MATH 101 V01 – ASSIGNMENT 8

Solutions

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1. (a) Find a power series representation for $f(x) = x\sin(x/2)$ and determine the interval of convergence.
   (b) Find the first four nonvanishing terms in the alternating series representation of $\int_0^{1/2} \arctan(x^3)\,dx$.
   (c) Evaluate $\lim_{x\to 0} \frac{-x+\sin(x)}{x^4}$, or determine that the limit does not exist.
   (d) Evaluate $\lim_{x\to 0} \frac{x^2-2+2\cos(x)}{x^4}$, or determine that the limit does not exist.
   (e) If $f(x) = 2\sin(x)\cos(x)$, find $f^{(101)}(0)$.

Solution: For all parts, we use well known Taylor (or Maclaurin) series.

(a) Starting with the Maclaurin series for $\sin(x)$,

$$\begin{aligned}
\sin(x) &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \dots \quad \text{if } -\infty < x < \infty,\\
\sin\left(\tfrac{x}{2}\right) &= \left(\tfrac{x}{2}\right) - \tfrac{1}{3!}\left(\tfrac{x}{2}\right)^3 + \tfrac{1}{5!}\left(\tfrac{x}{2}\right)^5 - \tfrac{1}{7!}\left(\tfrac{x}{2}\right)^7 + \dots \quad \text{if } -\infty < \tfrac{x}{2} < \infty, \text{ i.e. if } -\infty < x < \infty,\\
x\sin\left(\tfrac{x}{2}\right) &= x\left[\tfrac{1}{2}x - \tfrac{1}{3!2^3}x^3 + \tfrac{1}{5!2^5}x^5 - \tfrac{1}{7!2^7}x^7 + \dots\right]\\
&= \tfrac{1}{2}x^2 - \tfrac{1}{3!2^3}x^4 + \tfrac{1}{5!2^5}x^6 - \tfrac{1}{7!2^7}x^8 + \dots\\
&= \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n+1)!2^{2n+1}} x^{2n+2}
\end{aligned}$$

with interval of convergence $(-\infty, \infty)$.

(b) We start with the Maclaurin series for $\arctan(x)$,

$$\begin{aligned}
\arctan(x) &= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \dots \quad \text{with radius of convergence 1},\\
\arctan(x^3) &= x^3 - \tfrac{1}{3}x^9 + \tfrac{1}{5}x^{15} - \tfrac{1}{7}x^{21} + \dots \quad \text{with radius of convergence } \sqrt[3]{1} = 1,\\
\int_0^{1/2} \arctan(x^3)\,dx &= \int_0^{1/2} \left(x^3 - \tfrac{1}{3}x^9 + \tfrac{1}{5}x^{15} - \tfrac{1}{7}x^{21} + \dots\right) dx\\
&= \left(\tfrac{1}{4}x^4 - \tfrac{1}{3\cdot 10}x^{10} + \tfrac{1}{5\cdot 16}x^{16} - \tfrac{1}{7\cdot 22}x^{22} + \dots\right)\Big|_0^{1/2}\\
&= \tfrac{1}{4\cdot 2^4} - \tfrac{1}{3\cdot 10\cdot 2^{10}} + \tfrac{1}{5\cdot 16\cdot 2^{16}} - \tfrac{1}{7\cdot 22\cdot 2^{22}} + \dots\\
&= \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n+1)(6n+4)2^{6n+4}}.
\end{aligned}$$

(c) We use the Maclaurin series for $\sin(x)$,

$$\begin{aligned}
\lim_{x\to 0} \tfrac{-x+\sin(x)}{x^4} &= \lim_{x\to 0} \frac{-x + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)}{x^4}\\
&= \lim_{x\to 0} \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots}{x^4}\\
&= \lim_{x\to 0} \left(-\tfrac{1}{3!}\,\tfrac{1}{x} + \tfrac{1}{5!}x + \dots\right)
\end{aligned}$$

does not exist.

(d) We use the Maclaurin series for $\cos(x)$,

$$\begin{aligned}\lim_{x\to 0}\tfrac{x^2-2+2\cos x}{x^4} &= \lim_{x\to 0}\frac{x^2-2+2\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\dots\right)}{x^4}\\ &= \lim_{x\to 0}\frac{x^2-2+2-x^2+\frac{2}{4!}x^4-\frac{2}{6!}x^6+\dots}{x^4}\\ &= \lim_{x\to 0}\left(\tfrac{2}{4!}-\tfrac{2}{6!}x^2+\dots\right)\\ &= \tfrac{2}{4!}=\tfrac{1}{12}\end{aligned}$$

