④ Mon 2020-01-13

Webwork assignment W1 due Fri.

(Ex. 1.4.B cont.)

$$\frac{dx}{dt} + \frac{1}{20}x = \frac{1}{4}, \quad x(0) = 10$$

Integrating factor

$$r(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{20}t} = e^{\frac{1}{20}t}$$

Multiply

$$\underbrace{e^{\frac{1}{20}t}x' + e^{\frac{1}{20}t}\frac{1}{20}x}_{\text{(Prod. rule)}} = e^{\frac{1}{20}t}\frac{1}{4}$$

$$\frac{d}{dt}\left[e^{\frac{1}{20}t}x\right] = \frac{1}{4}e^{\frac{1}{20}t}$$

$$e^{\frac{1}{20}t}x = \int \frac{1}{4}e^{\frac{1}{20}t}\,dt$$

$$= 5e^{\frac{1}{20}t} + C$$

$$x = 5 + Ce^{-\frac{1}{20}t}$$

At $t = 0$, $x = 10$:

$$10 = 5 + Ce^{-\frac{1}{20}0}$$

$$C = 5$$

Soln of IVP is

$$x(t) = 5 + 5e^{-\frac{1}{20}t}$$

After 1 hr (= 60 min) there is

$$x(60) = 5 + 5e^{-3} \text{ kg} \quad (\text{"calculator ready"})$$

of salt in the tank, and after a long time there is (approximately)

$$\lim_{t\to\infty}\left(5 + 5\underbrace{e^{-\frac{1}{20}t}}_{\to 0}\right) = 5 \text{ kg}$$

(Input concentration · Volume of tank =
$0.05\ \frac{\text{kg}}{\text{L}}$ · 100 L $= 5\text{ kg}$)

"Newton's law of cooling" problems also give 1st order linear ODE: Example 1.3.3, Exercise 1.4.10

Read textbook: how to derive

$$r(x) = e^{\int p(x)\,dt}$$

1.6 ~~Aut~~ Autonomous equations

A 1st order ODE is autonomous if it can be written as

$$\frac{dx}{dt} = f(x)$$

(right hand side of equ. does not depend explicitly on the independent variable)

An autonomous equ. is separable

$$\frac{1}{f(x)}\,dx = dt$$

but there are also useful graphical/~~qual~~ qualitative methods available (in addition to the slope field)

Example 1.6.A

$$\frac{dx}{dt} = r x \left(1 - \frac{x}{K}\right)$$

"Logistic" equ., models population growth

$r > 0$ "intrinsic per capita growth rate"

$K > 0$ "carrying capacity"

<u>Exercise</u>: solve explicitly, with $x(0) = x_0$.

(Hint: partial fractions for integral) see online MATH 101 textbook.

Notice there are constant solutions:

$$x(t) \equiv \text{constant} \quad \left(x(t) = \text{constant for all } t \right)$$

$$x' = 0$$

$$0 = r x \left(1 - \frac{x}{K}\right)$$

$$\Leftrightarrow \quad x = 0 \quad \text{or} \quad x = K$$

A <u>critical point</u> of $\frac{dx}{dt} = f(x)$ is a solution of $0 = f(x)$

In this example there are exactly two critical points

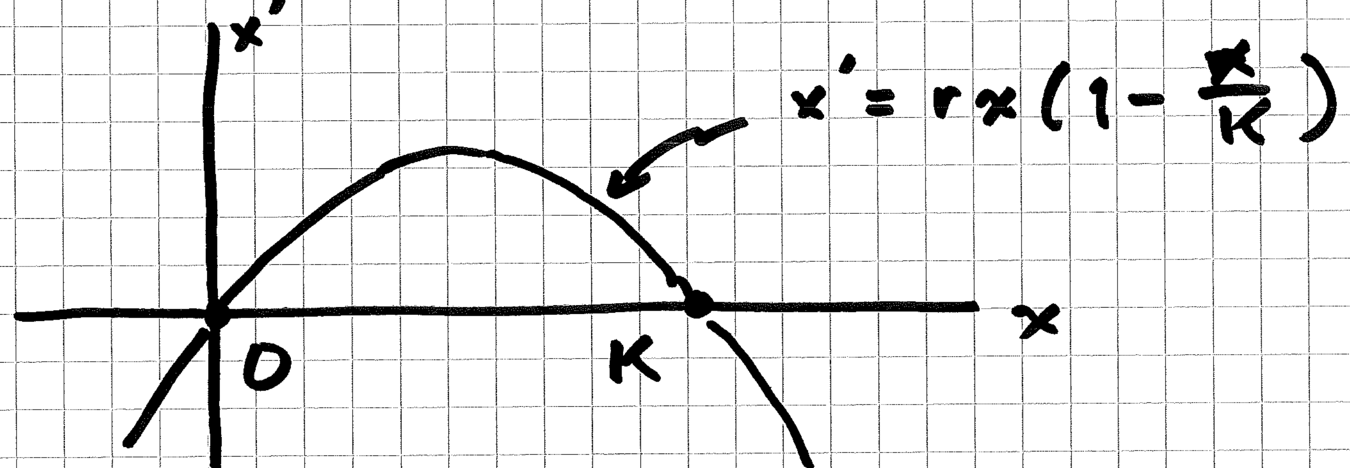
$$x = 0, \qquad x = K$$

Critical points correspond to constant solutions, or <u>equilibrium solutions</u>

