② Wed 2020-01-08

Separable equs. cont.

<u>Example</u> 1.3.A   Solve the IVP

$$y' = xy \quad , \quad y(0) = -1$$

and determine the largest open interval in which the solution exists.

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}\, dy = x\, dx$$

$$\int \frac{1}{y}\, dy = \int x\, dx$$

$$\ln|y| = \frac{1}{2}x^2 + c \qquad \text{(Implicit soln)}$$

$$|y| = e^{(\frac{1}{2}x^2 + c)}$$

$$= \underbrace{e^c\, e^{\frac{1}{2}x^2}}_{>0}$$

$$y = \pm e^c\, e^{\frac{1}{2}x^2} \qquad \left(\begin{array}{l} + \text{ if } y > 0 \\ - \text{ if } y < 0 \end{array}\right)$$

At $x=0$, $y(0) = -1$:

$$-1 = \pm \underbrace{e^c}_{>0}\, \underbrace{e^{\frac{1}{2}0^2}}_{1} \qquad \text{take } - \text{ sign}$$

$$-1 = -e^{c}$$

$$c = \ln(1) = 0$$

Soln. of IVP is

$$y(x) = -e^{\frac{1}{2}x^2}$$

This is continuous for $-\infty < x < +\infty$, so the largest open interval in which the solution exists is

$$-\infty < x < +\infty \quad (\text{or } (-\infty, \infty))$$

Summary

$$y(x) = -e^{\frac{1}{2}x^2}, \quad -\infty < x < +\infty$$

<u>Exercise</u> Solve $\frac{dy}{dx} = xy$, $y(1) = 0$

Method: Guess (look at slope field) and check

<u>Example</u> 1.3.B Solve the IVP

$$xy' + y^2 = 0, \quad y(-1) = 1,$$

and determine the largest open interval in which the solution exists.

$$\frac{dy}{dx} = -\frac{y^2}{x}$$

$$-y^{-2}\,dy = \frac{1}{x}\,dx$$

$$\int(-y^{-2})\,dy = \int \frac{1}{x}\,dx$$

$$y^{-1} = \ln|x| + c \qquad \text{(implicit solu.)}$$

$$y = \frac{1}{c + \ln|x|}$$

At $x=-1$, $y=1$:

