Undetermined coeffs:

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

<u>if $\omega \neq \omega_0$</u>

<u>Exercise</u>: $A = \dfrac{F_0}{m(\omega_0^2 - \omega^2)}$, $B = 0$

<u>Example</u> 2.6.A $m = 1$, $c = 0$, $k = 9$

$F_0 = 80$, $\omega = 5$ (all SI units)

$$\begin{cases} x'' + 9x = 80\cos(5t) \\ x(0) = 0, \quad x'(0) = 0 \end{cases}$$

Exercise: $\omega_0 = \sqrt{9} = 3$

$$x_c(t) = C_1\cos(3t) + C_2\sin(3t)$$

General soln:

$$x(t) = C_1\cos(3t) + C_2\sin(3t) - 5\cos(5t)$$

then

$$x'(t) = -3C_1\sin(3t) + 3C_2\cos(3t) + 25\sin(5t)$$

At $t=0$:

$$\left.\begin{aligned} x(0) &= C_1 - 5 = 0 \\ x'(0) &= 3C_2 = 0 \end{aligned}\right\} \Leftrightarrow C_1 = 5,\ C_2 = 0$$

Soln. of IVP is

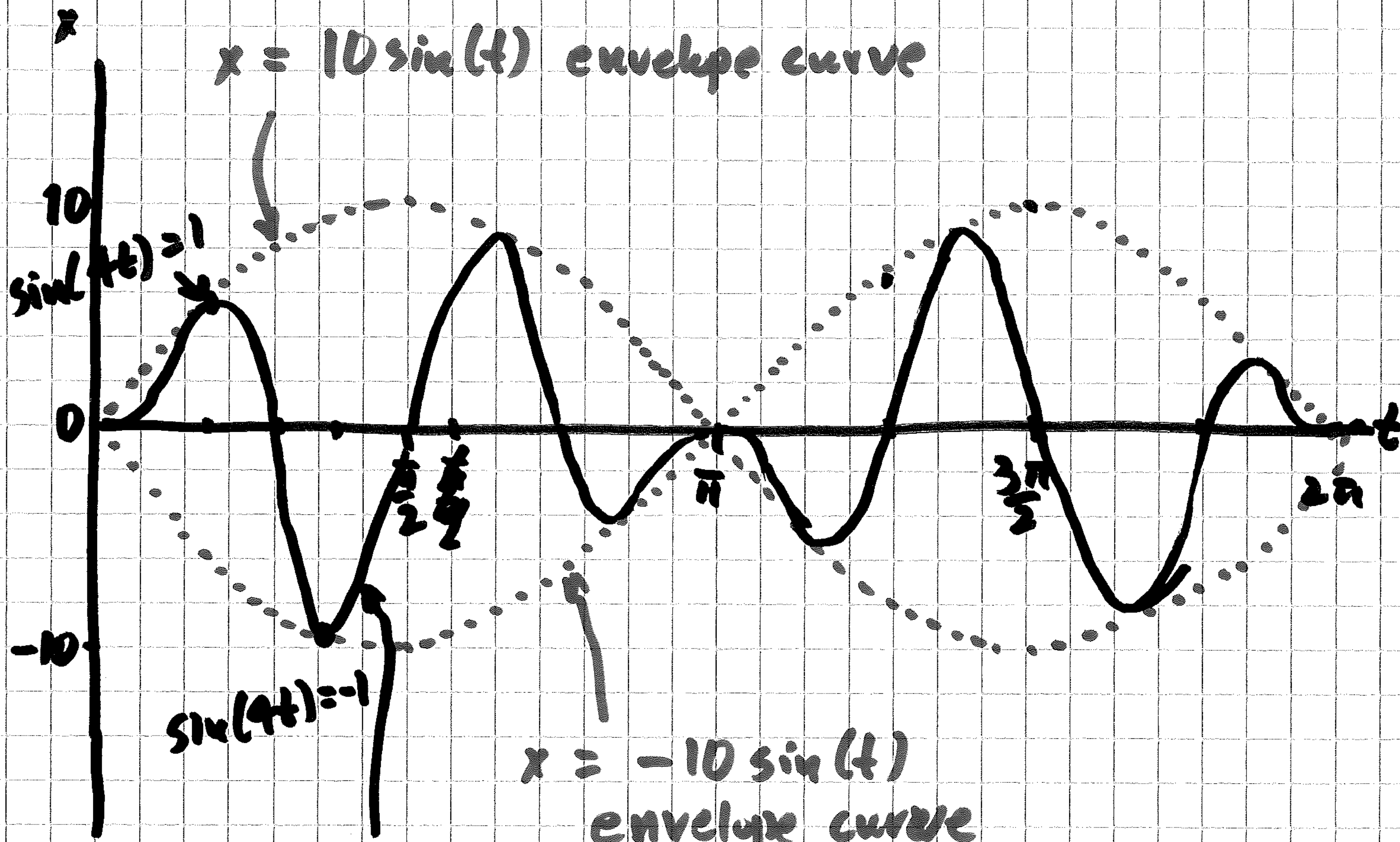
$$x(t) = 5\cos(3t) - 5\cos(5t)$$

$$\boxed{\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin(\alpha)\sin(\beta)}$$