In this example there are two equilibrium solutions

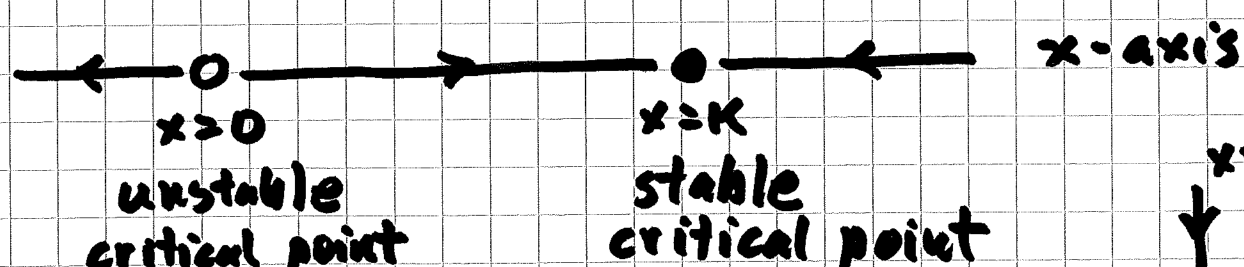
$$x(t) \equiv 0 \quad , \quad x(t) \equiv K$$

Plot the "vector field" x vs $f(x)$

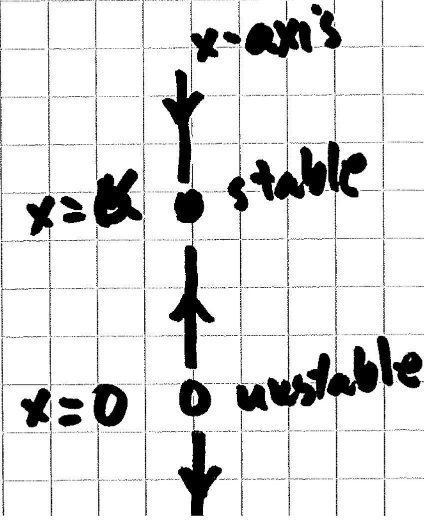


$x' < 0$ $x(t)$ decr. | $x' = 0$ $x(t) \equiv$ const | $x' > 0$ $x(t)$ incr. | $x' = 0$ $x(t) \equiv$ const | $x' < 0$ $x(t)$ decr.

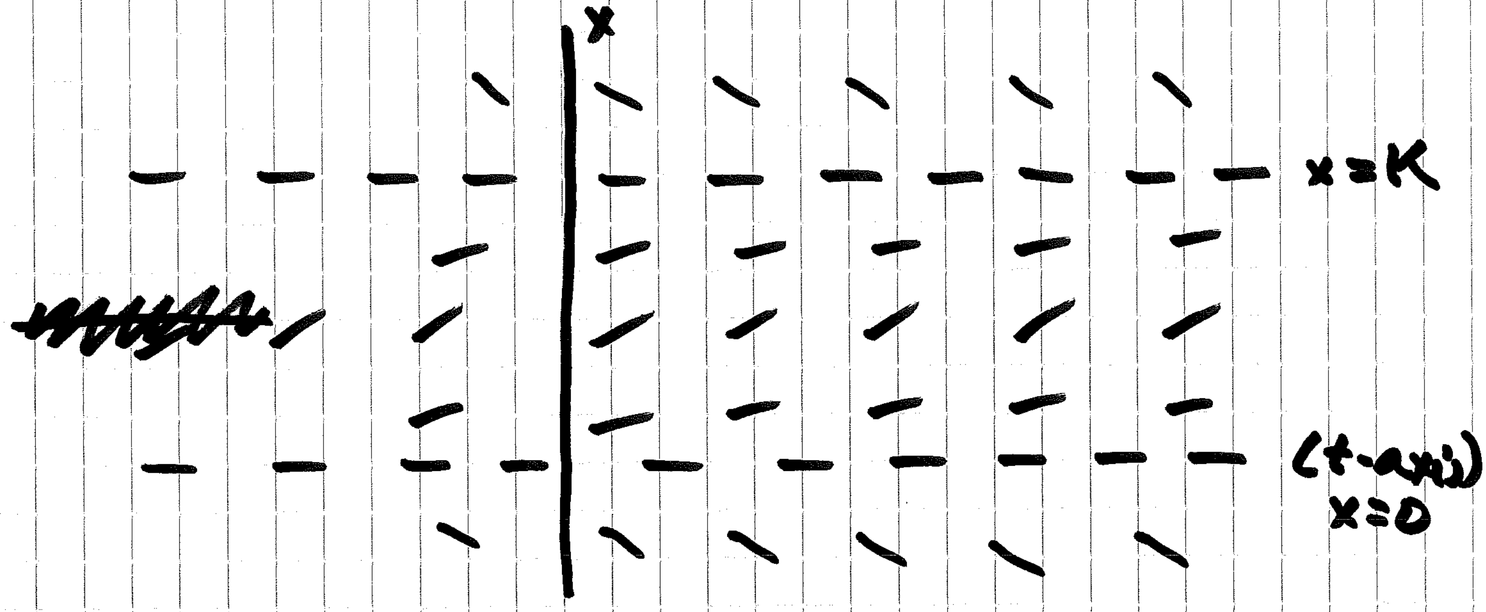
"Phase portrait"



