W4 Webwork Wed Mar 25 29 Fri 2020-03-20

Evaluate the Jacobian matrix at each (x_0, y_0):

i) at $(x_0, y_0) = (0,0)$ the matrix becomes $\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$,

this is diagonal, eigenvalues are $1, 1/2$

real, distinct, both positive $\Rightarrow$ $(0,0)$ is a (nodal) <u>source</u>, unstable.

ii) at $(x_0, y_0) = (0,2)$ we get $\begin{bmatrix} -1 & 0 \\ -3/2 & -1/2 \end{bmatrix}$ which is

(lower) triangular, eigenvalues are $-1, -1/2$

real, distinct, both negative $\Rightarrow$ $(0,2)$ is a (nodal) <u>sink</u>, asymptotically stable

<u>Exercise</u> Prove that the eigenvalues of $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ and of $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ are a, d in both cases.

iii) at $(x_0, y_0) = (1,0)$ we get $\begin{bmatrix} -1 & -1 \\ 0 & -1/4 \end{bmatrix}$ which is

(upper) triangular, eigenvalues are $-1, -\frac{1}{4}$

real, distinct, both negative $\Rightarrow$ $(1,0)$ is a (nodal) <u>sink</u>, asymptotically stable

iv) at $(x_0, y_0) = (\frac{1}{2}, \frac{1}{2})$ we get $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{8} \end{bmatrix}$, now we

actually need to calculate

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\frac{1}{2} - \lambda & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{8} - \lambda \end{vmatrix} = \lambda^2 + \frac{5}{8}\lambda + \underbrace{\frac{1}{16} - \frac{3}{16}}_{-1/8}$$

$$= \frac{1}{8}(8\lambda^2 + 5\lambda - 1), \quad \lambda = \frac{-(5) \pm \sqrt{(5)^2 - 4(8)(-1)}}{2(8)}$$

Eigenvalues are $\lambda_1 = \frac{-5 - \sqrt{57}}{16} < 0$, $\lambda_2 = \frac{-5 + \sqrt{57}}{16} > 0$

$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ is a <u>saddle</u>, unstable (note $57 > 49$)

(c) <u>Draw the phase portrait</u> ($x, y \geq 0$)

First, look at nullclines:

$x' = x(1 - x - y) > 0 \Leftrightarrow x$ and $1 - x - y$ have same signs.

$x < 0$ is irrelevant: $x' > 0$ if $x > 0$ and $\underbrace{1 - x - y > 0}_{y < 1 - x}$

Similarly (<u>Exericse</u>)

