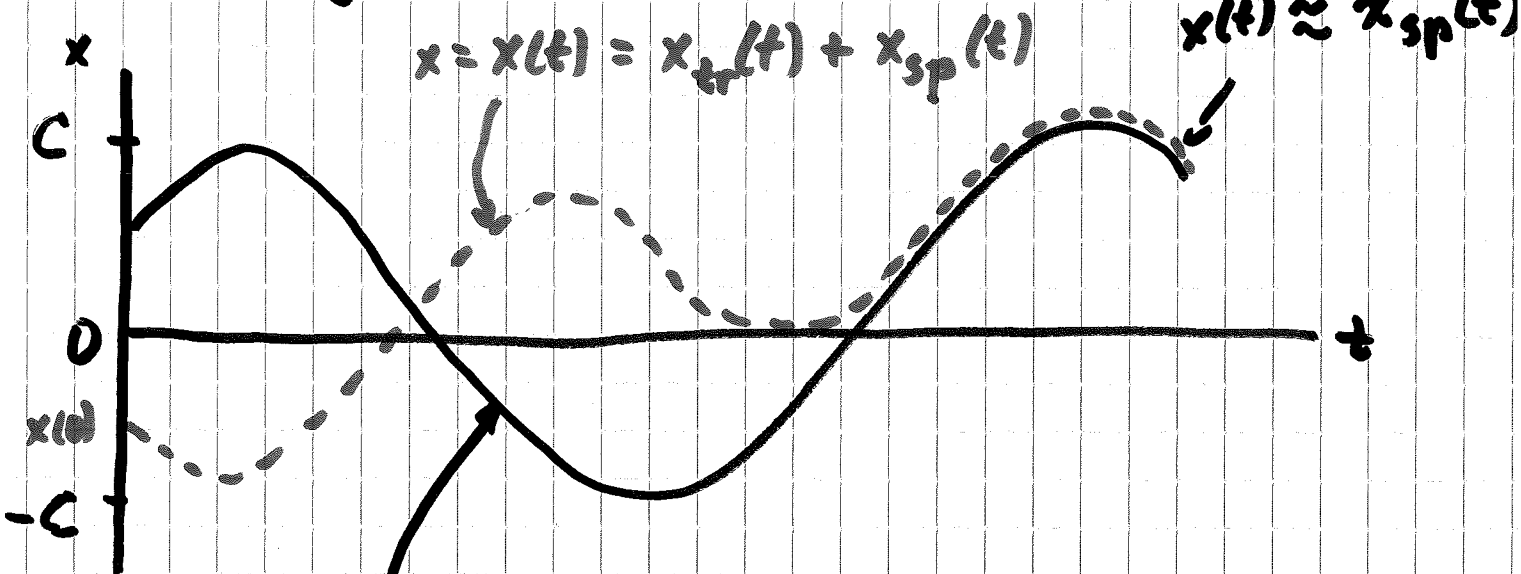
HW2 11:59 pm
HW3 Fri Feb 28

(17) Fri 2020-02-14

Typical graph

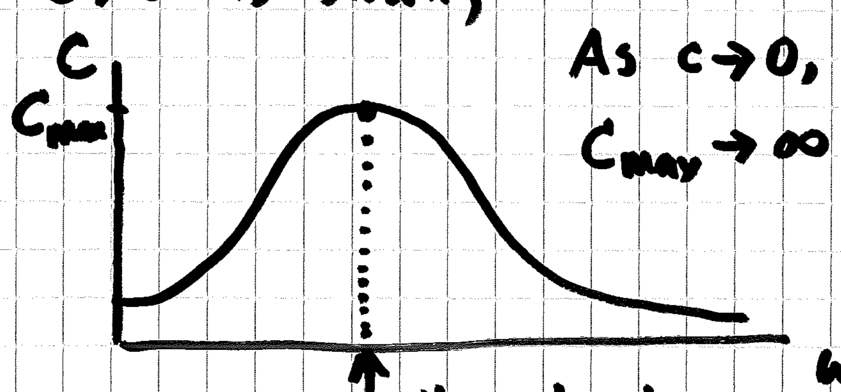


$x = x_{sp}(t) = C\cos(\omega t - \gamma)$

If m, c, k, F_0 are fixed, and consider $\omega > 0$ as a parameter, the A, B will be expressions involving ω : $A = A(\omega)$, $B = B(\omega)$

$$C = C(\omega) = \sqrt{A(\omega)^2 + B(\omega)^2}$$

If $c > 0$ is small,



As $c \to 0$, $C_{max} \to \infty$

near (but not =) $\omega_0 = \sqrt{\frac{k}{m}}$

"practical resonance frequency"

Chapter 3 Systems of ODEs

3.1 Introduction to systems of ODEs

First order systems

$$\begin{cases} x_1' = g_1(x_1, x_2, \dots, x_n, t) \\ x_2' = g_2(x_1, x_2, \dots, x_n, t) \\ \quad\vdots \\ x_n' = g_n(x_1, x_2, \dots, x_n, t) \end{cases}$$

in vector notation

$$\vec{x}' = \vec{g}(\vec{x}, t) \quad , \quad \vec{x} \in \mathbb{R}^n$$

A solution $\vec{x}(t) = (x_1(t) \dots x_n(t))^T$ is continuous, and satisfies the system of ODEs when it is substituted in.

Example 3.1. A Write $y'' + p(t)y' + q(t)y = f(t)$ as an (equivalent) 1st order system.

Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$. Then

$$x_1' = y' = x_2$$

$$x_2' = y'' = -q(t)\,y - p(t)\,y' + f(t)$$
$$= -q(t)\,x_1 - p(t)\,x_2 + f(t)$$

1st order system is

$$\begin{cases} x_1' = x_2 \\ x_2' = -q(t)\,x_1 - p(t)\,x_2 + f(t) \end{cases}$$

or

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\underbrace{\begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}}_{P(t)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}}_{\vec{g}(\vec{x},\, t)}$$

A system

$$\vec{x}\,' = \vec{g}(\vec{x})$$

is <u>autonomous</u> if $\vec{g}(\vec{x})$ does not depend explicitly on (the indep. variable) t

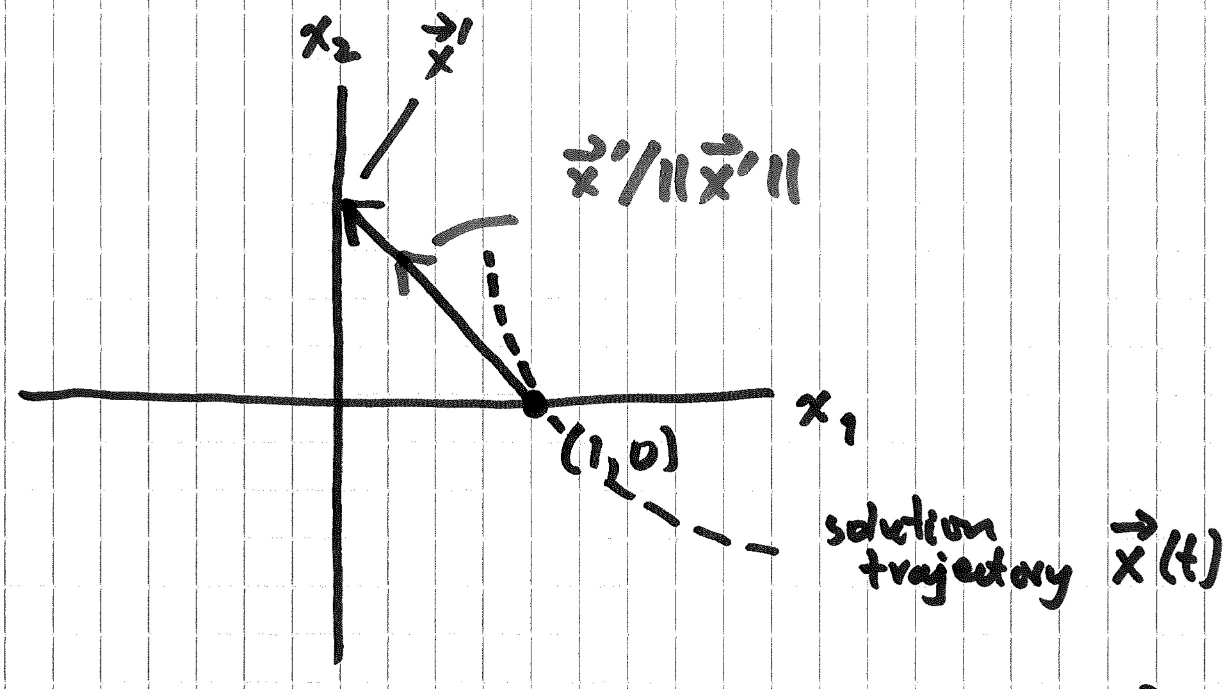
For autonomous systems, it is meaningful to plot a vector field or direction field : at each point $\vec{x}$, the vector $\vec{g}(\vec{x})$ specifies the "phase velocity" (tangent) vector $\vec{x}'$ ~~and~~ and

"direction" vector $\dfrac{\vec{x}'}{\|\vec{x}'\|}$ that is tangent to the solution trajectory $\vec{x}(t)$ in $\mathbb{R}^n$

Example 3.1.B
$$\begin{cases} x_1' = -x_1 + 2x_2 \\ x_2' = x_1 \end{cases}$$

is autonomous. At the point $\vec{x} = (1,0)$ the vector is
$$\vec{x}' = \vec{g}(\vec{x}) = \begin{bmatrix} -1 + 2(0) \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left(\text{direction is } \frac{\vec{x}'}{\|\vec{x}'\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right)$$



Do this for many points $\vec{x}$ (e.g. in a grid) to get an overall picture of the vector field or direction field.

3.3 Linear sytems of ODEs

$$\vec{x}\,' = P(t)\vec{x} + \vec{f}(t)$$

e.g. $(n=2)$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2t & e^t \\ 1/t & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

Homogeneous if $\vec{f}(t) \equiv 0$

$$(*) \qquad \vec{x}' = P(t)\,\vec{x}$$

Theorem 3.3.1 If $P(t)$ is a continuous $n \times n$ matrix and if $(*)$ has n linearly independent solutions $\vec{x}_1(t), \dots, \vec{x}_n(t)$ then the general solution of $(*)$ is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + \dots + c_n \vec{x}_n(t)$$

where $c_1, \dots, c_n$ are arbitrary constants.

Vector functions $\vec{x}_1(t), \dots, \vec{x}_n(t)$ are linearly independent in an interval I if

$$c_1 \vec{x}_1(t) + \dots + c_n \vec{x}_n(t) \underbrace{= 0 \text{ for all } t \text{ in } I}_{\text{i.e. } \equiv 0}$$

has only the solution $c_1 = 0, \dots, c_n = 0$

If $n=2$ and $\vec{x}_1(t), \vec{x}_2(t)$ are solutions of $(*)$, they are linearly indep. if and only if one is not a constant multiple of the other.