(28) Wed 2020-03-18

($\theta' = \omega, \quad \omega' = -\frac{g}{L}\sin(\theta)$ continued)

(5) Draw the phase portrait:

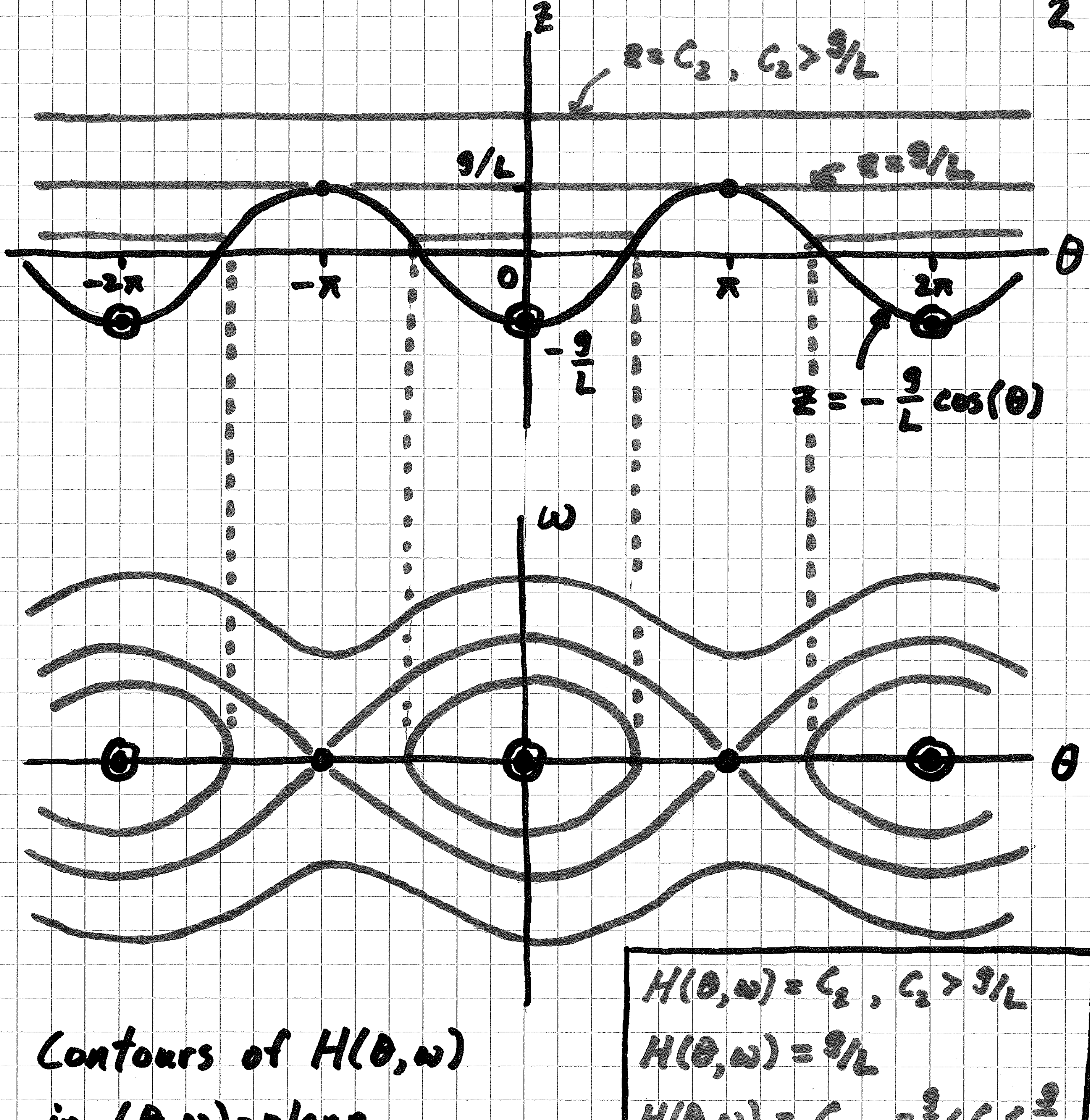
Because $\theta'' + \frac{g}{L}\sin(\theta) = 0$ is <u>conservative</u>, there are extra steps.

