HW 3 Fri 11:59 pm

19 Wed 2020-02-26

Ex. 3.4.A, cont.

At $t=0$

$$\vec{x}(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

i.e. $\begin{cases} -2c_1 + 2c_2 = 0 \\ c_1 + c_2 = 1 \end{cases}$

$\iff c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}$

Soln. of IVP is

$$\vec{x}(t) = \frac{1}{2} \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t}$$

with components

$$x(t) = -e^{-5t} + e^{7t}$$

$$y(t) = \frac{1}{2} e^{-5t} + \frac{1}{2} e^{7t}$$

3.4.3 Complex eigenvalues

$$(**) \qquad \vec{x}' = A\vec{x}$$

Theorem 3.4.2 If A is a constant real square matrix and if it has a complex (and nonreal) eigenvalue $\lambda_1 = a + ib$, a, b real, $b \neq 0$, with corresponding complex eigenvector $\vec{v}_1$, then it also has an eigenvalue $\lambda_2 = \overline{\lambda_1} = a - ib$, with corresponding eigenvector $\vec{v}_2 = \overline{\vec{v}_1}$. Furthermore, ~~the~~ two lin. indep. real solutions of $(**)$ are

$$\vec{x}_1(t) = \mathrm{Re}\left(\vec{v}_1 e^{\lambda_1 t}\right)$$

$$\vec{x}_2(t) = \mathrm{Im}\left(\vec{v}_1 e^{\lambda_1 t}\right)$$

(use Euler's formula)

<u>Example</u> 3.4.B

Write $u'' + 2u' + 2u = 0$, $u(0) = 0$, $u'(0) = 1$

as an IVP for a 1^{st}-order system, then solve the IVP for the system

(Compare Example 2.2.C ⑨ 2020·07·27)

Let $x_1 = u$, $x_2 = u'$. Then

$$x_1' = u' = x_2 \quad , \quad x_2' = u'' = -2u - 2u'$$
$$= -2x_1 - 2x_2$$

$$\begin{cases} x_1' = x_2 & x_1(0) = 0 \\ x_2' = -2x_1 - 2x_2 & x_2(0) = 1 \end{cases}$$

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{x} \quad , \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characteristic equation:

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix}$$

$$= \lambda(2+\lambda) + 2$$
$$= \lambda^2 + 2\lambda + 2$$

Eigenvalues

$$\lambda_1 = -1 + i \ , \quad \lambda_2 = -1 - i$$

By Thm. 3.4.2, just find eigenvector $\vec{v}_1$ for $\lambda_1 = -1+i$:

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

$$\begin{bmatrix} 1-i & 1 \\ -2 & \cancel{-1+i} \ -1-i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (1-i)\, v_{11} + v_{12} = 0 \\ -2 v_{11} + (\cancel{-1+i} \ -1-i) v_{12} = 0 \end{cases}$$

must be lin. dependent (why?)

Exercise. Show that one equ. is a complex scalar multiple of the other equ.

One nonzero solu. is

$$v_{11} = 1 \ , \quad v_{12} = -1+i$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$$

By Thm. 3.4.2 : find Re, Im of

$$\vec{v}_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} e^{(-1+i)t}$$

$$= e^{-t} \begin{bmatrix} 1 \\ -1+i \end{bmatrix} \overbrace{\{\cos(t) + i\sin(t)\}}^{e^{it}}$$

$$= e^{-t} \begin{bmatrix} \cos(t) + i\sin(t) \\ -\cos(t) - i\sin(t) + i\cos(t) + i^2\sin(t) \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \cos(t) + i\sin(t) \\ -\cos(t) - \sin(t) - i\sin(t) + i\cos(t) \end{bmatrix}$$

$$= \underbrace{e^{-t} \begin{bmatrix} \cos(t) \\ -\cos(t) - \sin(t) \end{bmatrix}}_{Re(\vec{v}_1 e^{\lambda_1 t})} + i\, \underbrace{e^{-t} \begin{bmatrix} \sin(t) \\ -\sin(t) + \cos(t) \end{bmatrix}}_{\substack{Im(\vec{v}_1 e^{\lambda_1 t}) \\ (\text{without the } i)}}$$

Gen. soln. is

$$\vec{x}(t) = c_1 \begin{bmatrix} e^{-t}\cos(t) \\ -e^{-t}\{\cos(t)+\sin(t)\} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\sin(t) \\ e^{-t}\{\cos(t)-\sin(t)\} \end{bmatrix}$$

where c_1, c_2 are arbitrary <u>real</u> constants.

At $t=0$

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1+c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Longleftrightarrow \quad c_1 = 0, \quad c_2 = 1$$

Soln. of IVP for system is

$$\vec{x}(t) = \begin{bmatrix} e^{-t}\sin(t) \\ e^{-t}\cos(t) - e^{-t}\sin(t) \end{bmatrix} \begin{matrix} \leftarrow u(t) \\ \leftarrow u'(t) \end{matrix}$$

Ex. 2.2.C

3.5 <u>Two-dimensional systems and their vector fields</u>

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 real const. matrix

Solution trajectories or "orbits"

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$ are parametrized curves in $\mathbb{R}^2$

The diagram in $\mathbb{R}^2$, of all trajectories is called the phase portrait of the system.

In practice, enough representative trajectories to show all typical solution behaviour.