Example 8.2.C: Draw the phase portrait for the system of 1st order equations that is equivalent to

$$x'' + x - x^2 = 0$$

Recognize as conservative equation.

Write as system $\begin{cases} x' = y \\ y' = -x + x^2 \end{cases}$

Exercise Find all critical points: $(0,0)$, $(1,0)$

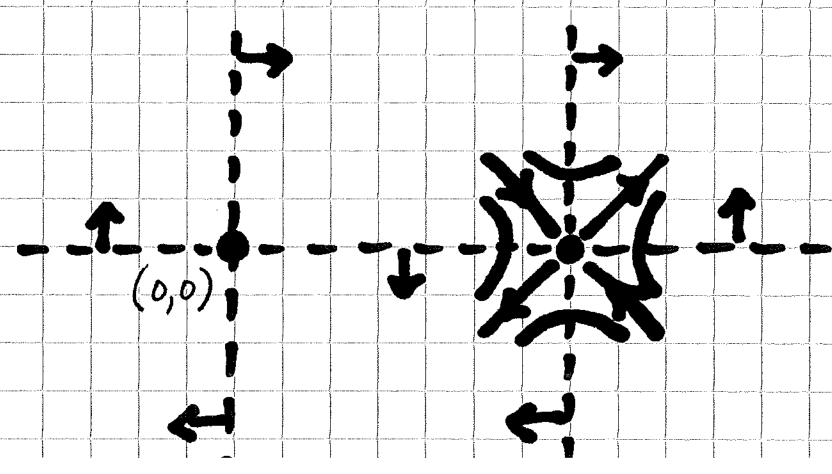
Linearize at ~~at~~ each critical point and determine

$(1,0)$ is a saddle, unstable

$(0,0)$ has purely imaginary eigenvalues, "linear centre"

linearization fails to determine behaviour or stability

Draw the nullclines and direction field, draw the local phase portrait near $(1,0)$ only.



Now using the fact that this system comes from a conservative equation

$$x'' + \underbrace{x - x^2}_{f(x)} = 0$$

we know the trajectories of the system $(x(t), y(t))$ lie on curves given implicitly by

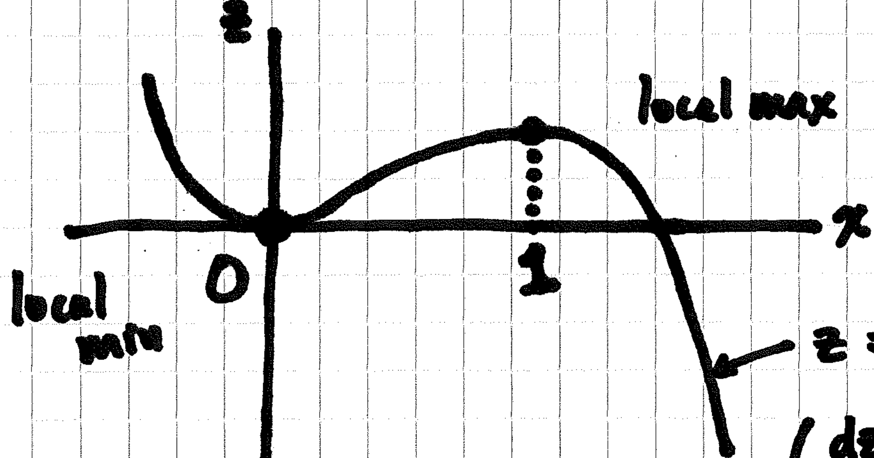
$$\frac{1}{2}y^2 + \int (x - x^2)\,dx = C$$

$$\underbrace{\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3}_{H(x,y)} = C$$

$H(x,y)$ "Hamiltonian" or "energy function"

Need to draw $H(x,y) = C$ in (x,y) plane.

