Webwork W2 Fri 11:59pm

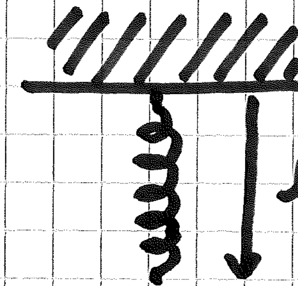
10 Wed 2020-01-29

2.4 Mechanical vibrations

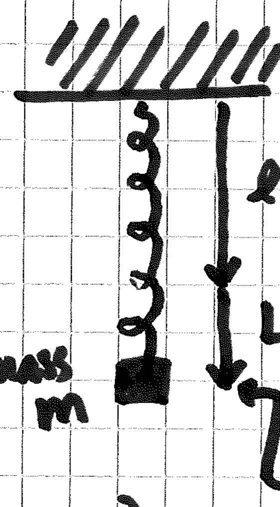
2.4.1 Some examples

(a) Mass-spring systems

Unstretched spring (no force)



statically stretched (gravity, no motion)



Measure displacement (in m) positive <u>downwards</u>

$$\text{Force of gravity} \quad F_{gravity} = \underset{\text{in kg}}{m}\,\underset{9.8\ \text{m}\cdot\text{s}^{-2}}{g} \quad \text{in N} = \text{kg}\cdot\text{m}\cdot\text{s}^{-2}$$

positive

Hooke's "law" for spring

$$F_{spring} \propto \text{displacement}$$

$$F_{spring} = -k \cdot \text{displacement}$$

negative (upwards)

$k > 0$ constant

in $N \cdot m^{-1} = kg \cdot s^{-2}$

e.g. if a force of 8 N stretches a spring 4 m, then the "spring constant" $k = \dfrac{8\,N}{4\,m} = 2\ N \cdot m^{-1}$

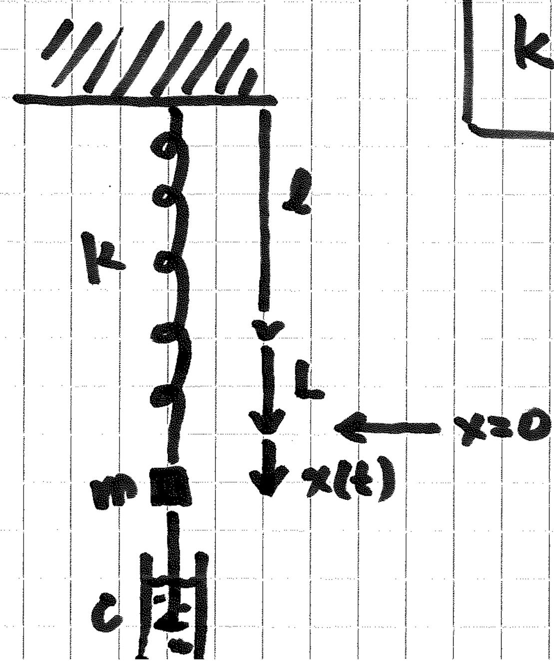
In our example, at equilibrium (no motion) the forces balance

$$F_{gravity} + F_{spring} = 0$$

$$mg - kL = 0$$

$$\boxed{k = \frac{mg}{L}}$$

In motion



L + x displ. from natural length

Model for damping (friction)

$$F_{damping} = -c\,\frac{dx}{dt}$$

$c \geqslant 0$ constant

in $\cancel{N\cdot m\cdot s^{-1}}$ $N\cdot s\cdot m^{-1}$
$= kg\cdot s^{-1}$

Dynamic force balance (Newton's "law")

