At $x=0$:

$$\frac{1}{2}v_0^2 = gR^2\frac{1}{R+0} + c$$

$$c = \frac{1}{2}v_0^2 - gR$$

Then

$$\frac{1}{2}v(x)^2 = gR^2\frac{1}{(R+x)} + \frac{1}{2}v_0^2 - gR$$

At maximum altitude $x=H$, $v(H)=0$,

$$0 = gR^2\frac{1}{R+H} + \frac{1}{2}v_0^2 - gR$$

Solve for v_0

$$v_0 = \pm\sqrt{\frac{2gRH}{R+H}}$$ take + sign (wlog)

The value of v_0 required ~~to reach~~ for the satellite to reach a max. alt. of H is

$$v_0 = \sqrt{\frac{2gRH}{R+H}}$$

and the escape velocity is

$$v_{escape} = \lim_{H\to\infty}\sqrt{\frac{2gRH}{R+H}} = \sqrt{2gR}\ \mathrm{m\cdot s^{-1}}$$

1.4 Linear equations and the integrating factor

A first order ODE is linear if it can be written (in "standard form") as

$$\boxed{y' + p(x)\,y = f(x)}$$

(linear in y' and y). To solve, multiply the entire equation by the "integrating factor"

$$\boxed{r(x) = e^{\int p(x)\,dx}}$$

$$\underbrace{e^{\int p(x)\,dx}\,y' + e^{\int p(x)\,dx}\,p(x)\,y}_{\frac{d}{dx}\left[e^{\int p(x)\,dx}\,y\right]\ \text{(product rule)}} = e^{\int p(x)\,dx} f(x)$$

Integrate both sides with respect to x, solve for y:

$$y(x) = e^{-\int p(x)\,dx}\left[\int e^{\int p(x)\,dx} f(x)\,dx + C\right]$$

Or, more precisely:

$$y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left[\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt + y_0\right]$$

(use FTC)

$$(\; y(x_0) = y_0 \;)$$

<u>Example</u> 1.4. A $\qquad y' = xy \;,\; y(0) = -1$

Write in std. form

$$y' - xy = 0$$

Integrating factor

$$r(x) = e^{\int p(x)\,dx} = e^{\int(-x)\,dx} = e^{-\frac{1}{2}x^2}$$

(~~Exercise~~ <u>Exercise</u> Try adding arb. constant to integral.)

$$\underbrace{e^{-\frac{1}{2}x^2} y' - x e^{-\frac{1}{2}x^2} y}_{\frac{d}{dx}\left[e^{-\frac{1}{2}x^2} y\right]} = \underbrace{e^{-\frac{1}{2}x^2} \cdot 0}_{0}$$

$$e^{-\frac{1}{2}x^2} y = \int 0\,dx = C$$

$$y = Ce^{\frac{1}{2}x^2}$$ ← For a linear eqn. this is the general solution: every soln. has this form for some C.

At $x=0$, $y=-1$:

$$-1 = C\underbrace{e^{\frac{1}{2}0^2}}_{1}, \quad C=-1$$

Soln. of IVP is

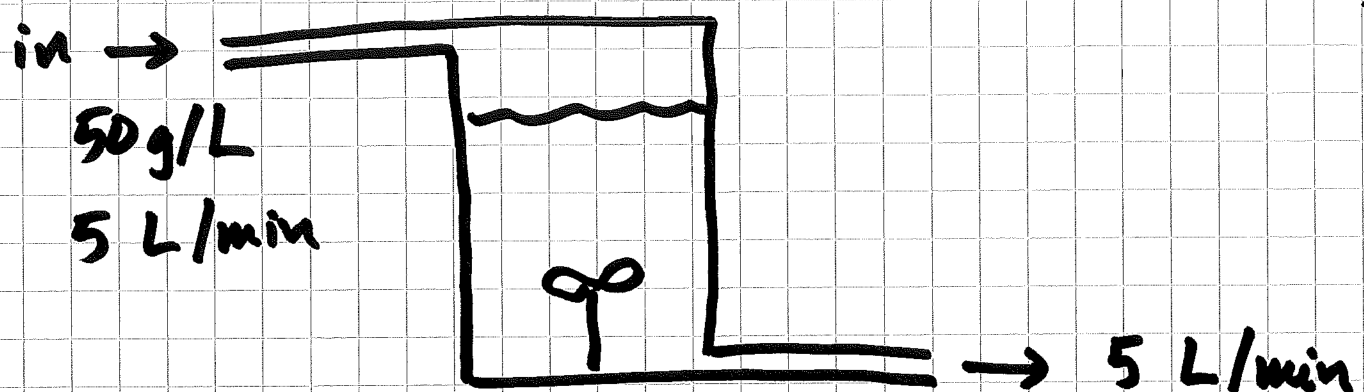
$$~~y(x)~~ \quad y(x) = -e^{\frac{1}{2}x^2} \quad (-\infty < x < \infty)$$

<u>Exercise</u> Same method works for

$$y' = xy, \quad y(1) = 0$$

<u>Example</u> 1.4.B (read p.44 Example 1.4.10 very carefully)

A 200 L tank ~~initially~~ initially contains 10 kg of salt dissolved in 100 L of water. Brine (= salt dissolved in water) of concentration 50 g/L is added at a rate of 5 L/min and the <u>well-stirred</u> mixture is removed at the same rate. Determine how much salt is in the tank after 1 hr, and after a long time.



Volume of brine in tank $V(t)$ L at t min

$$\frac{dV}{dt} = \underset{\text{(volume)}}{\text{Rate in}} - \underset{\text{(volume)}}{\text{Rate out}}$$

$$= 5 \text{ L/min} - 5 \text{ L/min}$$

$$= 0$$

$$V(0) = 100 \text{ L}$$

$V = 100$ L for all t

Amount (mass) of salt

$x(t)$ kg at t min

$$\frac{dx}{dt} = \underset{\text{(mass)}}{\text{Rate in}} - \underset{\text{(mass)}}{\text{Rate out}}$$

$$\text{Rate in} = \left(0.05 \frac{\text{kg}}{\text{L}}\right)\left(5 \frac{\text{L}}{\text{min}}\right)$$

$$\text{Rate out} = \left(\frac{x}{100} \ \frac{\text{kg}}{\text{L}} \right) \left(5 \ \frac{\text{L}}{\text{min}} \right)$$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{4} - \dfrac{1}{20} x & \dfrac{\text{kg}}{\text{min}} \\ \\ x(0) = 10 \ \text{kg.} \end{cases}$$