HW 4 Fri Mar 13

24 Mon 2020-03-09

8.1.2 Linearization

If (x_0, y_0) is a critical point, define $u(t), v(t)$ by

$$x(t) = x_0 + u(t) \qquad (u = x - x_0)$$
$$y(t) = y_0 + v(t) \qquad (v = y - y_0)$$

where $(x(t), y(t))$ ~~ar~~ is a solution of (8.1)

Find new ODEs for u, v assuming they are near 0

i) $u' = x' = f(x, y) = f(x_0+u, y_0+v)$

$$= \underbrace{f(x_0, y_0)}_{\substack{=0 \\ (\text{why?})}} + \frac{\partial f}{\partial x}(x_0, y_0)\, u + \frac{\partial f}{\partial y}(x_0, y_0)\, v + \underbrace{\ldots}_{\substack{\text{remainder} \\ \text{of Taylor} \\ \text{series}}}$$

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)\, u + \frac{\partial f}{\partial y}(x_0, y_0)\, v + \ldots$$

ii) $v' = \frac{\partial g}{\partial x}(x_0,y_0)u + \frac{\partial g}{\partial y}(x_0,y_0)v + \dots$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}}_{\text{Jacobian matrix (or derivative) of } \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}\text{, evaluated at } (x_0,y_0)} \begin{bmatrix} u \\ v \end{bmatrix} + \dots$$

By ignoring all the remainder terms we get linear approximation of (8.1), for (x,y) near (x_0,y_0), called the linearization of (8.1) at (x_0,y_0):

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Example 8.1. A, cont.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -y - x + x^2 \end{bmatrix}$$

② Find the linearization at each critical point.

Compute the Jacobian matrix:

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1+2x & -1 \end{bmatrix}$$

(a) At the critical point $(x_0, y_0) = (0, 0)$

$(-1 + 2x_0 = -1 + 2(0) = -1)$

the linearization is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

(b) At the critical point $(x_0, y_0) = (1, 0)$

$(-1 + 2(1) = 1)$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

8.2 Stability and classification of isolated critical points

8.2.1 Isolated critical points and almost linear systems

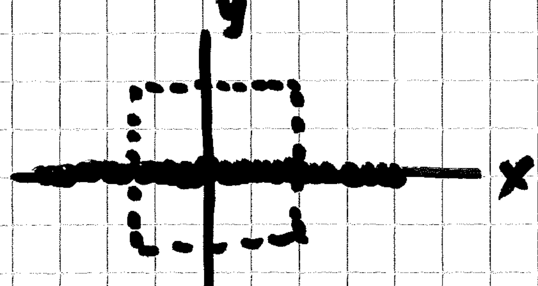
A critical point (x_0, y_0) is *isolated* if it is the only critical point in some sufficiently small open rectangle that contains (x_0, y_0).

e.g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

has a critical point $(x_0, y_0) = (0,0)$

But, all crit. pts. are sols. of $\begin{cases} 0 = y \\ 0 = 0 \end{cases}$

The entire x-axis consists of crit. pts.



$(0,0)$ is not an isolated critical point

A system (8.1) is almost linear at a critical point (x_0, y_0) if (x_0, y_0) is isolated and the Jacobian matrix evaluated at (x_0, y_0) is invertible (i.e. nonsingular i.e does not have a 0 eigenvalue) e.g. p. 357

8.2.2 Stability and classification of isolated critical points

Let (x_0, y_0) be a critical point of (8.1).

(a) (x_0, y_0) is (Lyapunov) stable if every solution $(x(t), y(t))$, that starts at $t=0$ with initial value $(x(0), y(0))$ sufficiently close to (x_0, y_0), remains arbitrarily close to (x_0, y_0) for all $t \geq 0$.

(b) (x_0, y_0) is unstable if it is not (Lyapunov) stable (i.e. at least one solu. starts at $t=0$ with initial value arbitrarily close to (x_0, y_0) but does not remain sufficiently close for all $t \geq 0$.

(c) (x_0, y_0) is asymptotically stable if every solution, that starts at $t=0$ with initial value sufficiently close to (x_0, y_0), approaches (x_0, y_0) as $t \to \infty$

$$\lim_{t\to\infty} x(t) = x_0$$

$$\lim_{t\to\infty} y(t) = y_0$$

In "most" cases, for (8.1) the local behavior and stability at crit. pts. can be determined by the eigenvalues of the linearization.

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Eigenvalues	(local) Behaviour	Stability
real distinct positive	(nodal) source/ unstable node	unstable
real distinct negative	(nodal) sink/ stable node	asymptotically stable