$$1 = \frac{1}{c + \underbrace{\ln|-1|}_{=\ln(1)=0}}$$

$$c = 1$$

$$y(x) = \frac{1}{1+\ln|x|}$$

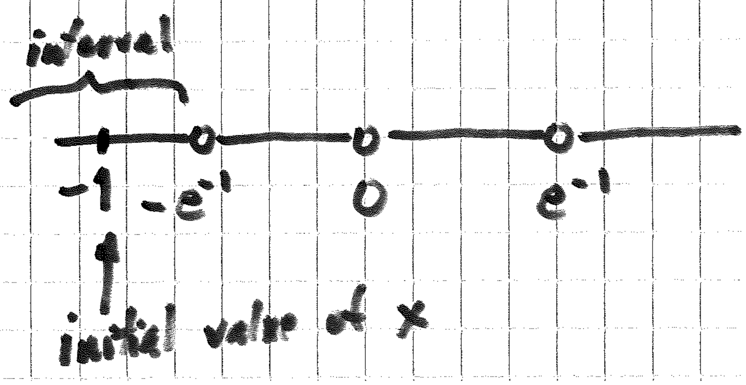
Note: $x \neq 0$, $1 + \ln|x| \neq 0$

$$\ln|x| \neq -1$$

$$|x| \neq e^{-1}$$

$$x \neq \pm e^{-1} = \pm\frac{1}{e}$$

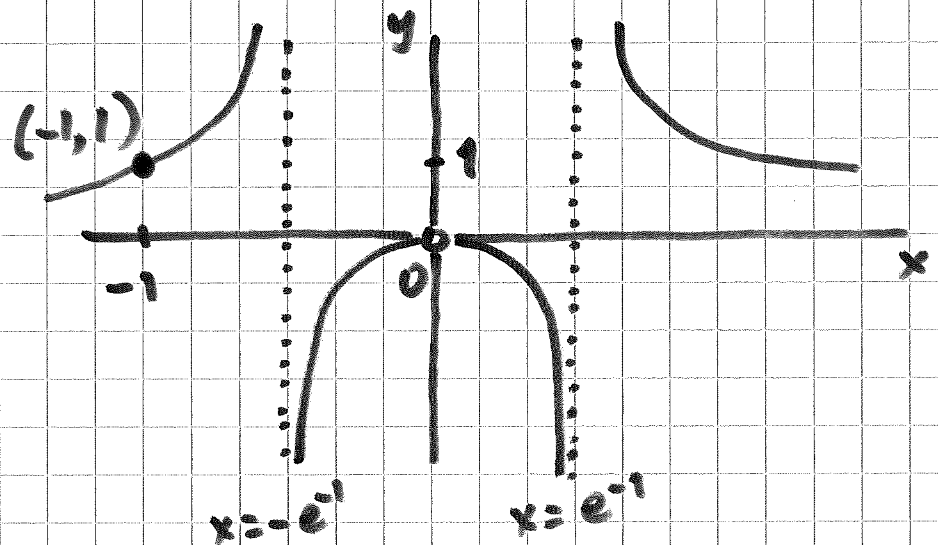
$$(e \approx 2.7)$$



Largest open interval in which soln. exists is

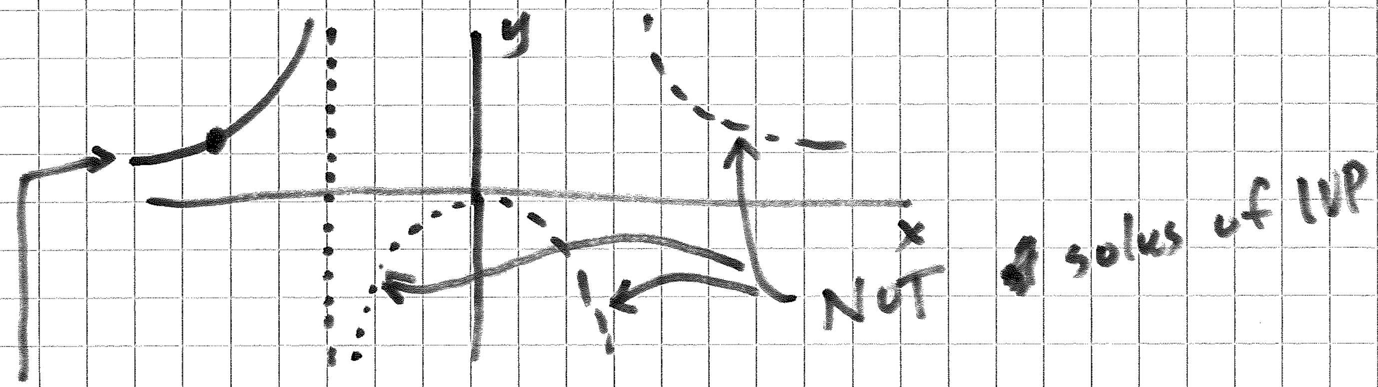
$$-\infty < x < -e^{-1}$$

Graph of $y = \dfrac{1}{1 + \ln|x|}$



Solutions of ODEs must be <u>continuous</u>

and solns of IVP pass through a specified point



Solution of IVP

$$\cancel{y =}\ y(x) = \frac{1}{1 + \ln|x|}, \quad -\infty < x < -e^{-1}$$

<u>Example</u> 1.3.C A satellite is sent directly up away from the earth's surface with initial velocity v_0. Considering gravity but neglecting friction, determine the value of v_0 required ~~to~~ for the satellite to reach a maximum altitude of H. Also find the escape velocity ($H = \infty$).

Hint: acceleration due to gravity is proportional to the inverse square ~~to~~ of the distance between the centres of the earth and satellite, with constant of proportionality gR^2,

$g = 9.8 \text{ m}\cdot\text{s}^{-2}$ (accel. due to gravity at surface of earth)

$R = 6.4 \times 10^6$ m (radius of earth)

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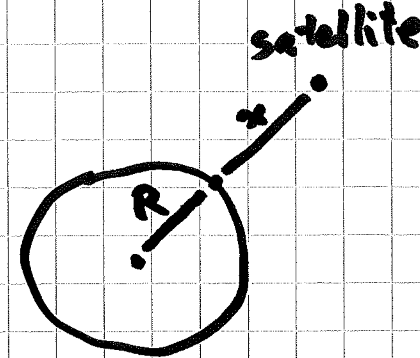
Let

$x = x(t)$ = altitude of satellite (~~with~~ satellite as point mass)

$v = \frac{dx}{dt}$ = upwards velocity

$a = \frac{dv}{dt}$ = upwards acceleration

Interpreting the hint



$$\underset{\frac{dv}{dt}}{\underset{\uparrow}{a}} = - gR^2 \frac{1}{(R+x)^2}$$

Trouble: three variables v, t, x

Think of v as a function of altitude x

$$v(x)$$

Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Now the DE becomes

$$v\frac{dv}{dx} = - gR^2 \frac{1}{(R+x)^2}$$

IC is

$$\underset{x=0}{v(\underset{\uparrow}{0})} = v_0$$

Separable eqn:

$$v\,dv = - gR^2 \frac{1}{(R+x)^2}\,dx$$

$$\int v\,dv = \int (-gR^2)\frac{1}{(R+x)^2}\,dx$$

$$\frac{1}{2} v^2 = g R^2 (R + x)^{-1} + c$$