MATH 101 V01 – ASSIGNMENT 9

Solutions

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1. Solve the initial-value problems:

   (a) $y' = y^2, \quad y(0) = 1.$

   (b) $y' = y^2, \quad y(0) = 0.$

   (c) $\frac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2.$

   Solution:

   (a) The first-order differential equation $dy/dx = y^2$ is separable: write it as

$$y^{-2}\,dy = 1\,dx$$

then integrate (i.e. find the antiderivatives of) both sides,

$$\int y^{-2}\,dy = \int 1\,dx$$

to get

$$-\frac{1}{y} = x + C,$$

where C is an arbitrary constant. It is convenient (but not necessary) to use the initial condition at this point (rather than waiting until after solving for y) to solve for C: when $x = 0$ we must have $y = 1$, so substituting in these values gives

$$-\frac{1}{1} = 0 + C,$$

so

$$C = -1 \quad \text{and} \quad -\frac{1}{y} = x - 1.$$

Now solve for y to get

$$y(x) = \frac{1}{1-x}.$$

(b) The differential equation is the same as in (a), but the separation of variables method *doesn't work* to find C (division by 0 is not defined). By plotting the direction field, or from plotting the phase portrait (this differential equation is autonomous), we "guess" there is a constant solution $y(x) = 0$ for all x. Then we must check that it satisfies the given initial condition $y(0) = 0$, which is obvious, also that the function $y(x) = 0$ satisfies the differential equation $y' = y^2$ when it is substituted in, which it clearly does ($0 = 0^2$). So the solution to the initial-value problem is

$$y(x) = 0 \quad \text{for } -\infty < x < \infty.$$

Another way to find this solution is to look for the *equilibrium*, solution(s) of the differential equation by setting $y' = 0$ (this is what y' would be if $y(x) = \text{constant}$) and then solving for y.

(This differential equation is *nonlinear*, so there is no guarantee that there is a *general solution*, i.e. an expression for the solution that is guaranteed to give *all* solutions of the differential equation that exist. In this example, $y(x) = -1/(x+C)$ is not a general solution, because not every solution of the differential equation can be expressed in this form.)

(c) The 1st-order differential equation $dP/dt = \sqrt{Pt} = \sqrt{P}\sqrt{t}$ is separable: write it as

$$P^{-1/2}\,dP = t^{1/2}\,dt$$

then integrate (or find the antiderivatives of) both sides,

$$\int P^{-1/2}\, dP = \int t^{1/2}\, dt$$

to get

$$2\sqrt{P} = \tfrac{2}{3}\, t^{3/2} + C,$$

where C is an arbitrary constant. It is convenient to use the initial condition at this point (rather than after solving for P) to solve for C: when $t = 1$ we must have $P = 2$, so substituting these values into the line above give

$$2\sqrt{2} = \tfrac{2}{3}\, 1^{3/2} + C,$$

so

$$C = 2\left(\sqrt{2} - \tfrac{1}{3}\right) \quad \text{and} \quad 2\sqrt{P} = \tfrac{2}{3}\, t^{3/2} + 2\left(\sqrt{2} - \tfrac{1}{3}\right).$$

Now solve for P to get

$$P(t) = \left(\tfrac{1}{3}\, t^{3/2} + \sqrt{2} - \tfrac{1}{3}\right)^2.$$

(notice we must have $0 < t < \infty$ for this solution to be valid).

2. A room containing 1000 m^3 of air is originally free of carbon monoxide (CO). Beginning at time $t = 0$, smoke containing 4% CO (by volume) is blown into the room at the rate of 0.1 m^3/min, and the well circulated mixture leaves the room at the same rate. Find the time when the CO concentration in the room reaches 0.012%.

   Solution: Let $y(t)$ be the volume of CO in the room, in m^3, where t is measured in min. We want to know: for what t does $y(t) = (0.00012)(1000 \text{ m}^3) = 0.12 \text{ m}^3$? (Other approaches would be to let $y(t)$ be the percentage of CO in the room, or the fraction of the total volume that is CO. The corresponding initial value problems will be slightly different.)
   The differential equation for y is

$$\begin{aligned}\frac{dy}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= \left(\tfrac{4}{100}\right)(0.1 \text{ m}^3/\text{min}) - \left(\frac{y}{1000}\right)(0.1 \text{ m}^3/\text{min}) \\ &= 0.004 - 0.0001y \quad \text{m}^3/\text{min}\end{aligned}$$

