Webwork
W3: Fri Mar 6

21 Mon 2020-03-02

Example 3.5.C $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Exercise Eigenvalues $\lambda_1 = 1+2i$, $\lambda_2 = 1-2i$

$\Rightarrow$ spiral source , unstable

Can't plot complex eigenvectors

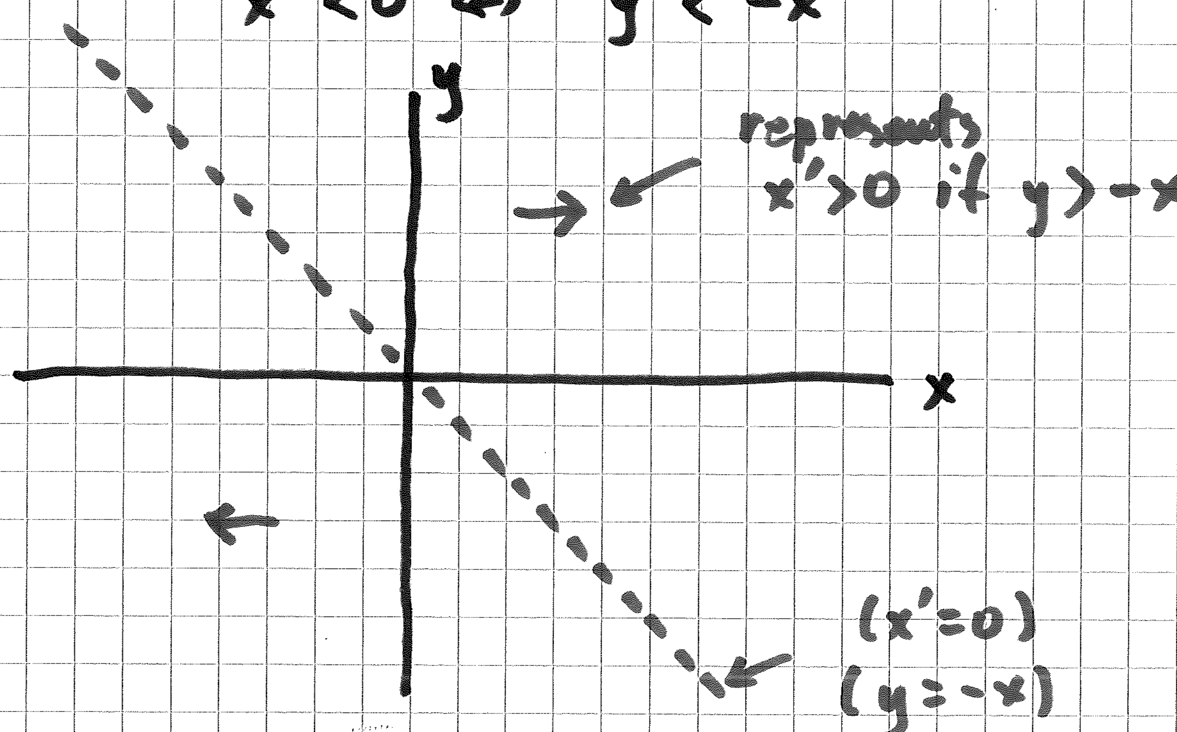
Nullclines, and some directions.

$x' = x+y$ is the horizontal "velocity" component.

