n+1)}(s)|\,|(0.4)^{n+1}| = \frac{(0.4)^{n+1}}{(n+1)(1+s)^{n+1}}.$$

For any positive integer n the expression $(1+s)^{n+1}$ is positive and increasing as s increases from 0 to 0.4, its reciprocal $1/(1+s)^{n+1}$ is positive and decreasing as s increases from 0 to 0.4 (you could show that the derivative is negative for $n \ge 1$ and any s in the interval), so the maximum possible value of $|E_n(0.4)|$ would occur at the left endpoint of the interval, if $s = 0$. We get the upper bound

$$|E_n(0.4)| \le \frac{(0.4)^{n+1}}{(n+1)(1+0)^{n+1}} = \frac{(0.4)^{n+1}}{n+1}.$$

Trying different values of n, we get

$$|E_1(0.4)| \le \frac{(0.4)^2}{2} = 0.08,\ |E_2(0.4)| \le \frac{(0.4)^3}{3} = 0.0213333,\ |E_3(0.4)| \le \frac{(0.4)^4}{4} = 0.0064,$$

$$|E_4(0.4)| \le \frac{(0.4)^5}{5} = 0.002048, \quad |E_5(0.4)| \le \frac{(0.4)^6}{6} = 0.000682667,$$

and we can stop at

$$n = 5$$

since the upper bound for $|E_5(0.4)|$ is less than $10^{-3} = 0.001$. Therefore the degree 5 Maclaurin polynomial $T_5(x)$, of $f(x) = \log(1+x)$, is guaranteed to be within 10^{-3} of $\log(1+0.4)$.
(In fact, checking with a calculator we get $T_5(0.4) = 0.3369813333$ and $\log(1.4) = 0.3364722366$, so the actual error is $\log(1.4) - T_5(0.4) = -0.0005090967$, whose absolute value is indeed no greater than 0.001, as guaranteed.)

3. Let R be the region between the y-axis and the curve $x = (16+y)^{1/4}$, with $-16 \le y \le 0$, and both x and y measured in metres. The region R is rotated around the y-axis, creating a volume. This volume is filled with a fluid that has volume density 888 kg/m^3. Determine the work done (in joules) pumping all the fluid up to $y = 0$. Use $g = 9.8$ m/s^2 for the acceleration due to gravity. (You must evaluate the integral and a calculator-ready answer is sufficient.)

Solution:
An infinitesimally thin slice of the fluid at level y is circular with radius $x = (16+y)^{1/4}$ and thickness dy. It has infinitesimal volume $\pi x^2\, dy = \pi\sqrt{16+y}\, dy$, infinitesimal mass $\rho\pi\sqrt{16+y}\, dy$, where $\rho = 888$ kg/m^3. The infinitesimal amount of work done in lifting this slice from level y up to 0, against gravity, is

$$dW = \rho\pi\sqrt{16+y}\, dy \cdot g \cdot (0-y), \quad -16 \le y \le 0,$$

where $g = 9.8$ m/s^2. Notice that $dW \ge 0$. Then the total amount of work lifting all the slices up to $y = 0$, with the fluid occupying slices at levels from $y = -16$ to $y = 0$, is

$$W = \int_{y=-16}^{y=0} dW = \pi\rho g \int_{-16}^{0} (-y)\sqrt{16+y}\, dy.$$

This integral can be evaluated by making a substitution

$$u = 16 + y, \quad du = dy,$$

then

$$\begin{aligned} W &= \pi\rho g \int_{-16}^{0} (-y)\sqrt{16+y}\, dy \\ &= \pi\rho g \int_{0}^{16} (16-u)\sqrt{u}\, du \\ &= \pi\rho g \int_{0}^{16} (16u^{1/2} - u^{3/2})\, du \\ &= \pi\rho g \left(\tfrac{32}{3}\, u^{3/2} - \tfrac{2}{5}\, u^{5/2}\right)\Big|_0^{16} \end{aligned}$$

and for a calculator-ready answer (i.e. a fully numerical expression that could be evaluated with a calculator) we must substitute in the numerical values for ρ and g. The work done is

$$W = \pi(888)(9.8)\left(\tfrac{32}{3}\, 16^{3/2} - \tfrac{2}{5}\, 16^{5/2}\right) \quad \text{J.}$$

On a test or exam, this need not be simplified any further, but if you have the time (like when doing homework) you can use $16^{1/2} = 4$ and simplify to

$$W = \pi(888)(9.8)\frac{4^6}{15} = \pi(888)(9.8)\frac{4096}{15} \approx 7465477.71 \quad \text{J.}$$