W2 Fri 01-31

9 Mon 2020-01-27

Theorem 2.2.2 (Euler's formula)

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)} \quad \text{for all } \theta$$

2.2.3 Complex roots of the characteristic equ.

Case iii. $b^2 - 4ac < 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \overbrace{\sqrt{-1}}^{i}\sqrt{4ac - b^2}}{2a}$$

$$= \underbrace{-\frac{b}{2a}}_{\alpha} \pm i\, \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\beta \neq 0,\ \text{real}}$$

$(4ac - b^2 > 0)$

(2.3) $ay'' + by' + cy = 0$

Let $y_1(x) = e^{r_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$

$$= e^{\alpha x}[\cos(\beta x) + i\sin(\beta x)]$$

Let $y_2(x) = e^{r_2 x} = e^{(\alpha - i\beta)x} = \ldots$

$$= e^{\alpha x}[\cos(\beta x) - i\sin(\beta x)]$$

Then

$$y_3(x) = \frac{y_1(x) + y_2(x)}{2} = e^{\alpha x} \cos(\beta x)$$

$$y_4(x) = \frac{y_1(x) - y_2(x)}{2i} = e^{\alpha x} \sin(\beta x)$$

Exercise Verify $y_3(x)$, $y_4(x)$ are solutions of (2.3) in this case, and they are linearly independent.

Theorem 2.2.3 If $r_1, r_2 = \alpha \pm i\beta$, $\beta \neq 0$ then the general solution of (2.3) is

$$\boxed{\begin{aligned} y(x) &= C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x) \\ &= e^{\alpha x} \left[C_1 \cos(\beta x) + C_2 \sin(\beta x) \right] \end{aligned}}$$

(real solution, no "i" appears)

where C_1, C_2 are arbitrary constants

$$\left(\alpha = -\frac{b}{2a}, \ \beta = \left| \frac{\sqrt{4ac - b^2}}{2a} \right| \right)$$

Example 2.2.C

Solve the IVP $\quad y'' + 2y' + 2y = 0,$

$$y(0) = 0, \quad y'(0) = 1$$

Characteristic equation

$$r^2 + 2r + 2 = 0$$

$$r_1, r_2 = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm i\sqrt{4}}{2} = \frac{-2 \pm i2}{2}$$

$$= -1 \pm i$$

$$(\alpha = -1, \beta = 1)$$

General solution:

$$y(t) = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$$

$$y'(t) = C_1 \left[-e^{-t} \cos(t) - e^{-t} \sin(t) \right]$$

$$+ C_2 \left[-e^{-t} \sin(t) + e^{-t} \cos(t) \right]$$

At $t = 0$:

$$\left. \begin{aligned} y(0) &= C_1 &= 0 \\ y'(0) &= -C_1 + C_2 &= 1 \end{aligned} \right\} \Leftrightarrow \begin{aligned} C_1 &= 0 \\ C_2 &= 1 \end{aligned}$$

Solution of IVP is

$$y(t) = e^{-t}\sin(t)$$

Example 2.2.D Find the general solution of

$$x'' + \omega_0^2 x = 0$$

where $\omega_0 > 0$ is a constant.

Characteristic equation:

$$r^2 + \omega_0^2 = 0$$

$$r^2 = -\omega_0^2$$

$$r = \pm\sqrt{-\omega_0^2} = \pm i\omega_0$$

"purely imaginary"

$r_1 = i\omega_0$, $r_2 = -i\omega_0$ $\quad (\alpha = 0, \beta = \omega_0)$

General solution is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

where A, B are arbitrary constants.

By a trig. identity,

$$x(t) = C\cos(\omega_0 t - \gamma)$$

where C, γ arbitrary constants, but related to A, B.

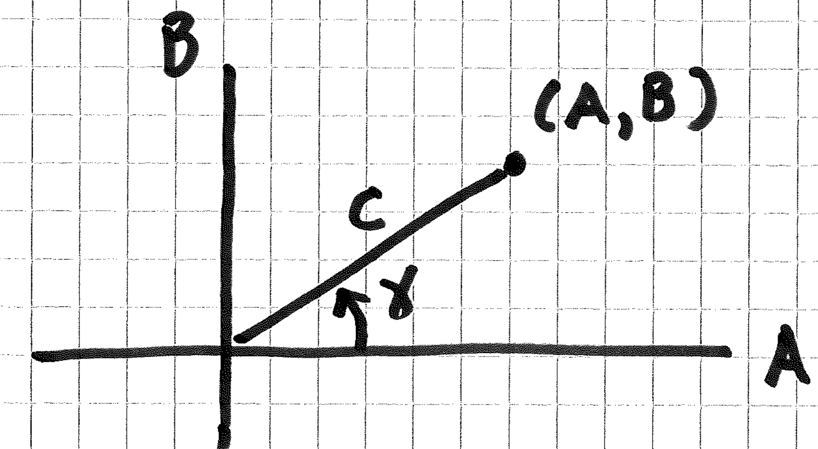
<u>Relationship between A, B and C, γ</u> :

$$x(t) = C\cos(\omega_0 t - \gamma)$$

$$= C\left[\cos(\omega_0 t)\cos(\gamma) + \sin(\omega_0 t)\sin(\gamma)\right]$$

$$= \underbrace{C\cos(\gamma)}_{A}\cos(\omega_0 t) + \underbrace{C\sin(\gamma)}_{B}\sin(\omega_0 t)$$

$$\boxed{A = C\cos(\gamma), \quad B = C\sin(\gamma)}$$



$$A^2 + B^2 = C^2 \cos^2(\gamma) + C^2 \sin^2(\gamma)$$

$$= C^2 \left[\cos^2(\gamma) + \sin^2(\gamma)\right]$$

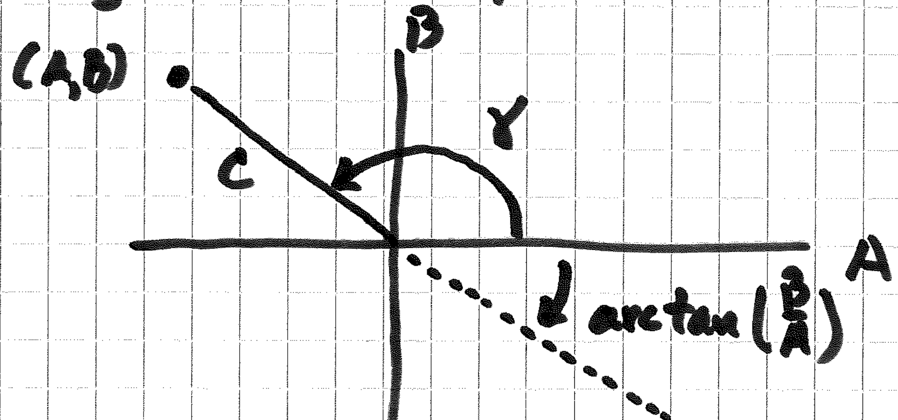
$$= C^2$$

$$\frac{B}{A} = \frac{C \sin(\gamma)}{C \cos(\gamma)} = \tan(\gamma)$$

$$\boxed{C = \sqrt{A^2 + B^2}, \quad \tan(\gamma) = \frac{B}{A}}$$

γ must be in the <u>correct quadrant</u>

e.g. if $A < 0$, $B > 0$



In this case

$$\gamma = \arctan\left(\frac{B}{A}\right) + \pi$$