   and the initial condition is

$$y(0) = 0.$$

   The differential equation is both separable and linear, and can be solved with either of two methods. If we solve the differential equation $dy/dt = 0.004 - 0.0001y$ by separating variables,

$$\frac{dy}{0.004 - 0.0001y} = dt$$

   and integrating

$$\int \frac{dy}{0.004 - 0.0001y} = \int dt,$$

   we get

$$-10,000 \log(|0.004 - 0.0001y|) = t + C$$

   where C is an arbitrary constant, or

$$\log(|0.004 - 0.0001y|) = -0.0001t + B$$

where B is another arbitrary constant ($B = -C/10,000$ but this doesn't matter). Apply the exponential function

$$|0.004 - 0.0001y| = e^B e^{-0.0001t}$$

and remove the absolute value sign to get

$$0.004 - 0.0001y = Ae^{-0.0001t}$$

where $A = \pm e^B$. Use the initial condition to solve for A: $y = 0$ when $t = 0$ gives

$$A = 0.004.$$

Substituting this value in for A and then solving for y, we get

$$y(t) = 40 - 40e^{-0.0001t}.$$

Alternatively, we can use the integrating factor method for linear equations. Writing the differential equation as

$$\frac{dy}{dt} + 0.0001\,y = 0.004,$$

we let $P(t) = 0.0001$ and $Q(t) = 0.004$. Then an integrating factor is

$$I(t) = e^{\int P(t)\,dt} = e^{0.0001\,t}$$

(or any nonzero constant multiple of this will also give a suitable integrating facror) so we multiply the differential equation by $I(t)$ and write the left-hand side as the derivative of a product (this is the reason we find an integrating factor!):

$$e^{0.0001\,t}\,\frac{dy}{dt} + 0.0001\,e^{0.0001\,t}\,y = 0.004\,e^{0.0001\,t},$$

$$\frac{d}{dt}[\,e^{0.0001\,t}\,y\,] = 0.004\,e^{0.0001\,t}.$$

Now integrate (i.e. find the antiderivatives, the left-hand side is now easy) to get

$$e^{0.0001\,t}\,y = 40\,e^{0.0001\,t} + A,$$

where A is an arbitrary constant. Solving for y gives the general solution

$$y = 40 + A\,e^{-0.0001\,t}.$$

Use the initial condition at $t = 0$

$$0 = 40 + A,$$

to get $A = -40$ and the solution of the initial value problem is the same as obtained by separation of variables,

$$y(t) = 40 - 40e^{-0.0001t}.$$

Now setting $y(t) = 0.12$ and solving the equation

$$0.12 = 40 - 40e^{-0.0001t}$$

for t, we get

$$t = -10,000\log\left(\tfrac{39.88}{40}\right) \approx 30.05 \quad \text{min.}$$

3. (a) Evaluate the integral $\int_0^1 x^2(1-x)^7\,dx$.

   (b) Evaluate the integral $\int_0^\pi x^2\sin(x)\,dx$.

   (c) Find the antiderivative (indefinite integral) $\int \cos(\theta)\cos^5(\sin(\theta))\,d\theta$.

   (d) Find the antiderivative (indefinite integral) $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$.

   (e) Find the antiderivative (indefinite integral) $\int \frac{3}{(x-1)^2(x+2)}\,dx$.

Solution:

(a) It is possible to expand the integrand $x^2(1-x)^7$ into a 9th-degree polynomial, but this would be slow. Instead, make the substitution

$$u = 1-x; \qquad du = -dx,$$

and transform the integral into something easier to evaluate,

$$\begin{aligned}\int_0^1 x^2(1-x)^7\,dx &= -\int_1^0 (1-u)^2u^7\,du\\ &= \int_0^1 (u^7 - 2u^8 + u^9)\,du\\ &= \left(\tfrac{1}{8}u^8 - \tfrac{2}{9}u^9 + \tfrac{1}{10}u^{10}\right)\Big|_0^1\\ &= \tfrac{1}{8} - \tfrac{2}{9} + \tfrac{1}{10} \quad (= \tfrac{1}{360}).\end{aligned}$$

(b) Integrate by parts,

$$u = x^2, \quad dv = \sin(x)\,dx; \qquad du = 2x\,dx, \quad v = -\cos(x),$$

then

$$\int_0^\pi x^2\sin(x)\,dx = -\,x^2\cos(x)\big|_0^\pi + 2\int_0^\pi x\,\cos(x)\,dx.$$

