HW3 11:59 pm
W3 Fri Mar 6

(20) Fri 2020-02-28

... Draw arrows on trajectories to inicate increasing t.

Trajectories must be consistent with vector field, or direction field

There is always (at least) one critical point (soln. of $A\vec{x} = 0$), namely $\vec{x} = \vec{0}$, corresponding to an equilibrium solution

$$\vec{x}(t) \equiv \vec{0}$$

"Case 3" Distinct real eigenvalues with opposite signs $\lambda_1 < 0 < \lambda_2$

Classification: saddle point Stability: unstable

General solution:

$$\vec{x}(t) = c_1 \vec{v}_1 \underbrace{e^{\lambda_1 t}}_{\substack{\to 0 \\ \text{as } t \to \infty}} + c_2 \vec{v}_2 \underbrace{e^{\lambda_2 t}}_{\substack{\to \infty \\ \text{as } t \to \infty}}$$

(pos, neg, zero) $\nearrow$ c_1

If $c_1 = 0$, exp. growth along the line $c_2 \vec{v}_2$

If $c_2 = 0$, exp. decay along the line $c_1 \vec{v}_1$

<u>Example</u> 3.5.A $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(Triangular matrix) Eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$

Know this $(\vec{0})$ is a saddle point, unstable

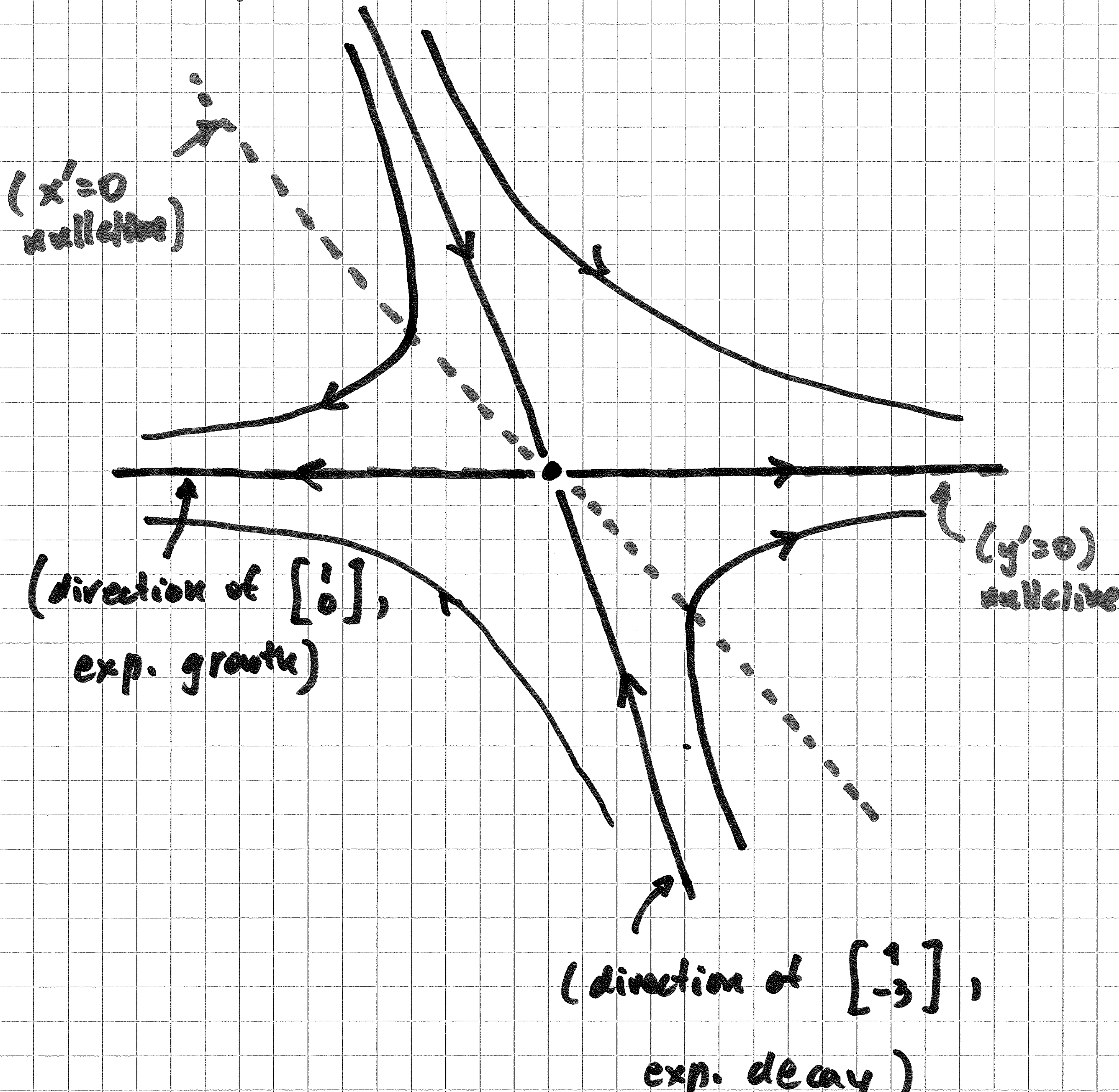
Exercise: Gen. solu. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}$

