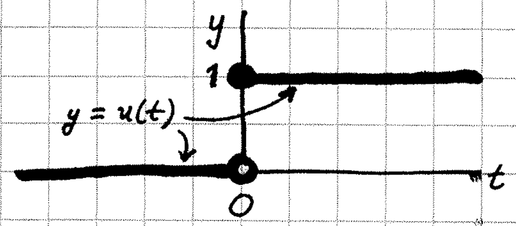
W4 Wed Mar 25 — 30 Mon 2020-03-23

6.1.1, continued

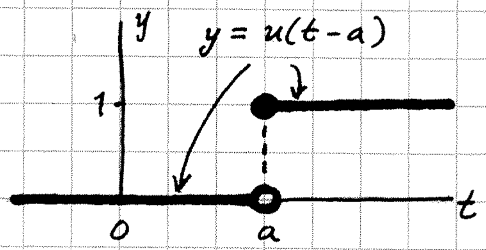
The unit step function (or Heaviside function) is

$$u(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



Example 6.1.B. Use the definition to find $\mathcal{L}\{u(t-a)\}$, where $a \geq 0$ is a constant. (Jump discontinuity at $t=a$)

$$u(t-a) = \begin{cases} 0 & \text{if } t-a<0 \text{ i.e. if } t<a \\ 1 & \text{if } t-a \geq 0 \text{ i.e. if } t \geq a \end{cases}$$



$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

(2) $$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_a^\infty$$

$$= \lim_{b\to\infty} \left[-\frac{1}{s} e^{-sb} \right] - \left[-\frac{1}{s} e^{-sa} \right]$$

$$= \frac{1}{s} e^{-as} \quad \text{(provided } s>0 \text{ so the integral converges)}$$

(In the first integral of (2), we have $0 \leq t \leq a$, and $u(t-a)=0$ for all $0 \leq t < a$. The value of $u(t-a)$ at the endpoint $t=a$ does not affect the value of $\int_0^a e^{-st} u(t-a)\, dt$.)

See Table 6.1 (p.295). Exercise Verify some of it.

Theorem 6.1.1. (Linearity)

$$\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$$

for constants A, B and functions $f(t), g(t)$.

Exercise Prove it. (Use the definition)

For example,

$$\mathcal{L}\{4e^{-3t} - 2u(t-\pi)\} = \frac{4}{s+3} - \frac{2e^{-\pi s}}{s}$$

6.1.2. Existence and uniqueness

Theorem 6.1.2 says, roughly: as long as $f(t)$ doesn't grow faster than an exponential function, $F(s) = \mathcal{L}\{f(t)\}$ exists (converges) for all sufficiently large s.

(All the functions we consider in this course will not grow faster than an exponential function i.e. will be of exponential order)

Theorem 6.1.3 says, roughly: if $f(t)$ and $g(t)$ both have the same Laplace transform, then they must be the same function.

6.1.3. The inverse transform

By Thm. 6.1.3, it makes sense to define the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\} = f(t)$ to be the function whose Laplace transform is $F(s)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$.

For example,

$$\mathcal{L}^{-1}\left\{ \frac{4}{s+3} - \frac{2e^{-\pi s}}{s} \right\} = 4e^{-3t} - 2u(t-\pi)$$

$$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)$$

In practice, we use Table 6.1 to find $\mathcal{L}^{-1}\{F(s)\}$.

<u>Example</u> 6.1.C. Find $\mathcal{L}^{-1}\left\{ \frac{3s+4}{s^2+s-2} \right\}$

Can't find anything like this $F(s)$ in Table 6.1

Does s^2+s-2 factor (over the reals)? (In quadratic formula, is "$b^2-4ac > 0$"?)

$$s^2+s-2 = (s+2)(s-1)$$

Partial fractions:

$$\frac{3s+4}{s^2+s-2} = \frac{3s+4}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$3s+4 = A(s-1) + B(s+2)$$

Substitute $s=1 \Rightarrow B = 7/3$

Substitute $s=-2 \Rightarrow A = 2/3$

$$\frac{3s+4}{s^2+s-2} = \frac{2}{3}\left(\frac{1}{s+2}\right) + \frac{7}{3}\left(\frac{1}{s-1}\right)$$

$$\mathcal{L}^{-1}\left\{ \frac{3s+4}{s^2+s-2} \right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{7}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{2}{3} e^{-2t} + \frac{7}{3} e^{t}$$