HW 2: Fri Feb 10

15 Mon 2020-02-10

2.5.3 Variation of parameters

General method for nonhomog. linear equs.: const. or variable coeff. $Ly = f(x)$

f "nice" or not. But put equ. is in **standard form**

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Idea: if

$$y_c(x) = C_1\,y_1(x) + C_2\,y_2(x)$$

then guess

$$y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

is a soln. of $Ly = f(x)$, then check.

Example 2.5.E Find a particular solution to $y'' + y = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Complementary soln. (Exercise)

$$y_c(x) = C_1 \underbrace{\cos(x)}_{y_1(x)} + C_2 \underbrace{\sin(x)}_{y_2(x)}$$

C_1, C_2 arb. constants.

Guess soln. of nonhomg. equ. is

$$y(x) = u_1(x)\,\underset{\substack{\uparrow\\ \cos(x)}}{y_1(x)} + u_2(x)\,\underset{\substack{\nwarrow\\ \sin(x)}}{y_2(x)}$$

$$y = u_1 y_1 + u_2 y_2$$

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

Assume we can find u_1, u_2 so that

$$u_1' y_1 + u_2' y_2 = 0$$

Then

$$y' = u_1 y_1' + u_2 y_2'$$

and

$$y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

so

$$y'' + y = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + u_1 y_1 + u_2 y_2$$

$$= u_1' y_1' + u_2' y_2' + u_1 (\underbrace{y_1'' + y_1}_{=0}) + u_2 (\underbrace{y_2'' + y_2}_{=0})$$

$\stackrel{?}{=} f(x) = \tan(x)$

Yes, if $\underline{u_1' y_1' + u_2' y_2' = f(x)}$

So, $y = u_1 y_1 + u_2 y_2$ is a solution if we find u_1, u_2 so that

$$\boxed{\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= f(x) \end{aligned}}$$

i.e.

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

Solve for u_1', u_2'

$$\begin{cases} \cos(x)\, u_1' + \sin(x)\, u_2' = 0 \\ -\sin(x)\, u_1' + \cos(x)\, u_2' = \tan(x) \end{cases}$$

$u_2' = -\dfrac{\cos(x)}{\sin(x)} u_1'$,

$$-\sin(x)\, u_1' - \frac{\cos^2(x)}{\sin(x)} u_1' = \frac{\sin(x)}{\cos(x)}$$

$$-\cancel{\left[\sin^2(x) + \cos^2(x)\right]}$$

$$-\left[\underbrace{\sin^2(x)+\cos^2(x)}_{=1}\right] u_1' = \frac{\sin^2(x)}{\cos(x)}$$

$$u_1' = -\frac{\sin^2(x)}{\cos(x)}, \quad u_2' = \sin(x)$$

Integrate:

$$u_1(x) = \int u_1'(x)\,dx = -\int \frac{\sin^2(x)}{\cos(x)}\,dx$$

$$= \int \frac{\cos^2(x) - 1}{\cos(x)}\,dx$$

$$= \int \left[\cos(x) - \frac{1}{\cos(x)}\right] dx$$

$$= \int \cos(x)\,dx - \int \sec(x)\,dx$$

$$= \sin(x) - \ln\left|\sec(x) + \tan(x)\right| + C_1$$

$$u_2(x) = \int u_2'(x)\,dx = \int \sin(x)\,dx$$

$$= -\cos(x) + C_2$$

So we get solutions

$$y(x) = \left[\cancel{\sin(x)} - \ln \left| \sec(x) + \tan(x) \right| + C_1 \right] \cos(x)$$

$$+ \left[-\cancel{\cos(x)} + C_2 \right] \sin(x)$$

For a particular soln. choose values for C_1, C_2

e.g. $C_1 = 0$, $C_2 = 0$

$$y_p(x) = -\ln \left| \sec(x) + \tan(x) \right| \cos(x)$$

Another way to solve for u_1', u_2'

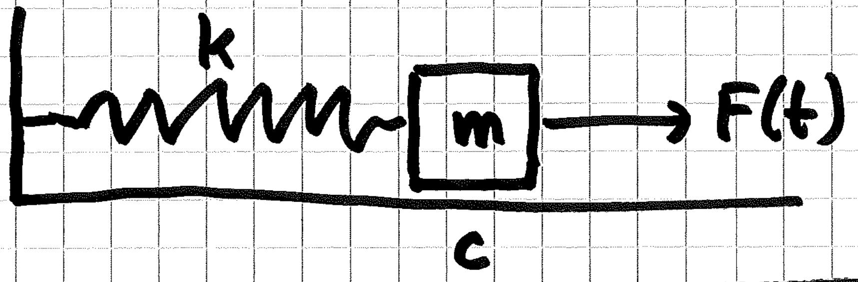
$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$= \frac{1}{\underbrace{y_1 y_2' - y_2 y_1'}_{W \text{ ("Wronskian")}}} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$= \frac{1}{W} \begin{bmatrix} -y_2 f(x) \\ y_1 f(x) \end{bmatrix}$$

i.e. $u_1' = -\dfrac{y_2 f(x)}{W}$, $u_2' = \dfrac{y_1 f(x)}{W}$

2.6 Forced oscillations and resonance



$$\boxed{m x'' + c x' + k x = F(t)}$$

$$m,\ k > 0\ ,\quad c \geq 0$$

$x(t)$ is displacement from equilibrium (rest position)

$c = 0$

Undamped forced motion; resonance

$$\underbrace{m x'' + k x}_{Lx} = \underbrace{F_0 \cos(\omega t)}_{F(t)} \qquad F_0, \omega > 0$$

$$m r^2 + k = 0$$

$$r^2 = -\frac{k}{m} \qquad r_1, r_2 = \pm i \omega_0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

Complementary soln.

$$x_c(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

Soln. $x(t) = x_c(t) + x_p(t)$

Undetermined coeffs:

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$