<u>Nullclines</u> $x' = 0$ or $y' = 0$

<u>$x' = 0$</u> $\iff$ $x + y = 0$ $\iff$ $y = -x$ (vector field is vertical)

<u>$y' = 0$</u> $\iff$ $-2y = 0$ $\iff$ $y = 0$ (vector field is horizontal)

Phase portrait



"Case 1" <u>Distinct real eigenvalues, both positive</u>

$0 < \lambda_1 < \lambda_2$

Classification: (nodal) source or unstable node

Stability: unstable

$$\vec{x}(t) = c_1 \vec{v}_1 \underbrace{e^{\lambda_1 t}}_{\to\infty} + c_2 \vec{v}_2 \underbrace{e^{\lambda_2 t}}_{\to\infty}$$

as $t \to \infty$

$\|\vec{x}(t)\| \to \infty$ as $t \to \infty$

More detail: eigenvalue with larger abs. value is "strong", other one is "weak": $e^{\lambda_2 t}$ grows faster than $e^{\lambda_1 t}$

$$\vec{x}(t) = e^{\lambda_1 t}\left[c_1 \vec{v}_1 + c_2 \vec{v}_2 \underbrace{e^{(\lambda_2-\lambda_1)t}}_{\to 0} \right]$$

as $t \to -\infty$

$\approx e^{\lambda_1 t} c_1 \vec{v}_1$ for t large negative

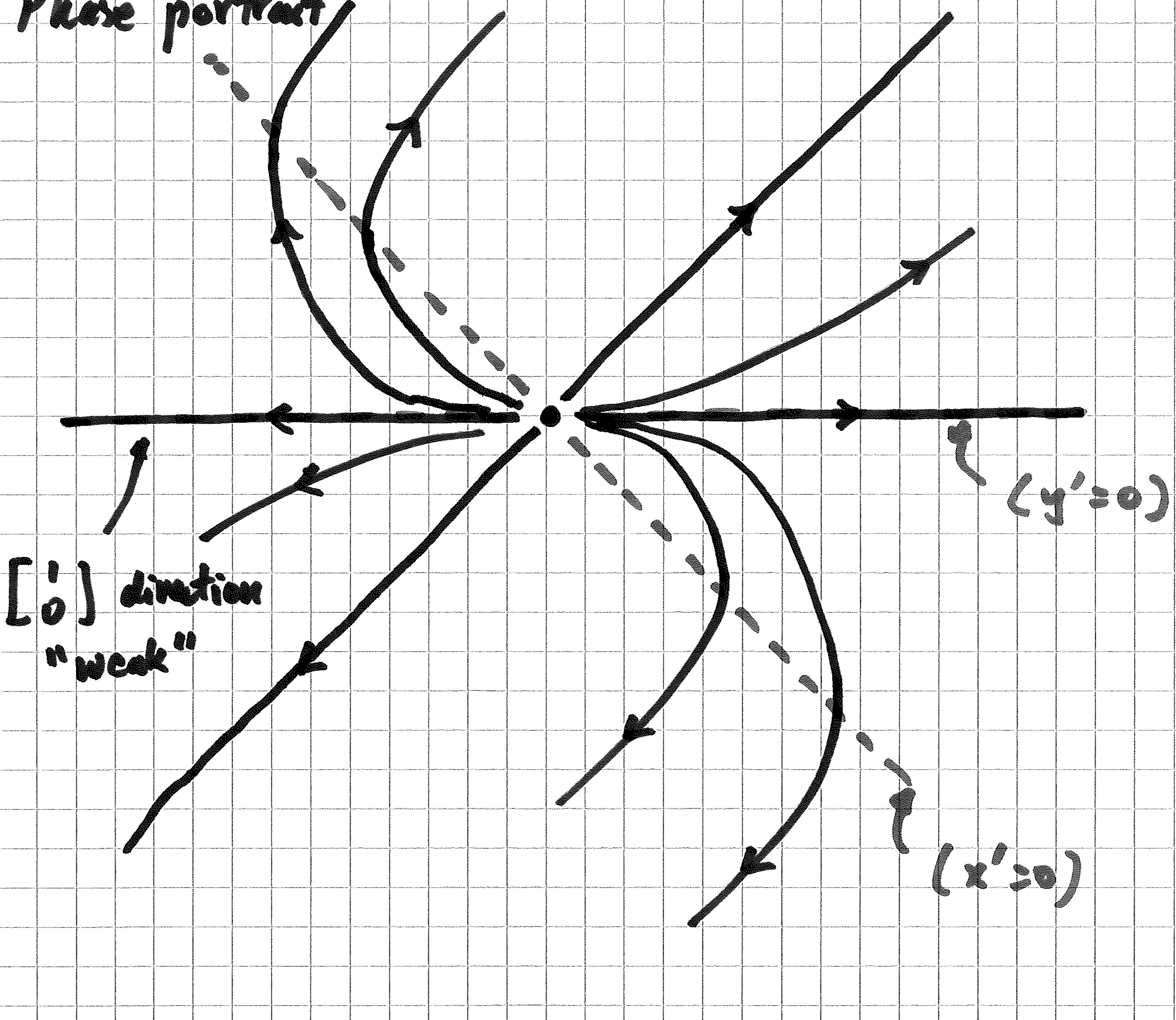
tangent to weak direction as $t \to -\infty$

<u>Example</u> 3.5.B $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Triangular matrix: eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$

Exercise: $\vec{x}(t) = \underbrace{c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}}_{\text{"weak"}} + \underbrace{c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}}_{\text{"strong"}}$

Phase portrait



"Case 2" <u>Distinct real eigenvalues, both negative</u>

$\lambda_1 < \lambda_2 < 0$

Classification: (nodal) sink or stable node

Stability: stable

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$\|\vec{x}(t)\| \to 0 \quad \text{as} \quad t \to \infty$$

Like source, but reverse direction of t.

"Case 5" <u>Complex (and nonreal) eigenvalues, positive real parts</u> $\quad \mathrm{Re}\, \lambda_{1,2} > 0$

Classification: spiral source or unstable focus

Stability: unstable

$\lambda_1 = a + ib, \ b \neq 0$
$(\lambda_2 = a - ib)$

$\mathrm{Re}\, \lambda_1 = a > 0$

$$e^{\lambda_1 t} = e^{(a+ib)t} = \underbrace{e^{at}}_{\substack{\to +\infty \\ \text{as } t \to \infty}} \underbrace{[\cos(bt) + i \sin(bt)]}_{\substack{\text{oscillates around} \\ 0 \text{ with per } \frac{2\pi}{b}}}$$