

Example 3.7.B Find "the" general solution:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} \textcircled{1} \quad 0 &= \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -3 \\ 3 & -1-\lambda \end{vmatrix} \\ &= (\lambda-5)(\lambda+1) + 9 = \lambda^2 - 4\lambda + 4 \\ &= (\lambda-2)^2 \end{aligned}$$

$\lambda_1 = 2$ is an eigenvalue with alg. mult. 2
("double" eigenvalue)

$$\begin{aligned} \textcircled{2} \quad (A - \lambda_1 I) \vec{v}_1 \\ = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Leftrightarrow v_{11} - v_{12} = 0$$

$$\vec{v}_1 = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c \text{ arbitrary}$$

$$\text{e.g. } c=1 (\neq 0) \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\dim \ker(A - 2I) = 1 \leftarrow$ geom. mult. of $\lambda_1 = 2$

$\lambda_1 = 2$ is defective.

$$(3) \quad (A - \lambda_1 I) \vec{v}_2 = \vec{v}_1$$

$$\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow 3v_{21} - 3v_{22} = 1$$

$$\text{e.g. } v_{22} = 0, \quad v_{21} = \frac{1}{3}$$

$$\vec{v}_2 = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$$

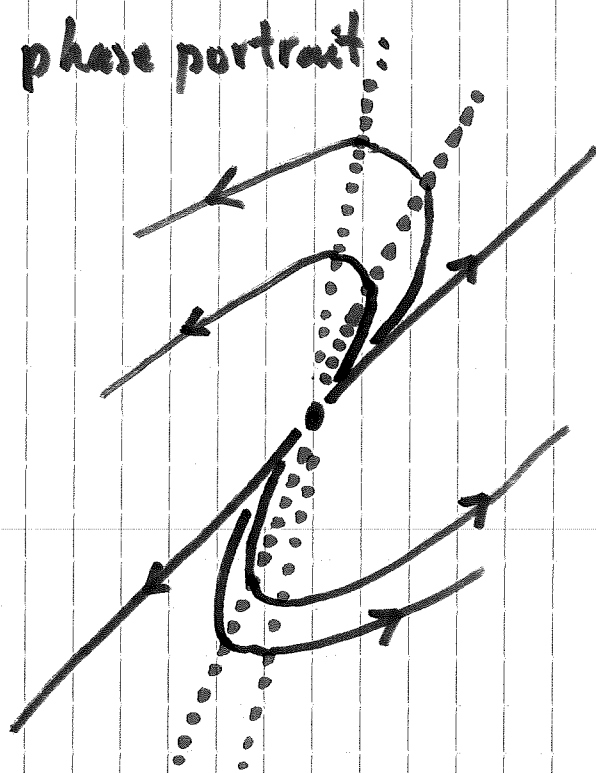
(4) Gen. solu. (not unique)

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 (\vec{v}_2 + t \vec{v}_1) e^{\lambda_1 t}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^{2t}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1/3 + t \\ t \end{bmatrix} e^{2t}$$

Exercise phase portrait:
(MATLAB is helpful)



3.9 Nonhomogeneous systems

$$(***) \quad \vec{x}' = P(t)\vec{x} + \vec{f}(t)$$

$P(t)$ is a real $n \times n$ matrix function.

Theorem 3.3.2 (p.138) The general solution of (***) is

$$\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$$

where the complementary solution $\vec{x}_c(t)$ is the gen. solu. of the corresp. homog. equ. $\vec{x}' = P(t)\vec{x}$ and $\vec{x}_p(t)$ is one particular solution of (***)

3.9.2 First order variable (or constant) coefficient

Variation of parameters

In $n=2$ dimensions, if the gen. solu. of
~~the~~ $\vec{x}' = P(t)\vec{x}$ is $\vec{x}_c(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$
 then a (2×2) fundamental matrix is

$$X(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix}$$

and $\vec{x}_c(t) = X(t) \vec{c}$, $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ arbitrary.
 "parameters" (constants)

"Guess" $\vec{x}_p(t) = X(t) \vec{u}(t)$

"Check" $\vec{x}_p'(t) \stackrel{?}{=} P(t) \vec{x}_p(t) + \vec{f}(t)$

$$\cancel{X'(t) \vec{u}(t)} + X(t) \vec{u}'(t) \stackrel{?}{=} P(t) \cancel{X(t) \vec{u}(t)} + \vec{f}(t)$$

$\vec{x}_p(t) = X(t) \vec{u}(t)$ is

because $X'(t) = P(t)X(t)$
 for a fund. mat.

actually a solution of (***)

if $X(t) \vec{u}'(t) = \vec{f}(t)$

$$\vec{u}'(t) = X(t)^{-1} \vec{f}(t)$$

$$\vec{u}(t) = \int X(t)^{-1} \vec{f}(t) dt$$

$$\boxed{\vec{x}_p(t) = X(t) \int X(t)^{-1} \vec{f}(t) dt}$$

Example 3.9.A Solve the initial value problem

$$x' = -5y + 2, \quad y' = 5x + 1;$$

$$x(0) = 0, \quad y(0) = 0.$$

$$\vec{x}' = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Exercise Gen. sol. of $\vec{x}' = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} \vec{x}$

$$\lambda_{1,2} = \pm i5$$

$$\vec{x}_c(t) = c_1 \underbrace{\begin{bmatrix} \cos(5t) \\ \sin(5t) \end{bmatrix}}_{\vec{x}_1(t)} + c_2 \underbrace{\begin{bmatrix} -\sin(5t) \\ \cos(5t) \end{bmatrix}}_{\vec{x}_2(t)}$$

Now a fund. matrix is

$$X(t) = \begin{bmatrix} \cos(st) & -\sin(st) \\ \sin(st) & \cos(st) \end{bmatrix}$$

$$\vec{x}_p(t) = X(t) \vec{u}(t)$$

$$\vec{u}'(t) = X(t)^{-1} \vec{f}(t)$$

$$= \begin{bmatrix} \cos(st) & -\sin(st) \\ \sin(st) & \cos(st) \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(or solve $X(t) \vec{u}'(t) = \vec{f}(t)$
for $\vec{u}'(t)$ directly)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\vec{u}'(t) = \frac{1}{\underbrace{\cos^2(st) + \sin^2(st)}_{=1}} \begin{bmatrix} \cos(st) & \sin(st) \\ -\sin(st) & \cos(st) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$