(e) By a trigonometric identity, $f(x)=2\sin(x)\cos(x)=\sin(2x)$, then

$$\begin{aligned}\sin(2x) &= (2x)-\tfrac{1}{3!}(2x)^3+\tfrac{1}{5!}(2x)^5-\dots\\ &= 2x-\tfrac{2^3}{3!}x^3+\tfrac{2^5}{5!}x^5-\dots\\ &= \sum_{n=0}^{\infty}\tfrac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{2n+1}\end{aligned}$$

The coefficient of x^{101} would be (positive)

$$\tfrac{2^{101}}{101!}$$

but we also know for any Maclaurin series that this coefficient is equal to

$$\tfrac{f^{(101)}(0)}{101!},$$

therefore

$$f^{(101)}(0)=2^{101}.$$

2. Let $f(x)=(1+x)^\alpha$, where α is any fixed real number.
   (a) Find the Maclaurin series of $(1+x)^\alpha$.
   (b) Find the radius of convergence of the Maclaurin series.
   (c) The length L of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a>b>0$, is (you do not have to show this)

$$L=4a\int_0^{\pi/2}\sqrt{1-\epsilon^2\sin^2(\theta)}\,d\theta,$$

   where $\epsilon=\frac{\sqrt{a^2-b^2}}{a}$ is the *eccentricity* of the ellipse. If ϵ is near 0, the ellipse is nearly a circle. Use part (a) to find the first three nonvanishing terms in the series representation of L, in powers of ϵ. Use the series to estimate the length of the ellipse with $a=1.01$, $b=0.99$.

Solution:

(a) We calculate derivatives of $f(x)=(1+x)^\alpha$ and evaluate them at the centre $x=c=0$:

$$\begin{aligned}f(x)&=(1+x)^\alpha & f(0)&=1\\ f'(x)&=\alpha(1+x)^{\alpha-1} & f'(0)&=\alpha\\ f''(x)&=\alpha(\alpha-1)(1+x)^{\alpha-2} & f''(0)&=\alpha(\alpha-1)\\ f'''(x)&=\alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} & f'''(0)&=\alpha(\alpha-1)(\alpha-2)\\ &\vdots & &\vdots\\ f^{(n)}(x)&=\alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n} & f^{(n)}(0)&=\alpha(\alpha-1)\cdots(\alpha-n+1)\end{aligned}$$

Then the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, f^{(n)}(0)\, x^n = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\, x^n.$$

(This series is known as "the binomial series.")

(b) If α is a nonnegative integer, then all the terms for n sufficiently large are 0, and so the series is finite. In this case the series converges for $-\infty < x < \infty$ and the radius of convergence is infinite.

Otherwise, none of the terms is 0, and we use the Ratio Test to test for absolute convergence. If $a_n = \frac{1}{n!}\, f^{(n)}(0)\, x^n$, we have

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)\, x^{n+1}}{(n+1)!} \cdot \frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)\, x^n} \right| \\ &= \frac{|\alpha-n|}{n+1}\, |x| \\ &= \frac{1-\frac{\alpha}{n}}{1+\frac{1}{n}}\, |x|, \end{aligned}$$

for all n sufficiently large (all $n > \alpha$), so the limit as $n \to \infty$ exists,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x|,$$

and by the Ratio Test the series is absolutely convergent (and convergent) if $|x| < 1$ and divergent if $|x| > 1$, so the radius of convergence is 1.

(c) We use the result of part (a) with $\alpha = \frac{1}{2}$ and $x = -\epsilon^2 \sin^2(\theta)$ to expand the integrand $[1-\epsilon^2\sin^2(\theta)]^{1/2}$ to the first three terms, obtaining

$$\begin{aligned} L &= 4a \int_0^{\pi/2} [1-\epsilon^2\sin^2(\theta)]^{1/2}\, d\theta \\ &= 4a \int_0^{\pi/2} \left[1 - \frac{\frac{1}{2}}{1!}\,\epsilon^2\sin^2(\theta) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\,\epsilon^4\sin^4(\theta) - \ldots \right] d\theta \\ &= 4a \int_0^{\pi/2} \left[1 - \tfrac{1}{2}\epsilon^2\sin^2(\theta) - \tfrac{1}{8}\epsilon^4\sin^4(\theta) - \ldots \right] d\theta \\ &= 4a \int_0^{\pi/2} 1\, d\theta \;-\; 2a\epsilon^2 \int_0^{\pi/2} \sin^2(\theta)\, d\theta \;-\; \tfrac{1}{2}a\epsilon^4 \int_0^{\pi/2} \sin^4(\theta)\, d\theta - \ldots. \end{aligned}$$