$x' = x+y > 0 \Leftrightarrow y > -x$

similarly $x' = 0 \Leftrightarrow y = -x$

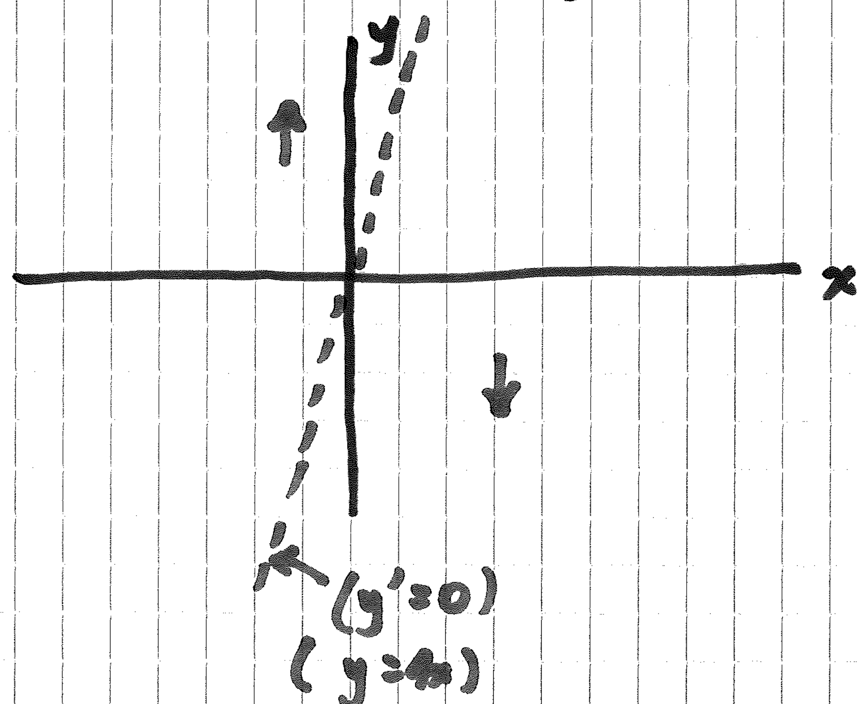
$x' < 0 \Leftrightarrow y < -x$



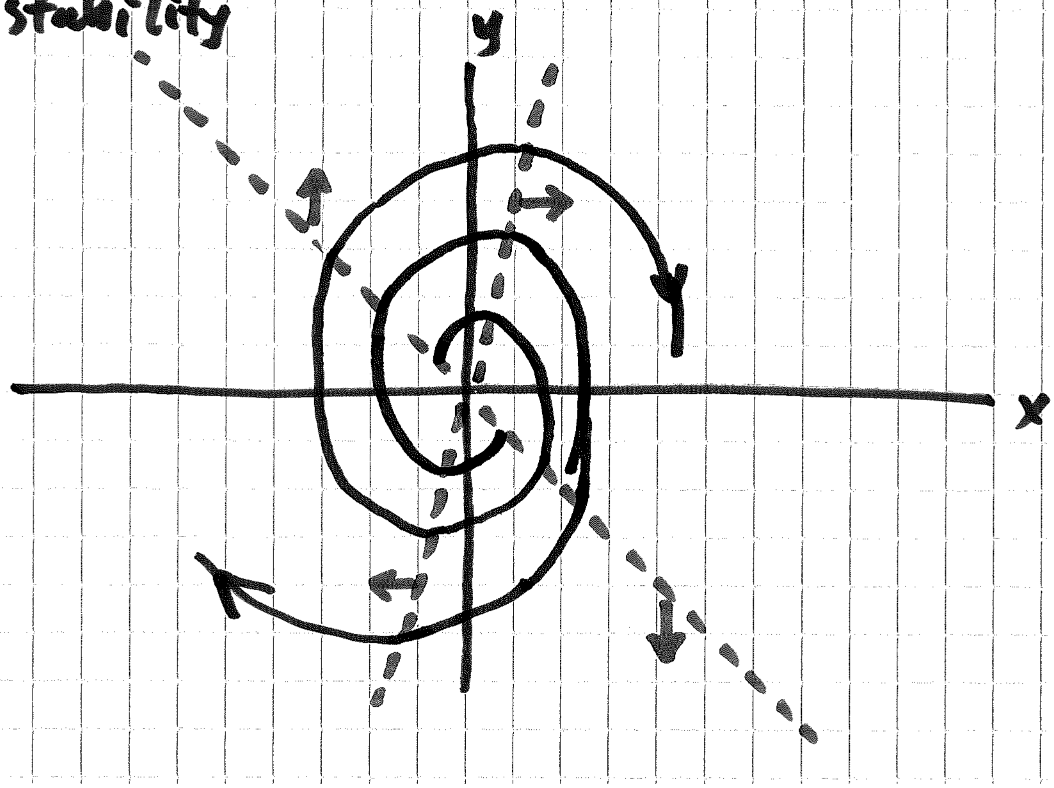
$y' = -4x + y > 0 \iff y > 4x$

$y' = 0 \iff y = 4x$

$y' < 0 \iff y < 4x$



Superimpose (or draw on same plot), use classification and stability



"Case 6" <u>Complex (and nonreal) eigenvalues with negative real parts</u> $\mathrm{Re}\,\lambda_{1,2} < 0$

