HW4 tonight 11:59 pm

MT2 Fri Mar 20

see Canvas

26 Fri 2020-03-13

Remarks.

1. At a saddle there are two special trajectories called "separatrices" that are <u>tangent</u> at the critical point to the stable eigenvector direction (the union of the saddle point and separatrices is called the "stable manifold" of the saddle point.)

2. Separatrices separate behaviour of $(x(t), y(t))$ as $t \to \infty$ depending on initial values $(x(0), y(0))$.

3. If $\mathrm{Re}\,\lambda_{1\text{ or }2} = 0$ i.e. if 0 is one of the eigenvalues, or if the eigenvalues are purely imaginary, then Table 8.1 gives no information: the linearization is <u>not a good enough approximation</u> to determine local behaviour and stability.

8.2.3 The trouble with centres

In a linear system if (0,0) has purely imaginary eigenvalues, we know the general solution exactly: the behaviour is a centre and (0,0) is (Lyapunov) stable but not asymptotically stable.

BUT in a nonlinear system, if (x_0, y_0) has purely imaginary eigenvalues, more work is needed to determine behaviour, stability.

Example 8.2.B

$$\begin{cases} x' = -y \pm (x^3 + xy^2) \\ y' = x \pm (x^2 y + y^3) \end{cases}$$

In both cases $(+,-)$: $(x_0, y_0) = (0,0)$ is a critical point.

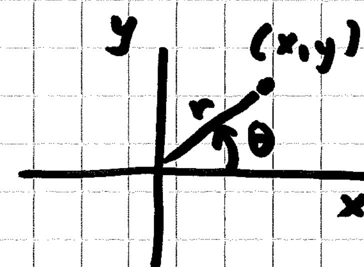
<u>**Exercise**</u> Linearization at $(0,0)$ is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{in } \underline{\text{both}} \text{ cases}$$

with eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$ which are purely imaginary.

To see the actual behaviour and stability, in this example we can express the ODE in polar coordinates

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$



If $(x(t), y(t))$ is a solution

$$r(t) = \sqrt{x(t)^2 + y(t)^2}$$

$$r^2 = x^2 + y^2$$

$$2rr' = 2xx' + 2yy'$$

$$r' = \frac{xx' + yy'}{r}$$

$$= \frac{x[-y \pm (x^3 + xy^2)] + y[x \pm (x^2 y + y^3)]}{r}$$

$$r' = \pm \frac{x^4 + x^2y^2 + x^2y^2 + y^4}{r}$$

$$= \pm \frac{(x^2+y^2)^2}{r}$$

$$= \pm \frac{(r^2)^2}{r}$$

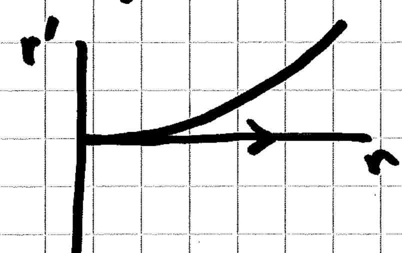
$$= \pm r^3$$

<u>Exercise</u> $\tan(\theta(t)) = \frac{y(t)}{x(t)} \Rightarrow \theta' = 1$

$$\theta(t) = t + \theta_0$$

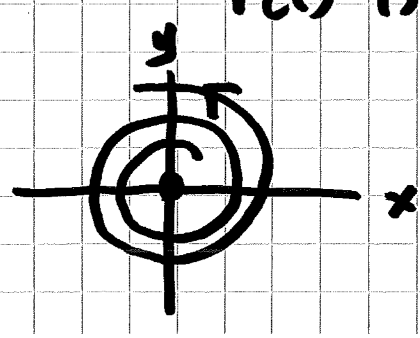
$r' = r^3$, $r' = -r^3$ can be solved explicitly (separable : Exerise) or graphically

$+$ case, $r' = r^3$



$r' > 0$ for all $r > 0$

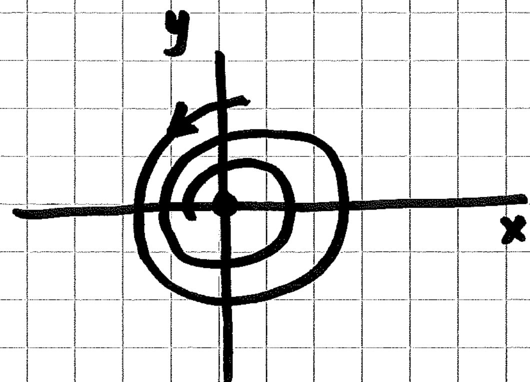
$r(t)$ is increasing



spiral source, unstable

– case, $r' = -r^3$

$r(t)$ decreasing



spiral sink,
asymptotically stable

Linearization is <u>same</u> in both cases.

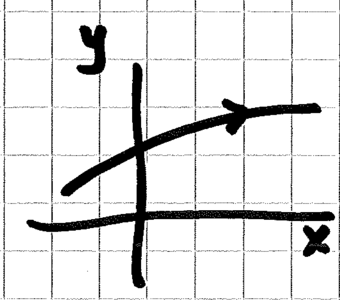
8.2.4 <u>Conservative equations</u>

A <u>conservative</u> equation is a 2nd order ODE of the form

$$\boxed{x'' + f(x) = 0}$$

Write as equivalent system

$$\begin{cases} x' = y \\ y' = -f(x) \end{cases}$$



Along a typical trajectory $(x(t), y(t))$

follows a curve $y = y(x)$

$$y(t) = y(x(t))$$

$$-f(x) = y' = \frac{dy}{dt} = \frac{dy}{dx} \underbrace{\frac{dx}{dt}}_{= x' = y} = \frac{dy}{dx} y$$

$$\frac{dy}{dx} y = -f(x)$$

$$y \, dy = -f(x) \, dx$$

$$\frac{1}{2} y^2 = -\int f(x) \, dx + C$$

$$\boxed{\frac{1}{2} y^2 + \int f(x) dx = C}$$

$(x(t), y(t))$ lies on a curve given implicitly by