HW3 Fri Feb 28

18 Mon 2020-02-24

For any $\vec{x}_j(t) \in \mathbb{R}^n$, $j = 1, .., n$ it is easy to see (Exercise) that

$$c_1 \vec{x}_1(t) + \dots + c_n \vec{x}_n(t) = X(t)\vec{c}$$

where $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and $X(t) = \begin{bmatrix} \vec{x}_1(t) & \dots & \vec{x}_n(t) \end{bmatrix}$

is the $n \times n$ matrix whose columns are $\vec{x}_j(t)$.

If $\vec{x}_1(t), \dots, \vec{x}_n(t)$ are linearly independent solutions of $(*)$ then the corresponding $X(t)$ is called a fundamental matrix (solution) of $(*)$

Exercise (easiest if $n=2$)

$$\det X(t) \neq 0 \quad \text{and} \quad X'(t) = \underbrace{P(t)\,X(t)}_{\text{mult. } n\times n \text{ matrices}}$$

Example 3.3.A

Exercise: verify that $\vec{x}_1(t) = \begin{bmatrix} e^t \\ \frac{1}{2}e^t \end{bmatrix}$, $\vec{x}_2(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$

are lin. indep. solutions of

$$\vec{x}\,' = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \vec{x}$$

Then

$$X(t) = \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2}e^{t} & e^{-t} \end{bmatrix}$$ is a fundamental matrix

Observe that

$$\det X(t) = (e^{t})(e^{-t}) - (\tfrac{1}{2}e^{t})(0) = 1 \neq 0$$

$$X'(t) = \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2}e^{t} & -e^{-t} \end{bmatrix}$$

$$P(t)\,X(t) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ \frac{1}{2}e^{t} & e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2}e^{t} & -e^{-t} \end{bmatrix}$$

$$= \checkmark$$

$$X(t)\vec{c} = \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2}e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ \frac{1}{2}c_1 e^{t} + c_2 e^{-t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} e^{t} \\ \frac{1}{2}e^{t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

3.4 Eigenvalue method

(for constant coefficient, i.e. autonomous, homogeneous linear systems)

$$(**) \qquad \vec{x}' = A\vec{x}$$

where A is a constant <u>real</u> $n \times n$ matrix.

Guess: a solution of $(**)$ has the form

$$\vec{x}(t) = \vec{v}\, e^{\lambda t}$$

for some constant vector $\vec{v}$ and constant scalar (i.e. number) λ

Check: plug into $(**)$ and see

$$\vec{x}'(t) = \vec{v}\,\lambda e^{\lambda t} = \lambda e^{\lambda t}\,\vec{v}$$

$$A\vec{x}(t) = A\vec{v}\,e^{\lambda t} = e^{\lambda t} A\vec{v}$$

$$\underbrace{\lambda \cancel{e^{\lambda t}}\,\vec{v}}_{\vec{x}'(t)} = \underbrace{\cancel{e^{\lambda t}}\,A\vec{v}}_{A\vec{x}(t)}$$

is true if (and only if) $\quad A\vec{v} = \lambda\vec{v}$

Conclusion: $\vec{x}(t) = \vec{v}\, e^{\lambda t}$

is a solution, provided that λ is an eigenvalue of A, and $\vec{v}$ is a corresponding eigenvector.

3.4.2 The eigenvalue method with distinct real eigenvalues

Theorem 3.4.1 If the $n \times n$ constant real matrix A has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ then there exist n linearly independent corresponding real eigenvectors $\vec{v}_1, \dots, \vec{v}_n$, and the general solution to $(**)$ is

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}$$

where $c_1, \dots, c_n$ are arbitrary constants.

Example 3.4.A

Solve the IVP $\quad x' = x + 12y, \quad y' = 3x + y;$

$$x(0) = 0, \quad y(0) = 1$$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$: $\quad \vec{x}' = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Characteristic equation:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - 36$$

$$= \lambda^2 - 2\lambda - 35 = (\lambda + 5)(\lambda - 7)$$

Eigenvalues: $\quad \lambda_1 = -5, \quad \lambda_2 = 7$

Eigenvector of A corresponding to $\lambda_1 = -5$

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$

i.e. $(A - \lambda_1 I)\vec{v}_1 = \vec{0} \qquad \vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$

$$\begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Any nonzero soln. of

$$v_{11} + 2 v_{12} = 0$$

e.g. $v_{11} = -2$, $v_{12} = 1$, $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(The 1-dim. eigenspace for $\lambda_1 = -5$ is

$$c_1 \vec{v}_1 = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1 \\ c_1 \end{bmatrix} \quad c_1 \text{ arbitrary } \neq 0 \;)$$

One solution is

$$\vec{x}_1(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} = \begin{bmatrix} -2e^{-5t} \\ e^{-5t} \end{bmatrix}$$

Eigenvector corresp. to $\lambda_2 = 7$

<u>Exercise</u>: $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\vec{x}_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t}$$

Gen. soln

$$\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t}$$