Integrate by parts a second time,

$$u = x, \quad dv = \cos(x)\,dx; \qquad du = dx, \quad v = \sin(x),$$

then we get

$$\begin{aligned}-\,x^2\cos(x)\big|_0^\pi + 2\int_0^\pi x\,\cos(x)\,dx &= -\,x^2\cos(x)\big|_0^\pi + 2\,x\,\sin(x)\big|_0^\pi - 2\int_0^\pi \sin(x)\,dx\\ &= -\,x^2\cos(x)\big|_0^\pi + 2\,x\,\sin(x)\big|_0^\pi + 2\,\cos(x)\big|_0^\pi\\ &= -\pi^2\cos(\pi) + 2\pi\sin(\pi) + 2\cos(\pi) - 2\cos(0)\\ &= \pi^2 - 4,\end{aligned}$$

where we have used $\cos(\pi) = -1$, $\sin(\pi) = 0$, $\cos(0) = 1$.

(c) Make the substitution

$$u = \sin(\theta); \qquad du = \cos(\theta)\, d\theta,$$

then

$$\begin{aligned}
\int \cos(\theta) \cos^5(\sin(\theta))\, d\theta &= \int \cos^5(u)\, du \\
&= \int \cos^4(u) \cos(u)\, du \\
&= \int [1 - \sin^2(u)]^2 \cos(u)\, du.
\end{aligned}$$

Make another substitution

$$v = \sin(u); \qquad dv = \cos(u)\, du,$$

then

$$\begin{aligned}
\int [1 - \sin^2(u)]^2 \cos(u)\, du &= \int (1 - v^2)^2\, dv \\
&= \int (1 - 2v^2 + v^4)\, dv \\
&= v - \tfrac{2}{3}\, v^3 + \tfrac{1}{5}\, v^5 + C \\
&= \sin(u) - \tfrac{2}{3}\, \sin^3(u) + \tfrac{1}{5}\, \sin^5(u) + C \\
&= \sin(\sin(\theta)) - \tfrac{2}{3}\, \sin^3(\sin(\theta)) + \tfrac{1}{5}\, \sin^5(\sin(\theta)) + C,
\end{aligned}$$

where C is an arbitrary constant.

(d) Make the trigonometric substitution

$$x = \tan(x); \qquad dx = \sec^2(x)\, dx, \quad \sqrt{1 + x^2} = \sec(\theta),$$

then we get

$$\begin{aligned}
\int \frac{1}{x^2 \sqrt{1 + x^2}}\, dx &= \int \frac{1}{\tan^2(\theta) \sec(\theta)}\, \sec^2(\theta)\, d\theta \\
&= \int \frac{1}{\sin^2(\theta)}\, \cos(\theta)\, d\theta.
\end{aligned}$$

Then we make the substitution

$$u = \sin(\theta); \qquad du = \cos(\theta)\, d\theta,$$

and get

$$\begin{aligned}
\int \frac{1}{\sin^2(\theta)}\, \cos(\theta)\, d\theta &= \int \frac{1}{u^2}\, du \\
&= -\frac{1}{u} + C \\
&= -\frac{1}{\sin(\theta)} + C \\
&= -\frac{\sqrt{1 + x^2}}{x} + C,
\end{aligned}$$

where C is an arbitrary constant.

(e) The integrand is a proper rational function, and we first make a partial fraction decomposition of the form

$$\frac{3}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

and multiply by $(x-1)^2(x+2)$ to get

$$3 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

for all x. To find A, B, C quickly, we substitute $x = 1$ to get $B = 1$, then substitute $x = -2$ to get $C = \frac{1}{3}$. Then we can look at the coefficient of, say, x^2 and get $0 = A + C$, so $A = -\frac{1}{3}$, and we have

$$\frac{3}{(x-1)^2(x+2)} = -\frac{1}{3}\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{3}\frac{1}{x+2},$$

and now we can calculate

$$\begin{aligned} \int \frac{3}{(x-1)^2(x+2)}\,dx &= -\frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx + \frac{1}{3}\int \frac{1}{x+2}\,dx \\ &= -\tfrac{1}{3}\log(|x-1|) - \frac{1}{x-1} + \tfrac{1}{3}\log(|x+2|) + C, \end{aligned}$$

where C is an arbitrary constant.