$$\boxed{\begin{aligned} \alpha &= \left(\frac{\omega_0+\omega}{2}\right)t \\ \beta &= \left(\frac{\omega_0-\omega}{2}\right)t \end{aligned}} \quad \begin{aligned} &= \frac{3+5}{2}t = 4t \\ &= \frac{3-5}{2}t = -t \end{aligned} \qquad \begin{aligned} \omega_0 &= 3 \\ \omega &= 5 \end{aligned}$$

Express $x(t)$ as

$$\begin{aligned} x(t) &= -5\,[\overset{\omega t}{\cos(5t)} - \overset{\omega_0 t}{\cos(3t)}] \\ &= -10\sin(4t)\sin(-t) \\ &= \underbrace{10\sin(t)}_{\substack{\text{slowly varying} \\ \text{"amplitude"}}}\ \underbrace{\sin(4t)}_{\text{rapidly varying between } \pm 1} \end{aligned}$$



$$x = 10\sin(t)\sin(4t) = 5\cos(3t) - 5\cos(5t)$$

"beats" or "amplitude modulation (AM)"

Case ii. $\underline{\omega_0 = \omega}$

Particular soln. now must be

$$x_p(t) = t\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

<u>**Example**</u> 2.6.B Same m, $c = 0$, k, F_0 as above, but now $\omega = 3$, same IC.

