HW3 11:59 pm tonight
HW4 Fri Mar 13

(23) Fri 2020-03-06

$$\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$$

$$\vec{x}_c(t) = X(t)\vec{c}$$

$$\vec{x}_p(t) = X(t)\vec{u}(t)$$

$$X(t)\vec{u}\,'(t) = \vec{f}(t)$$

$$\vec{u}\,'(t) = X(t)^{-1}\vec{f}(t)$$

$$= \begin{bmatrix} \cos(5t) & \sin(5t) \\ -\sin(5t) & \cos(5t) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos(5t) + \sin(5t) \\ -2\sin(5t) + \cos(5t) \end{bmatrix}$$

$$\vec{u}(t) = \int X(t)^{-1}\vec{f}(t)\,dt = \int \begin{bmatrix} 2\cos(5t) + \sin(5t) \\ -2\sin(5t) + \cos(5t) \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{2}{5}\sin(5t) - \frac{1}{5}\cos(5t) \quad (+c_1) \\ \frac{2}{5}\cos(5t) + \frac{1}{5}\sin(5t) \quad (+c_2) \end{bmatrix}$$

(Ignore constants of integration, because we are looking for just one particular solution, take $c_1 = 0$, $c_2 = 0$)

$$\vec{x}_p(t) = X(t)\,\vec{u}(t)$$

$$= \begin{bmatrix} \cos(5t) & -\sin(5t) \\ \sin(5t) & \cos(5t) \end{bmatrix} \begin{bmatrix} \frac{2}{5}\sin(5t) - \frac{1}{5}\cos(5t) \\ \frac{2}{5}\cos(5t) + \frac{1}{5}\sin(5t) \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{\frac{2}{5}\sin(5t)\cos(5t)} - \frac{1}{5}\cos^2(5t) - \cancel{\frac{2}{5}\sin(5t)\cos(5t)} - \frac{1}{5}\sin^2(5t) \\ \frac{2}{5}\sin^2(5t) - \cancel{\frac{1}{5}\sin(5t)\cos(5t)} + \frac{2}{5}\cos^2(5t) + \cancel{\frac{1}{5}\sin(5t)\cos(5t)} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

Gen. soln.

$$\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t) = c_1 \begin{bmatrix} \cos(5t) \\ \sin(5t) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(5t) \\ \cos(5t) \end{bmatrix} + \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

At $t=0$

$$\vec{x}(0) = \begin{bmatrix} c_1 - \frac{1}{5} \\ c_2 + \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{array}{l} c_1 = \frac{1}{5} \\ c_2 = -\frac{2}{5} \end{array}$$

Solu. of IVP

$$\vec{x}(t) = \frac{1}{5}\begin{bmatrix} \cos(5t) \\ \sin(5t) \end{bmatrix} - \frac{2}{5}\begin{bmatrix} -\sin(5t) \\ \cos(5t) \end{bmatrix} + \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$$

with components

$$x(t) = \frac{1}{5}\cos(5t) + \frac{2}{5}\sin(5t) - \frac{1}{5}$$

$$y(t) = \frac{1}{5}\sin(5t) - \frac{2}{5}\cos(5t) + \frac{2}{5}\ .$$

Note: if $P(t) = A$ is a constant matrix, and $\vec{f}(t)$ is "nice" it may be more efficient to guess the form of $\vec{x}_p(t)$ more directly (undetermined coefficients)

Exercise In Ex. 3.9.A, since $\vec{f}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ guess $\vec{x}_p(t)$ has the form $\vec{x}_p(t) = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, A_1, A_2 constants. Try this. $(A_1 = -\frac{1}{5}, A_2 = \frac{2}{5})$

Chapter 8 Nonlinear systems

8.1 Linearization, critical points, and equilibria

8.1.1 Autonomous systems and phase plane analysis

Autonomous systems in 2 dims.

$$(8.1) \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

where f, g do not depend explicitly on the independent variable t. Solutions: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ is a vector field: specifies a vector $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathbb{R}^2$ at every point (x,y) in $\mathbb{R}^2$

A critical point (or equilibrium) as found by setting $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e. a critical point is a solution (x_0, y_0) of

$$\begin{cases} 0 = f(x,y) \\ 0 = g(x,y) \end{cases}$$

A critical point corresponds to an equilibrium solution (constant soln.)

$$x(t) \equiv x_0 \ , \ y(t) \equiv y_0$$

Example 8.1.A Write $x'' + x' + x - x^2 = 0$ as an equivalent autonomous system of 1st order ODES, then find all equilibrium solutions.

Let $y = x'$. Then $x' = y$, $y' = -x' - x + x^2$
$= -y - x + x^2$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -y - x + x^2 \end{bmatrix}$$

Critical points

$$\begin{cases} y = 0 \\ -y - x + x^2 = 0 \end{cases}$$

$y = 0$, $-x + x^2 = x(x-1) = 0$

$x = 0$ or $x = 1$

$(x_0, y_0) = (0,0), (1,0)$

Exactly
Two equilibrium solutions

① $x(t) \equiv 0, \quad y(t)\ (= x'(t)) \equiv 0$

② $x(t) \equiv 1, \quad y(t)\ (= x'(t)) \equiv 0$

The phase plane is the (x, y)-plane.

"Phase plane analysis" refers to calculations that justify phase portraits in (x, y)-plane.

Step ① find critical points.

Step ② linearization ...