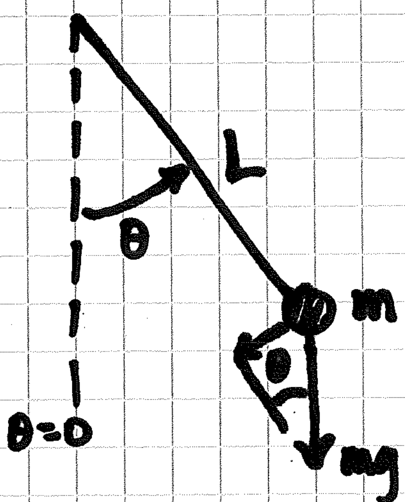
8.3 Applications of nonlinear systems

8.3.1 Pendulum (with no damping)



Newton's 2nd law:

$$mL\theta'' = -mg\sin(\theta)$$

(θ in radians)

$$\theta'' + \underbrace{\frac{g}{L}\sin(\theta)}_{f(\theta)} = 0 \qquad (*)$$

Conservative 2nd order ODE

Write as system: let $\omega = \theta'$ (angular velocity)

$$\begin{cases} \theta' = \omega \\ \omega' = -\frac{g}{L}\sin(\theta) \end{cases}$$

① Find all critical points:

$$\begin{cases} 0 = \omega \\ 0 = -\frac{g}{L}\sin(\theta) \end{cases} \quad \begin{aligned} &\Leftrightarrow \omega = 0 \\ &\Leftrightarrow \sin(\theta) = 0 \\ &\Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \dots \\ &\Leftrightarrow \theta = n\pi,\ n \in \mathbb{Z} \end{aligned}$$

All critical points are

$$(\theta_0, \omega_0) = (n\pi, 0),\ n \in \mathbb{Z} \text{ (i.e. } n = 0, \pm 1, \pm 2, \dots)$$

②③ Find linearizations at each critical point:

$$\begin{bmatrix} \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L}\sin(\theta) \end{bmatrix}$$

Jacobian matrix of vector field is

$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(\theta) & 0 \end{bmatrix}$$

Now evaluate at each critical point $(\theta_0, \omega_0) = (n\pi, 0)$

a) At $(\theta_0, \omega_0) = (n\pi, 0)$, n an even integer ($n = 0, \pm 2, \pm 4, \ldots$)

$\cos(n\pi) = 1$ and the linearization is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}}\begin{bmatrix} u \\ v \end{bmatrix}$$

__Exercise__: eigenvalues $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}}$

Since we know the system corresponds to a __conservative equation__, we can predict these critical points are "nonlinear centres", stable but not asymptotically stable.

b) At $(\theta_0, \omega_0) = (n\pi, 0)$, n an odd integer ($n = \pm 1, \pm 3, \pm 5, \cdots$)

$\cos(n\pi) = -1$ and the linearization is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Exercise: eigenvalues $\lambda_{1,2} = \pm\sqrt{\frac{g}{L}}$

So we know these critical points are saddles, they are unstable.

④ Nullclines, direction field, local phase portraits where possible. **Exercise**