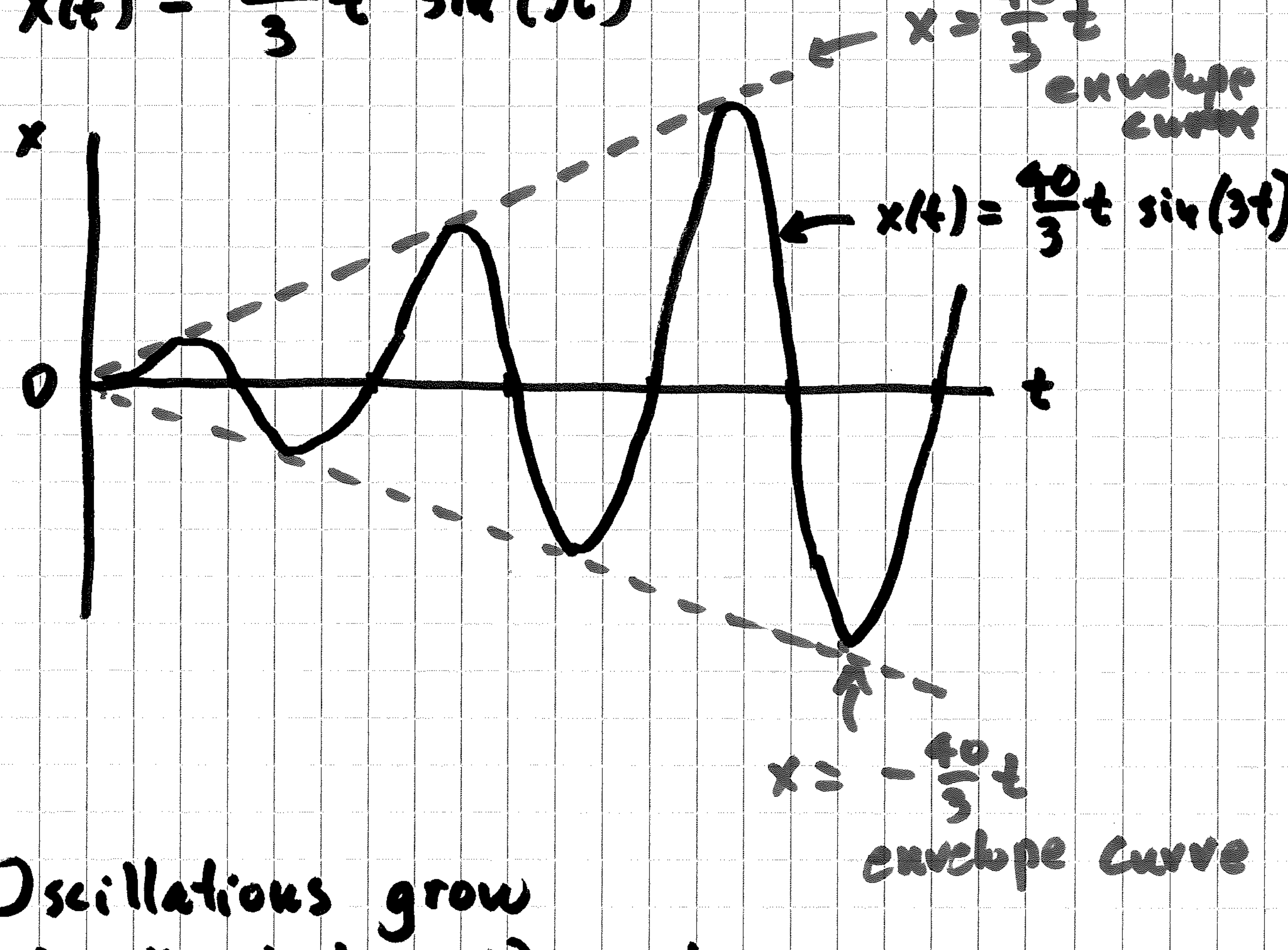
$$\begin{cases} x'' + 9x = 80\cos(3t) \\ x(0) = 0, \ x'(0) = 0 \end{cases}$$

Exercise: general solution is

$$x(t) = C_1\cos(3t) + C_2\sin(3t) + \frac{40}{3}t\sin(3t)$$

soln. of IVP

$$x(t) = \frac{40}{3}t\sin(3t)$$



Oscillations grow (without bound) as $t \to \infty$

"(pure) resonance"

no damping

2.6.2 Damped forced oscillation and "practical" resonance

In "real" life $c > 0$, sometimes small.

$$m x'' + c x' + k x = F_0 \cos(\omega_{\cancel{0}} t)$$

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Three cases for $x_c(t)$

i. $c^2 - 4mk > 0$

ii. $c^2 - 4mk = 0$

iii. $c^2 - 4mk < 0$

If $m, c, k > 0$ then in all three cases

$$x_c(t) \to 0 \quad \text{as} \quad t \to \infty$$

and in all three cases

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$= C\cos(\omega t - \gamma)$$

$$C = \sqrt{A^2 + B^2}, \quad \tan(\gamma) = \frac{B}{A}$$

Solu. of ODE + IVP looks like

$$x(t) = \underbrace{x_c(t)}_{\substack{\to 0 \text{ as } t\to\infty \\ \text{"transient solution"} \\ x_{tr}(t)}} + \underbrace{C\cos(\omega t - \gamma)}_{\substack{x_{sp}(t) \\ \text{"steady periodic solution"}}}$$