Trajectories $(\theta(t), \omega(t))$ of the corresponding system must lie on curves given implicitly by

$$\frac{1}{2}\omega^2 + \int \frac{g}{L}\sin(\theta)\,d\theta = C$$

$$H(\theta,\omega) = \frac{1}{2}\omega^2 - \frac{g}{L}\cos(\theta) = C$$

The 3-D graph $z = H(\theta, \omega)$ has parabolic cross-sections $z = \frac{1}{2}\omega^2 + \text{constant}$ with min. at $\omega = 0$, for any fixed $\theta = \theta_0$, we can find the contours $H(\theta,\omega) = C$ from a plot of $z = H(\theta, 0)$:

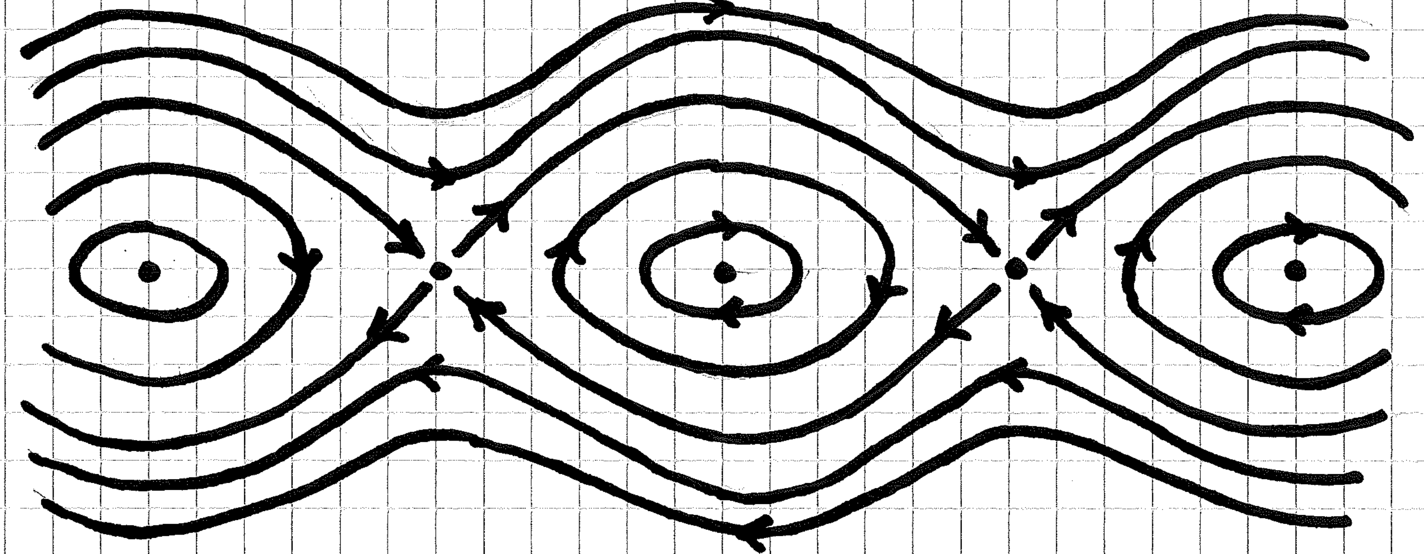


Contours of $H(\theta, \omega)$
in (θ, ω)-plane

Crit pts $(n\pi, 0)$, n even integer ⊙ are local minima of $z = H(\theta, \omega)$

Crit. pts. $(n\pi, 0)$, n odd integer • are saddle points

Phase portrait of $\theta' = \omega$, $\omega' = -\frac{g}{L}\sin(\theta)$ is (axes, nullclines not shown, see 27 p. 8)



Picture repeats in θ-direction with period 2π.

"Low energy" trajectories near stable crit. pt. $(0,0)$ are periodic, but period depends on amplitude (unlike the linear centre for $\theta' = \omega$, $\omega' = -\frac{g}{L}\theta$, where the period is constant)

See textbook pp. 366-367 esp. Fig. 8.8, but you don't need to know details of derivation.

These "low energy" trajectories ($H(\theta,\omega) < g/L$) represent back-and-forth oscillations of $\theta(t)$ about $\theta = 0$. What about if $H(\theta,\omega) > g/L$?