HW 2 Feb 14

MT 1 Feb 7 9:00 am

13 Wed 2020-02-05

Example 2.9.A cont.

Gen. soln. (by Thm.) is

$$y(x) = C_1 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{3}e^{2x}$$

Then

$$y'(x) = C_1[\cdots] + C_2[\cdots] + \frac{2}{3}e^{2x}$$

(Exercise)

and at $x = 0$

$$\left.\begin{aligned} y(0) &= C_1 + \frac{1}{3} = \frac{2}{3} \\ y'(0) &= \frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 + \frac{2}{3} = \frac{5}{6} \end{aligned}\right\} \Longleftrightarrow C_1 = \frac{1}{3},\ C_2 = 0$$

Soln. of IVP is

$$y(x) = \frac{1}{3}e^{\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{3}e^{2x}$$

<u>Note</u>: Find $y(x) = y_c(x) + y_p(x)$ <u>first</u>,

then solve C_1, C_2.

<u>Example</u> 2.5.B Find gen. soln.:

$$y'' - y' + y = \underbrace{\underbrace{e^{2x}}_{f_1(x)} + \underbrace{2\sin(3x)}_{f_2(x)}}_{f(x)}$$

Complementary solution:

$$y_c(x) = C_1 e^{\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

Particular soln.:

$$y_p(x) = \underbrace{y_{p_1}(x)}_{\frac{1}{3}e^{2x}} + y_{p_2}(x)$$

(see above)

Just need $y_{p_2}(x)$, corresponding to $f_2(x) = 2\sin(3x)$

Guess

$$y_{p_2}(x) = x^s\left[A\cos(3x) + B\sin(3x)\right]$$

$s = 0$ or 1 or 2.

Try $s=0$ first: is any term of

$$x^0\left[A\cos(3x)\right] + x^0\left[B\sin(3x)\right]$$

a soln. of $Ly = 0$? <u>No!</u> (Why?!)

So we should guess

$$y_{p_2}(x) = A\cos(3x) + B\sin(3x)$$

Check (find A, B): plug into

$$y_{p_2}''(x) - y_{p_2}'(x) + y_{p_2}(x) \stackrel{?}{=} f_2(x)$$

$$\left[-9A\cos(3x) - 9B\sin(3x)\right]$$

$$-\left[-3A\sin(3x) + 3B\cos(3x)\right]$$

$$+\left[A\cos(3x) + B\sin(3x)\right] \stackrel{?}{=} 2\sin(3x)$$

$$\underbrace{(-8A-3B)}_{=0}\cos(3x) + \underbrace{(3A-8B)}_{2}\sin(3x) \stackrel{?}{=} 0\cos(3x) + 2\sin(3x)$$

True, if and only if

$$\begin{cases} -8A - 3B = 0 \\ 3A - 8B = 2 \end{cases} \iff A = \frac{6}{73},\ B = -\frac{16}{73}$$

General solution:

$$y(x) = C_1 e^{\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$+ \frac{1}{3}e^{2x} + \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x),$$

where C_1, C_2 are arbitrary constants.

Example 2.5.C Find a particular solu.

$$y'' - 9y = -2e^{3x}$$

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Still start with compl. solu.:

$$r^2 - 9 = 0$$

$$r_1, r_2 = \pm 3$$

$$y_c(x) = C_1 e^{3x} + C_2 e^{-3x}$$

Particular solu.:

$$y_p(x) = x^s A e^{3x} \qquad s = 0 \text{ or } 1 \text{ or } 2$$

Try $s = 0$ first:

is $x^0 A e^{3x} = A e^{3x}$ a solution of $Ly = 0$?

Yes. (why?)

$y_p(x) = Ae^{3x}$ won't work!

Exercise: try it (wastes time)

Try $s = 1$ next:

is $x^1 A e^{3x} = A x e^{3x}$ a solu. of $Ly = 0$?

No. (why?)

So we should guess

$$y_p(x) = A x e^{3x}$$

Now determine A : plug into nonhomog. ODE

$$y_p''(x) - 9\,y_p(x) \stackrel{?}{=} -2e^{3x}$$

$$[Axe^{3x}]'' - 9[Axe^{3x}] \stackrel{?}{=} -2e^{3x}$$

$$A(\underbrace{3e^{3x} + 3e^{3x}}_{6e^{3x}} + \underline{9xe^{3x}}) - \underline{9Axe^{3x}} \stackrel{?}{=} -2e^{3x}$$

True, if $\quad 6A = -2$

$$A = -\tfrac{1}{3}$$

A particular soln. is

$$y_p(x) = -\tfrac{1}{3}\,x\,e^{3x}$$

(Another particular soln. is

$$y_p(x) = -\tfrac{1}{3}xe^{3x} + e^{3x} \quad . \quad \text{Why?}$$)

<u>Example</u> 2.5.D Set up the form of a particular soln but do not solve for the coefficients, for

$$y'' + 2y' + y = (1+5x)\,e^{-x}$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r_1, r_2 = -1$$

$$y_c(x) = C_1 e^{-x} + C_2 x e^{-x}$$

$$y_p(x) = x^s (A + Bx) e^{-x}$$

$s = 0$? $\quad Ae^{-x} + Bxe^{-x}$ so $s \neq 0$

$s = 1$? $\quad Axe^{-x} + Bx^2e^{-x}$ so $s \neq 1$

$s = 2$: $\quad y_p(x) = Ax^2e^{-x} + Bx^3e^{-x}$