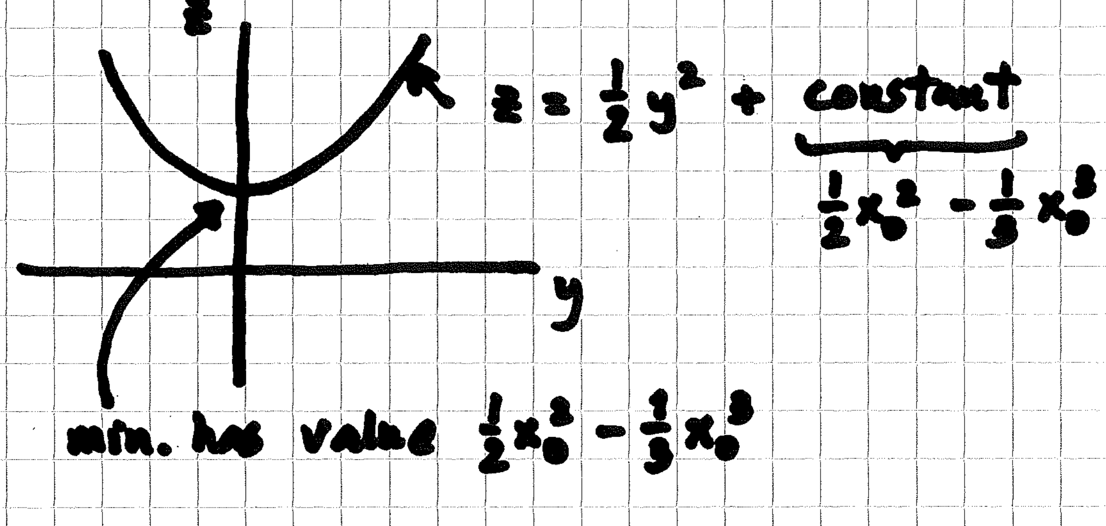
Set $y = 0$, draw 2-D graph of $z = H(x,0)$ in (x,z)-plane : "potential energy"



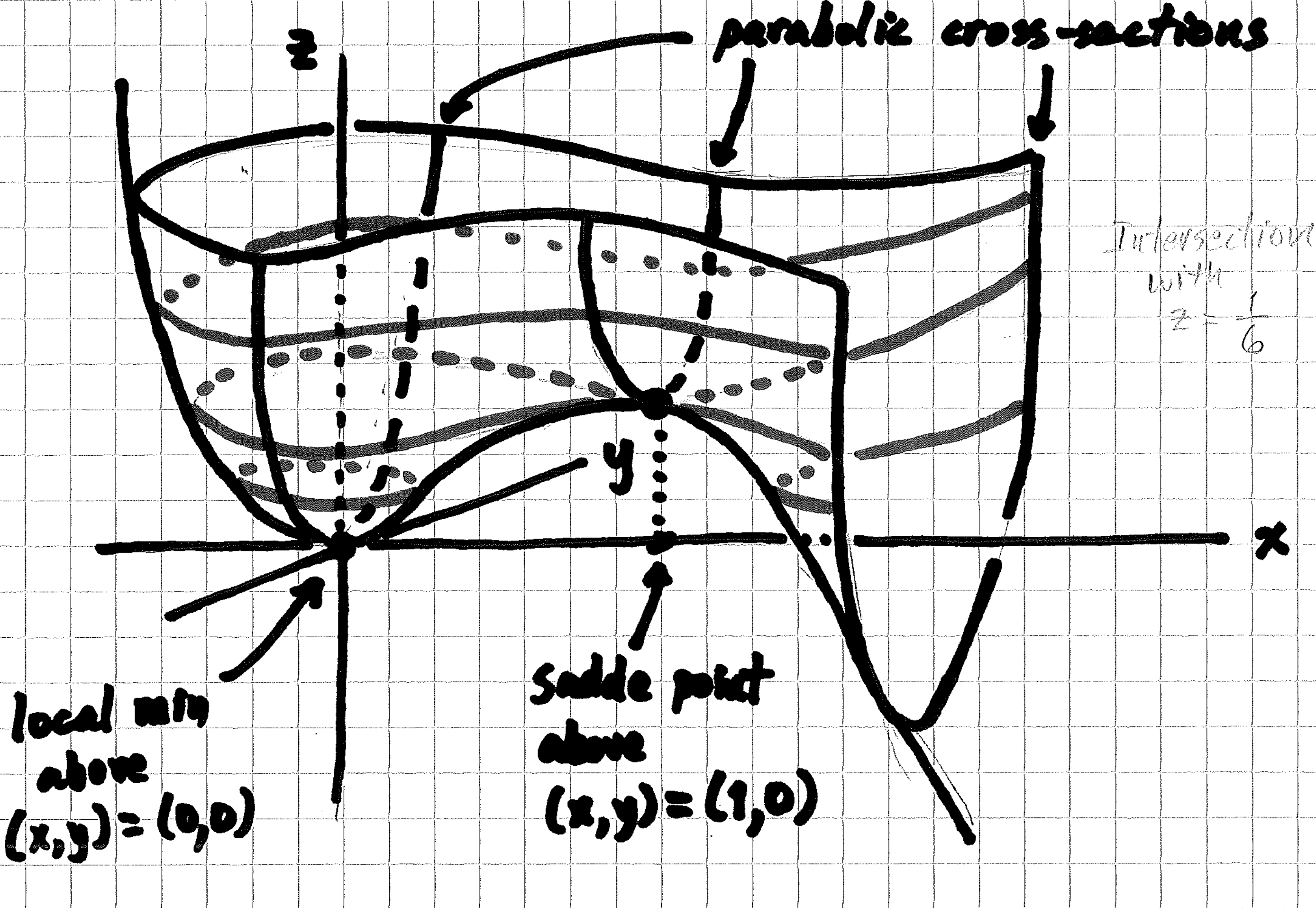
$$z = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