$x' = 0$ if $x = 0$ or $y = 1 - x$

$x' < 0$ if $x > 0$ and $y > 1 - x$

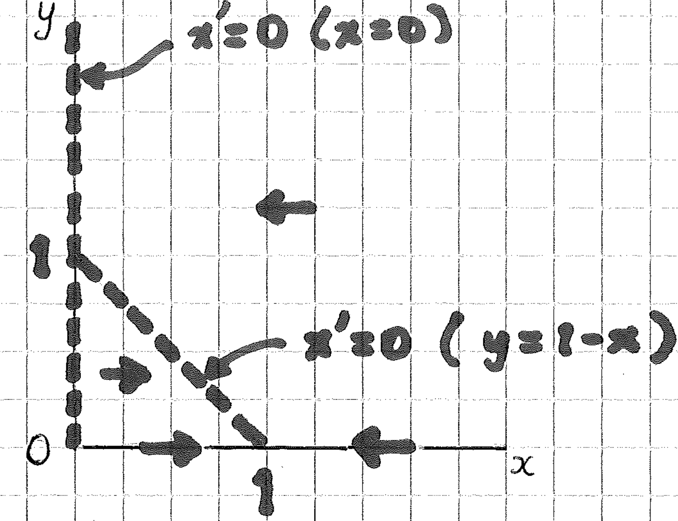
$y' > 0$ if $y > 0$ and $y < 2 - 3x$

$y' = 0$ if $y = 0$ or $y = 2 - 3x$

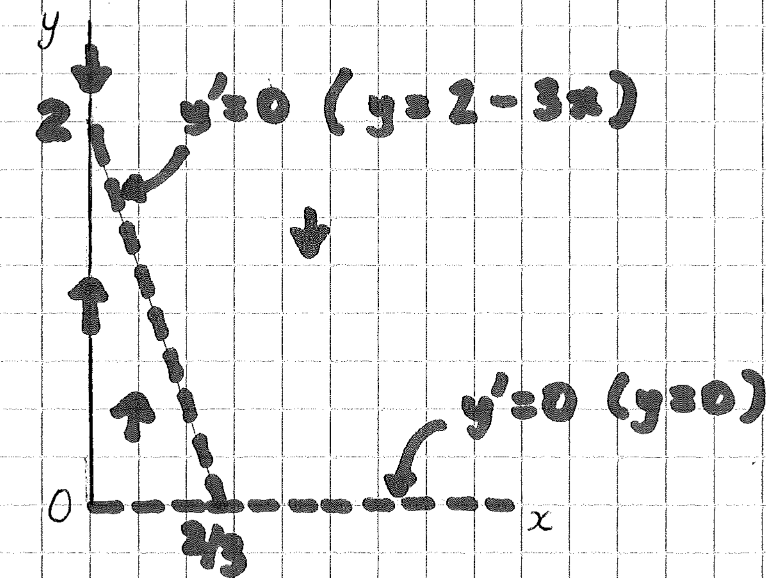
$y' < 0$ if $y > 0$ and $y > 2 - 3x$

We can draw diagrams that show where $x' > 0, = 0, < 0$ and $y' > 0, = 0, < 0$ separately for clarity or together to save time.

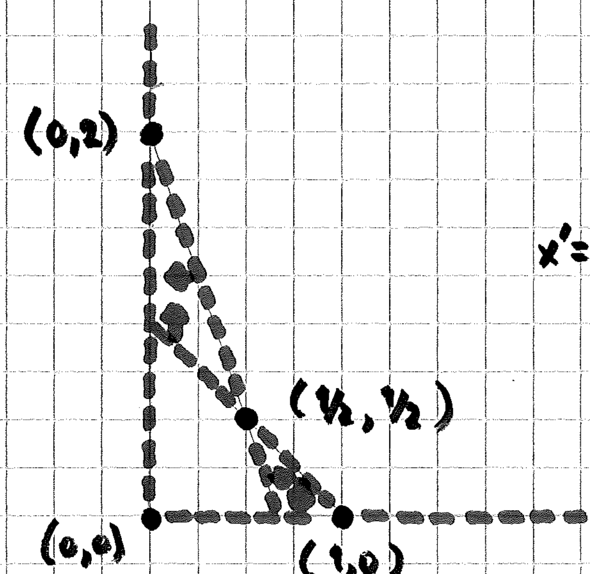
Sign of x'



Sign of y'



