Competing species

Let $x(t)$ = population of species 1 (e.g. x thousands of rabbits) at time t

$y(t)$ = population of species 2 (e.g. y thousands of sheep) at time t

$x(t) \geq 0$ and $y(t) \geq 0$: negative values of x, y not relevant

Assume $x(t), y(t)$ are continuous functions: this approximation is reasonable if populations are large

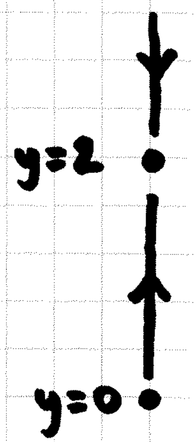
Consider when both species are confined to the same region, and they compete for the same food source (e.g. grass)

Our model is the system of ODEs

$$\begin{cases} x' = x(1 - x - y) \\ y' = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right) \end{cases} \qquad x, y \geq 0$$

Observe: if we put $y = 0$ then $y' = 0$, $y(t)$ is constant which must be 0. So there are solutions of the form $(x(t), 0)$ that remain on the x-axis. We say the x-axis is an "invariant set". Also, if $y = 0$ then $x' = x(1 - x)$, a logistic equ. (Chapt. 1) with 1-D phase portrait $\underset{x=0}{\bullet} \longrightarrow \underset{x=1}{\bullet} \longleftarrow$ along the x-axis ($x \geq 0$ only)

Similarly (Exercise) the y-axis is an invariant set, there are solutions of the form $(0, y(t))$ where $y(t)$ behaves according to the 1-D phase portrait



along the y-axis ($y \geq 0$ only).

(a) Find all critical points with $x, y \geq 0$

We put $x'=0$, $y'=0$ in the model and solve

$$\begin{cases} 0 = x(1-x-y) \\ 0 = y\left(\frac{1}{2}-\frac{1}{4}y-\frac{3}{4}x\right) \end{cases} \qquad x, y \geq 0$$

$$\begin{cases} x'=0 \iff x(1-x-y)=0 \iff x=0 \text{ or } 1-x-y=0 \\ y'=0 \iff y\left(\frac{1}{2}-\frac{1}{4}y-\frac{3}{4}x\right)=0 \iff y=0 \text{ or } \frac{1}{2}-\frac{1}{4}y-\frac{3}{4}x=0 \end{cases}$$

So there are four possibilities for critical points (and we have to check that $x \geq 0$, $y \geq 0$):

i) $x=0$ and $y=0$: $(x_0, y_0) = (0,0)$ is a crit. pt.

ii) $x=0$ and $\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x = 0$: substitute $x=0$ into the 2nd equ., solve for y, we get $(x_0, y_0) = (0,2)$

iii) $1-x-y=0$ and $y=0$: we get $(x_0, y_0) = (1,0)$

iv) $1-x-y=0$ and $\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x = 0$: solve

$$\begin{cases} x + y = 1 \\ \frac{3}{4}x + \frac{1}{4}y = \frac{1}{2} \end{cases}$$ to get $(x_0, y_0) = (\frac{1}{2}, \frac{1}{2})$

The first three critical points have essentially already been found, in the discussion about invariant sets. All the critical points we found satisfy $x_0 \geq 0$, $y_0 \geq 0$ so there are precisely four critical points to consider.

(b) <u>At each critical point, use linearization to classify the behaviour (if possible) and find the stability (if poss.)</u>

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - x^2 - xy \\ \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy \end{bmatrix},$$ compute the Jacobian matrix

$$\begin{bmatrix} 1-2x-y & -x \\ -\frac{3}{4}y & \frac{1}{2} - \frac{1}{2}y - \frac{3}{4}x \end{bmatrix}$$ then evaluate at (x_0, y_0) ...