$$\left(\frac{dz}{dx} = x - x^2\right)$$

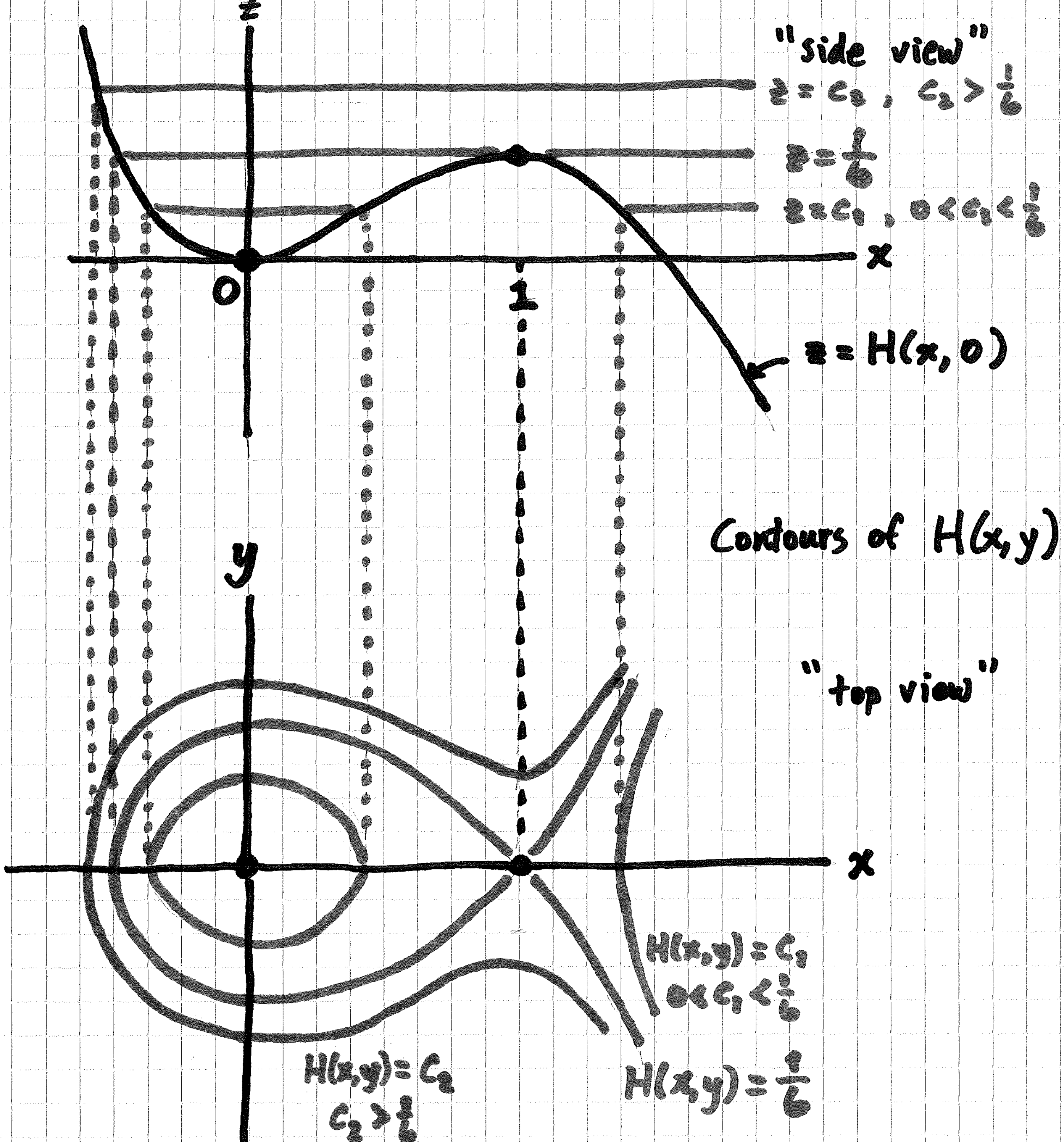
Optional: set $x = x_0$ (constant), draw 2-D graph of $z = H(x_0, y)$ in (y, z)-plane



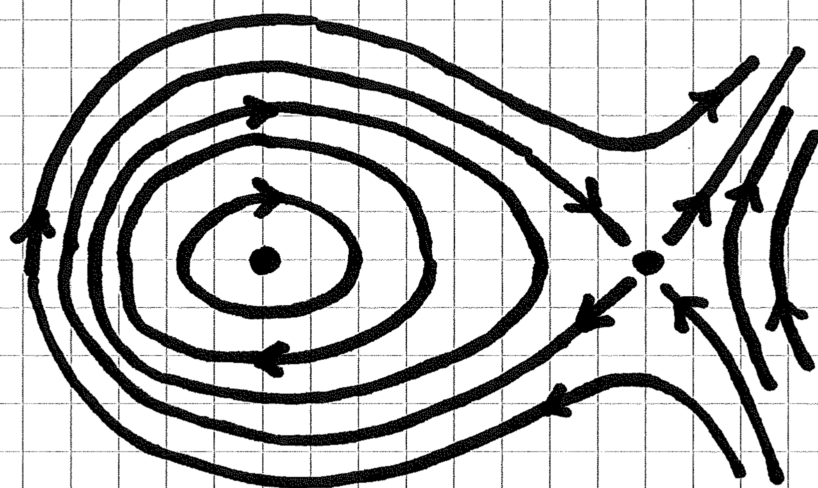
Optional: draw 3-D graph of $z = H(x, y)$ in (x, y, z)-space



Short cut to $H(x,y) = C$ graphs ("contours")



Phase portrait of nonlinear system:
must be consistent with nullclines, direction field, local phase portraits and graphs of contours $H(x,y) = C$



Now we see that $(0,0)$ is (Lyapunov) stable but not asymptotically stable: it is called a "nonlinear centre"

- corresponds to local min. of potential energy for conservative equ. $\quad z = \frac{1}{2}x^2 - \frac{1}{3}x^3$