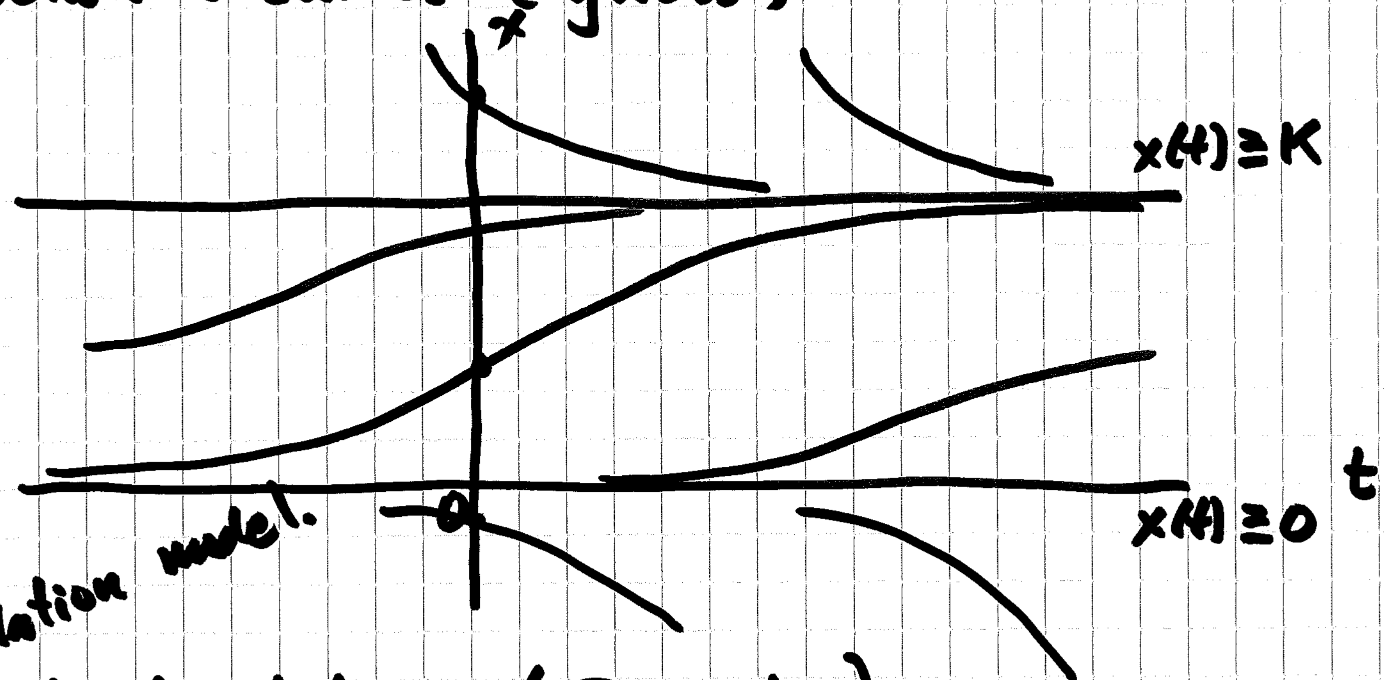
or



Corresponding slope field



Solution curves (guess)



ignore $x<0$ for population model.

Explicit solution (Exercise)

$$x(t) = \frac{x_0 K}{\underbrace{x_0 + (K - x_0)\,e^{-rt}}_{\text{possible vert. asymptote if denom.} = 0}}$$