$$m\,a = F_{gravity} + F_{damping} + F_{spring} + F_{external}$$

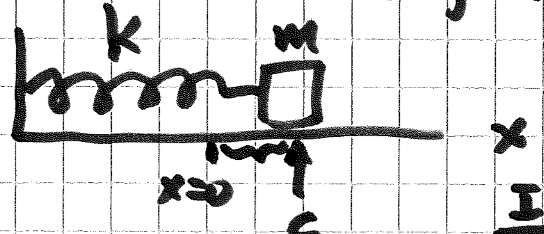
$$m x'' = mg - cx' - k(L+x) + F(t)$$

$$= \cancel{mg} - cx' - \cancel{kL} - kx + F(t)$$

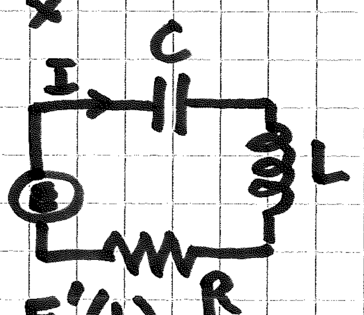
$$\boxed{m x'' + c x' + k x = F(t)}$$

same ODE for horizontal mass-spring system

$x(t)$ = displacement from equilibrium (rightward) (natural length)



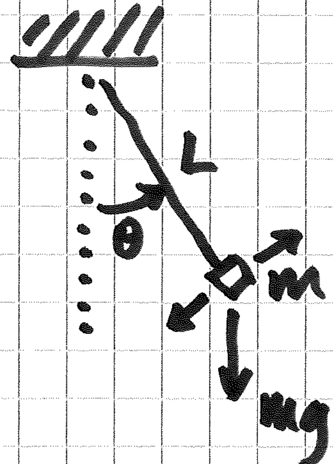
(b) Electric circuit



Current $I(t)$

$$L I'' + R I' + \frac{1}{C} I = E'(t)$$

(c) Pendulum



$$mL\theta'' = -mg\sin(\theta)$$

$$\theta'' + \frac{g}{L}\sin(\theta) = 0$$

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1 \quad \text{i.e. } \theta \approx \sin(\theta)$$

if θ is near 0

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$ (if θ is near 0)

(ignoring damping)

2.4.2 $\underbrace{\text{Free}}_{F(t)\equiv 0}$ $\underbrace{\text{undamped}}_{c=0}$ motion

$$mx'' + kx = 0 \qquad (m>0,\ k>0)$$

In standard form

$$x'' + \frac{k}{m}x = 0$$

Compare with Example 2.2.D $\qquad \omega_0^2 = \frac{k}{m}$

angular frequency of vibration $\qquad \boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$= \underbrace{C}_{\text{amplitude}} \cos(\omega_0 t - \underbrace{\gamma}_{\text{phase shift}})$$

<u>Example</u> 2.4.A A mass of 10 g hangs from a spring, stretching the spring 2.45 cm. The mass is lifted 2 cm from its equilibrium position and released from rest. Assume accel. due to gravity is $9.8\ m \cdot s^{-2}$. Find the <u>amplitude</u> and <u>phase shift</u> of the resulting motion.

ODE: $m x'' + k x = 0$

x in m
t in s
m in kg

$m = 0.01$ kg

$$k = \frac{mg}{L} = \frac{(0.01)(9.8)}{0.0245} = 4\ N \cdot m^{-1}$$

$$0.01\, x'' + 4\, x = 0$$

$$x'' + 400\, x = 0$$

Angular freq. $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{400} = 20\ s^{-1}$

Initial conditions:

$$x(0) = -0.02\ m, \quad x'(0) = 0\ m \cdot s^{-1}$$

Solu. (see Ex. 2.2.D)

$$x(t) = A\cos(20t) + B\sin(20t)$$

$$x'(t) = -20A\sin(20t) + 20B\cos(20t)$$

At $t = 0$:

$$x(0) = A = -0.02 \text{ m}$$

$$x'(0) = 20B = 0 \iff B = 0$$

$$x(t) = -0.02\cos(20t)$$

$$= \underset{\substack{\uparrow \\ \text{find}}}{C}\cos(20t - \underset{\substack{\uparrow \\ \text{find}}}{\gamma})$$