HW1 : 11:59pm

(8) Fri 2020-01-24

Today only : office hr 12 - 12:50 (not 1-2)

W2

n functions $y_1(x), \dots, y_n(x)$ are linearly independent in an interval I if

$$C_1 y_1(x) + \dots + C_n y_n(x) \equiv 0 \implies C_1 = \dots = C_n = 0$$

Theorem 2.1.3 (Gen. solu. of homog. solu.)
If $p(x)$, $q(x)$ are both continuous in an open interval I, and if $y_1(x)$, $y_2(x)$ are two linearly independent solutions of (2.2) $Ly = 0$, then the general solution of (2.2) is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

where C_1, C_2 are arbitrary constants.

2.2 Constant coefficient homogeneous linear second order ODEs

(2.3) $$ay'' + by' + cy = 0$$

a, b, c constants, $a \neq 0$

"Guess and check": let's guess there is a solution of the form $y(x) = e^{rx}$

Check by substituting into (2.3)

$$a\,y''(x) + b\,y'(x) + c\,y(x) \stackrel{?}{=} 0 \quad (\text{for all } x)$$

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} \stackrel{?}{=} 0$$

$$\underbrace{e^{rx}}_{>0}\left(ar^2 + br + c\right) \stackrel{?}{=} 0$$

$y(x) = e^{rx}$ is a solution if r satisfies

$$\underbrace{ar^2 + br + c = 0}_{\text{characteristic equation}} \quad \text{for (2.3)}$$

solutions $r = r_1, r_2$ where

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three cases

i. $b^2 - 4ac > 0 \iff r_1, r_2$ distinct and real

ii. $b^2 - 4ac = 0 \iff r_1, r_2$ real, not distinct

iii. $b^2 - 4ac < 0 \iff r_1, r_2$ distinct, not real

In case i.

$$y_1(x) = e^{r_1 x}, \quad y_2(x) = e^{r_2 x}$$

gen. soln. $\quad y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

<u>Example</u> 2.2.A Solve the IVP

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

Characteristic equ.

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r_1 = 2, \quad r_2 = -1$$

Gen. soln.

$$y(x) = C_1 e^{2x} + C_2 e^{-x}$$

(C_1, C_2 arbitrary)

$$y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$$

Initial conditions:

$$\left.\begin{aligned} y(0) &= C_1 + C_2 = 0 \\ y'(0) &= 2C_1 - C_2 = 1 \end{aligned}\right\}$$

Exercise $C_1 = \frac{1}{3}$, $C_2 = -\frac{1}{3}$

Solution to IVP

$$y(x) = \frac{1}{3} e^{2x} - \frac{1}{3} e^{-x}$$

In case ii. $b^2 - 4ac = 0$ "guess & check" method only gives one lin. indep. soln. $y_1(x) = e^{r_1 x}$

Theorem 2.2.1 (ii) If $r_1 = r_2$ (soln. of char. equ.), then the gen. soln. of (2.3) is

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x} = e^{r_1 x}(C_1 + C_2 x)}$$

where C_1, C_2 are arbitrary constants.

Exercise Check $y_2(x) = x e^{r_1 x}$ is a solu. in this case. Are $e^{r_1 x}$, $x e^{r_1 x}$ lin. indep.?

Example 2.2.B Find the gen. solu. of

$$2y'' + 4y' + 2y = 0$$

Char. equ.:

$$\begin{aligned} 2r^2 + 4r + 2 &= 0 \\ 2(r^2 + 2r + 1) &= 0 \\ 2(r+1)^2 &= 0 \\ r_1 = -1 \quad (&= r_2) \end{aligned}$$

Gen. solu. is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}$$

(C_1, C_2 arbitrary)

2.2.2 Complex numbers and Euler's formula

$$i = \sqrt{-1} \qquad \boxed{i^2 = -1}$$

Adding complex numbers

$$z_1 = 5 + 3i$$
$$z_2 = 2 - i$$

$$z_1 + z_2 = 5 + 3i + 2 - i = 5 + 2 + (3-1)i$$
$$= 7 + 2i$$

(exactly like adding in $\mathbb{R}^2$:

$$(5,3) + (2,-1) = (7,2) \;)$$

Multiplying complex numbers

$$z_1 z_2 = (5+3i)(2-i)$$
$$= (5)(2) + (5)(-i) + (3i)(2) + (3i)(-i)$$
$$= 10 - 5i + 6i - 3\underbrace{i^2}_{-1}$$
$$= 13 + i$$

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \cdots$$

$$= 1 + i\theta + \frac{1}{2!} i^2\theta^2 + \frac{1}{3!} i^3\theta^3 + \frac{1}{4!} i^4\theta^4 + \cdots$$

$$\begin{aligned}(i^2 = -1 \quad , \quad & i^3 = i^2 i = (-1) i = -i \\ & i^4 = i^3 i = (-i) i = -i^2 = 1 \\ & i^5 = i^4 i = 1 i = i \\ & i^6 = -1 \quad \text{etc.})\end{aligned}$$

$$e^{i\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3$$

$$+ \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 - \frac{1}{6!}\theta^6 - i\frac{1}{7!}\theta^7 + \cdots$$

$$= 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \cdots$$

$$+ i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \cdots \right)$$