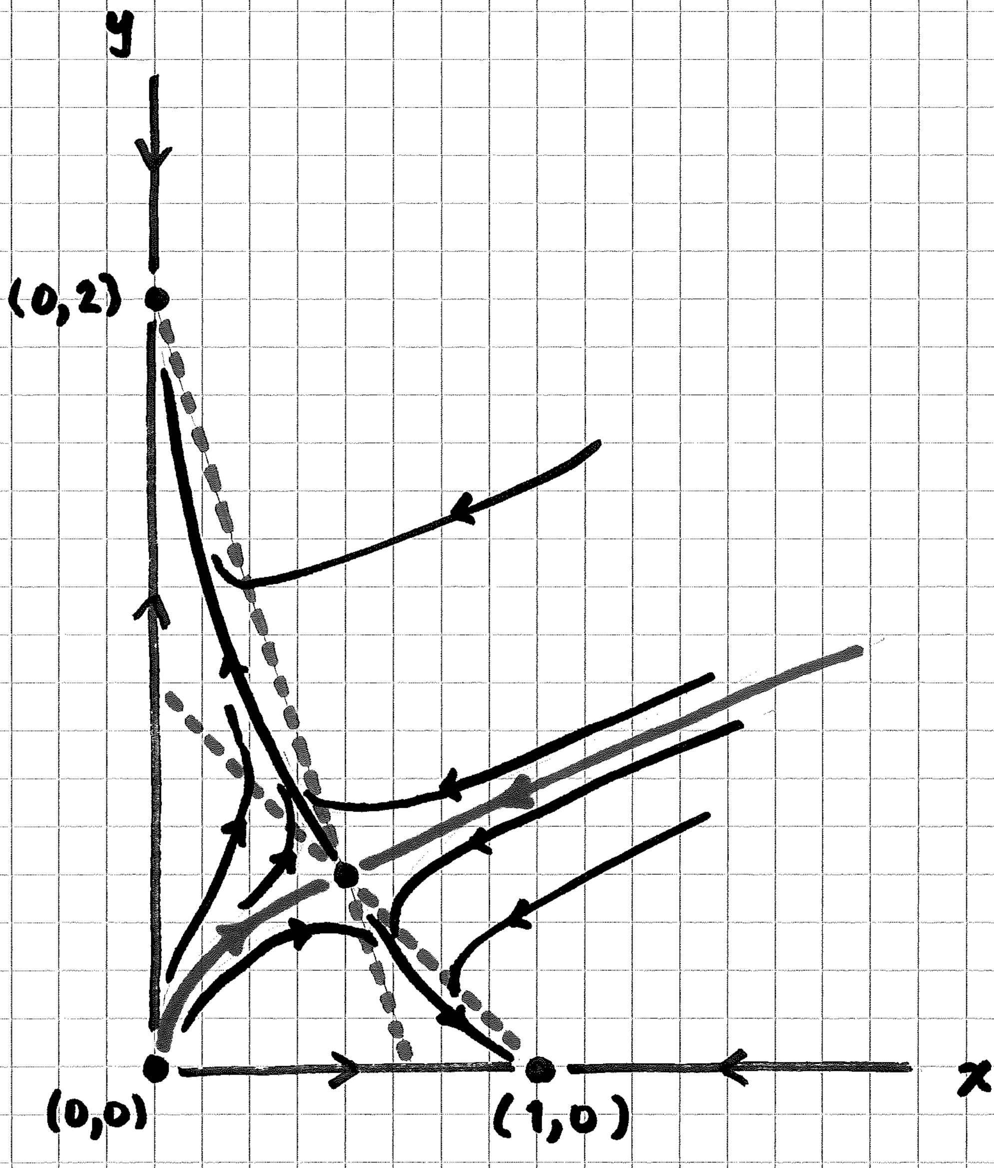
Both nullclines



Critical points at intersections of $x'=0$ nullcline (red) & $y'=0$ nullcline (green)

To save time, you can draw the phase portrait on top of the last picture.

Finally, the phase portrait (incorporating all above work, and local phase portraits near each critical point consistent with classification of behaviour & nullclines)



- For this model, only $x, y \geqslant 0$ are relevant
- Two separatices of the saddle (special trajectories shown in blue) separate behaviour of $(x(t), y(t))$ as $t \to \infty$

- Eigenvector directions for linearizations can be useful, if correctly calculated and correctly interpreted, but take more time so are optional

- MATLAB could plot a more accurate phase portrait but it is possible to do a reasonable job by hand

- Biological/ecological interpretation: in this particular example, almost all initial populations $(x(0), y(0))$ give solutions where one or the other species goes extinct as $t \to \infty$. This type of behaviour is called "competitive exclusion".