Now we need to calculate three integrals,

$$\int_0^{\pi/2} 1\, d\theta = \tfrac{\pi}{2},$$

$$\begin{aligned} \int_0^{\pi/2} \sin^2(\theta)\, d\theta &= \int_0^{\pi/2} \left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) \right] d\theta \\ &= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta) \right]\Big|_0^{\pi/2} \\ &= \tfrac{\pi}{4}, \end{aligned}$$

$$
\begin{aligned}
\int_0^{\pi/2} \sin^4(\theta)\, d\theta &= \int_0^{\pi/2} \left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)\right]^2\, d\theta \\
&= \int_0^{\pi/2} \left[\tfrac{1}{4} - \tfrac{1}{2}\cos(2\theta) + \tfrac{1}{4}\cos^2(2\theta)\right]\, d\theta \\
&= \int_0^{\pi/2} \left\{\tfrac{1}{4} - \tfrac{1}{2}\cos(2\theta) + \tfrac{1}{4}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos(4\theta)\right]\right\}\, d\theta \\
&= \int_0^{\pi/2} \left[\tfrac{3}{8} - \tfrac{1}{2}\cos(2\theta) + \tfrac{1}{8}\cos(4\theta)\right]\, d\theta \\
&= \left[\tfrac{3}{8}\theta - \tfrac{1}{4}\sin(2\theta) + \tfrac{1}{32}\sin(4\theta)\right]\Big|_0^{\pi/2} \\
&= \tfrac{3\pi}{16},
\end{aligned}
$$

and substituting these into the series for L, we obtain

$$L = 2a\pi - \tfrac{1}{2}a\pi\epsilon^2 - \tfrac{3}{32}a\pi\epsilon^4 - \dots.$$

Now if $a = 1.01$ and $b = 0.99$, then $\epsilon = 0.198019802$ and

$$L \approx 2(1.01)\pi - \tfrac{1}{2}(1.01)\pi(0.198019802)^2 - \tfrac{3}{32}(1.01)\pi(0.198019802)^4 = 6.283350026.$$

(For comparison, if $a = 1$ and $b = 1$, then we have a circle with circumference $2\pi = 6.283185308$.)

3. (a) Define, using Riemann sums, what it means for a function $f(x)$ to be integrable on a closed interval $[l, r]$, where $l < r$.

(b) Let

$$f(x) = \begin{cases} 1 & \text{if } x = j/2^k \text{ for integers } j \text{ and } k, \text{ with } k \text{ positive and } 0 \le j \le 2^k, \\ -1 & \text{otherwise.} \end{cases}$$

Prove that $f(x)$ is not integrable on $[0, 1]$.

Solution:

(a) For any positive integer n, subdivide the closed interval $[l, r]$ into n subintervals $[x_{i-1}, x_i]$ $(i = 1, 2, \dots, n)$ of equal width $\Delta x = \frac{r-l}{n}$, with

$$l = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = r,$$

and in each subinterval select a sample point

$$x_i^* \in [x_{i-1}, x_i].$$

Then form the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\, \Delta x.$$

The function $f(x)$ is integrable on $[l, r]$ if the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

exists, and has the same value for all choices of sample points.

(b) For any positive integer n, subdivide the closed interval $[0, 1]$ into n subintervals $[x_{i-1}, x_i]$ $(i = 1, 2, \dots, n)$ of equal width $\Delta x = \frac{1}{n}$, as described in part (a).

i) In each subinterval, select a sample point

$$x_i^* \in [x_{i-1}, x_i], \quad x_i^* = j/2^k,$$

for some integers j and k. This is possible because the spacing between numbers of the form $j/2^k$ for two consecutive values of j is $1/2^k$, and by choosing k sufficiently large (e.g. $k > \log(n)/\log(2)$) we can ensure the spacing $1/2^k$ between numbers of the form $j/2^k$ is less than the width $1/n$ of the subinterval $[x_{i-1}, x_i]$, so at least one of these numbers falls in the subinterval. Making this selection for x_i^* in every subinterval, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (1)\,(1/n) = \lim_{n\to\infty} (1/n) \sum_{i=1}^{n} (1) = 1.$$

ii) On the other hand, in each subinterval select a sample point

$$x_i^* \in [x_{i-1}, x_i], \quad x_i^* \neq j/2^k,$$

for any postive integer j and k. This is possible, for example, by taking x_i^* irrational. Making this selection for x_i^* in every subinterval, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (-1)\,(1/n) = \lim_{n\to\infty} (1/n) \sum_{i=1}^{n} (-1) = -1.$$

Since the limits in cases i) and ii) are different, for different choices of sample points, $f(x)$ is not integrable.