Classification: spiral sink or stable focus

Stability: stable

Similar to spiral source, but reverse direction of t

"Case 4" <u>Purely imaginary eigenvalues</u>

$\lambda_{1,2} = \pm ib$, $b \neq 0$

solutions are periodic with period $\frac{2\pi}{b}$

Classification: centre

[stability: "Lyapunov stable but not asymptotically stable"]

Trajectories are in general ellipses

3.7 Multiple eigenvalues

(**) $\qquad \vec{x}' = A\vec{x}$

3.7.1 Geometric multiplicity

The multiplicity of an eigenvalue λ_j as a root of the characteristic equation $\det(A-\lambda I)=0$ is called the algebraic multiplicity of λ_j.

The dimension of (also known as "null space" of $A-\lambda_j I$)

$$\ker(A-\lambda_j I) = \{ \vec{v} : (A-\lambda_j I)\vec{v} = \vec{0} \}$$

is called the geometric multiplicity of λ_j.

If algebraic multiplicity = geometric multiplicity then gen. soln. is similar to the distinct real eigenvalue case.

Example 3.7.A $\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

alg. mult.

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$$

$\lambda_1 = 3$, algebraic multiplicity 2
("double" eigenvalue)

$$(A - \lambda_1 I)\vec{v} = (A - 3I)\vec{v} = \begin{bmatrix} 3-3 & 0 \\ 0 & 3-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

true for any $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$

$\ker(A - \lambda_1 I) = \mathbb{R}^2$ with dimension 2

A basis for $\mathbb{R}^2$ is (for example)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\lambda_1 = 3$, geometric multiplicity 2

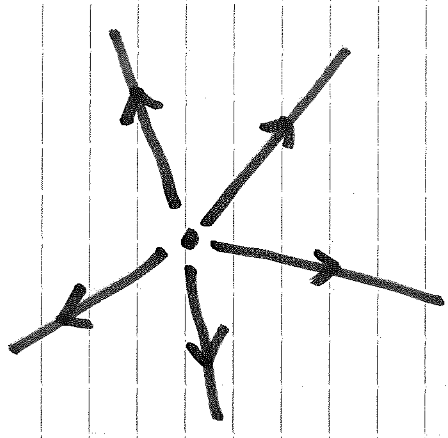
Gen. soln.

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_1 t}$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{3t}$$

Phase portrait



3.7.2 Defective eigenvalues

If geom. mult. < alg. mult. then λ_j is called **defective**.

~~One~~ There is always at least one lin. indep. eigenvector $\vec{v}_1$, so there is one solution $\vec{x}_1(t) = \vec{v}_1 e^{\lambda_j t}$

where $\boxed{(A - \lambda_j I)\vec{v}_1 = \vec{0}}$

In 2 dimensions, if λ_1 is defective we can "guess" another, lin. indep. soln.

$$\vec{x}_2(t) = (\vec{v}_2 + t\vec{v}_1)e^{\lambda_1 t}$$

"check"

$$\vec{x}_2'(t) = \vec{v}_1 e^{\lambda_1 t} + (\vec{v}_2 + t\vec{v}_1)\lambda_1 e^{\lambda_1 t}$$

$$A\vec{x}_2(t) = A(\vec{v}_2 + t\vec{v}_1)e^{\lambda_1 t}$$

$$\vec{v}_1 \cancel{e^{\lambda_1 t}} + \vec{v}_2 \lambda_1 \cancel{e^{\lambda_1 t}} + t\cancel{\vec{v}_1 \lambda_1} \cancel{e^{\lambda_1 t}}$$

$$\stackrel{?}{=} A\vec{v}_2 \cancel{e^{\lambda_1 t}} + t\cancel{A\vec{v}_1} \cancel{e^{\lambda_1 t}}$$

$\vec{x}_2(t)$ is actually a solution if

$$\boxed{(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1}$$

Gen. solu.

$$\boxed{\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2(\vec{v}_2 + t\vec{v}_1)e^{\lambda_1 t}}$$