Webwork W1 Jan 17 ~~⑤ Wed 2020-01-15~~

Homework HW1 Jan 24 ⑤ ⑥ Fri 2020-01-17

1.8 Exact Equations

Suppose $y(x)$ satisfies

$$F(x, y(x)) = c \qquad \text{(implicit soln.)}$$

Differentiate:

$$\frac{d}{dx} F(x,y(x)) = \boxed{\frac{\partial F}{\partial x}(x,y(x)) + \frac{\partial F}{\partial y}(x,y(x))\frac{dy}{dx}(x) = 0}$$

ODE that $y(x)$ satisfies

1.8.1 Solving exact equations

A 1st order ODE in the form

$$\boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0}$$

is exact if there exists $F(x,y)$ such that

$$\boxed{M = \frac{\partial F}{\partial x} \quad \text{and} \quad N = \frac{\partial F}{\partial y}}$$

In this case the solution is given

implicitly by

$$\boxed{F(x,y) = c}$$

where c is an arbitrary constant.

How can we tell if an ODE is exact?

Idea: $$\frac{\partial^2 F}{\partial x \, \partial y} = \frac{\partial^2 F}{\partial y \, \partial x}$$

Theorem 1.8.1 If $M(x,y)$ and $N(x,y)$ are continuously differentiable and if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

for all (x,y) near a point (x_0, y_0), then

$$M + N y' = 0$$

is exact near (x_0, y_0).

Example 1.8.A solve $\frac{dy}{dt} = -\frac{t^2+y^2}{2(t+1)y}$, $y(0)=-1$

Write as

$$\underbrace{t^2+y^2}_{M(t,y)} + \underbrace{2(t+1)y}_{N(t,y)} \frac{dy}{dt} = 0$$

Check if exact: is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$?

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(t^2+y^2) = 2y$$

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} 2(t+1)y = 2y$$

equal, so ODE is exact.

Look for $F(t,y)$ so that

$$\frac{\partial F}{\partial t} = M = t^2+y^2$$

and

$$\frac{\partial F}{\partial y} = N = 2(t+1)y$$

$$\frac{\partial F}{\partial t} = t^2+y^2 \Rightarrow \int (t^2+y^2)\,dt \quad (y \text{ constant})$$

$$F = \frac{1}{3}t^3 + ty^2 + A(y) \qquad (1)$$

a function of y only

$$\frac{\partial F}{\partial y} = 2ty+2y \Rightarrow F = \int (2ty+2y)\,dy$$

$$= (t+1)y^2 + B(t)$$

$$= ty^2 + y^2 + B(t) \qquad (2)$$

Compare (1) and (2) :

$$F = \frac{1}{3}t^3 + ty^2 + y^2 \quad (+\text{ const.})$$

Therefore

$$(t^2+y^2) + 2(t+1)y\frac{dy}{dt} = 0$$

$$\Leftrightarrow \quad \frac{d}{dt}\left(\frac{1}{3}t^3 + ty^2 + y^2\right) = 0$$

$$\Leftrightarrow \quad \frac{1}{3}t^3 + ty^2 + y^2 = \int 0\, dt$$
$$= c$$

Implicit solution is

$$\frac{1}{3}t^3 + ty^2 + y^2 = c \quad , \quad c \text{ arb.}$$

Convenient to find c now, using IC

$$y(0) = -1$$

(could be done after solving for y)

$t=0$, $y=-1$:

$$\frac{1}{3}0^3 + 0(-1)^2 + (-1)^2 = c$$
$$c = 1$$

Implicit soln. of IVP is

$$\frac{1}{3}t^3 + ty^2 + y^2 = 1$$

Explicit soln: solve for y

$$y = \pm\sqrt{\frac{1-\frac{1}{3}t^3}{t+1}}$$

Take − sign (why?)

$$y(t) = -\sqrt{\frac{1-\frac{1}{3}t^3}{t+1}}$$

Exercise: largest open interval in which soln. exists is $-1 < t < 3^{1/3}$

(contains $t=0$)

Example 1.8.B Solve

$$\underbrace{\frac{x^2+y^2}{x+1}}_{M(x,y)} + \underbrace{2y}_{N(x,y)}\frac{dy}{dx} = 0$$

Not separable, not linear; exact?

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x+1}\right) = \frac{2y}{x+1}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2y) = 0$$

(T.B.C.)

not equal, ODE is not exact

1.8.2 Integrating factors

$$M + N\frac{dy}{dx} = 0$$

Suppose

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

so ODE is not exact. There is a chance that by multiplying by an *integrating factor* $u(x,y)$, the resulting ODE would be exact:

$$uM + uNy' = 0$$

Exact?

Need $\dfrac{\partial}{\partial y}(uM) = \dfrac{\partial}{\partial x}(uN)$

$$\frac{\partial u}{\partial y}M + u\frac{\partial M}{\partial y} = \frac{\partial u}{\partial x}N + u\frac{\partial N}{\partial x}$$

Two special cases: i. $u = u(x)$ only

ii. $u = u(y)$ only

i. If $u = u(x)$ only then $\dfrac{\partial u}{\partial x} = u'$, $\dfrac{\partial u}{\partial y} = 0$

$$u\frac{\partial M}{\partial y} = u'N + u\frac{\partial N}{\partial x}$$

$$u' = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} u$$

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function <u>of x only</u>

then $u(x)$ satisfies a linear ODE, can solve.

ii. Similarly, if $u = u(y)$ only.